# Differential Renormalization of Supersymmetric Gauge Theories in Superspace

by

Yun S. Song

Submitted to the Department of Physics
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Author . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Department of Physics
May 10, 1996

Certified by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Daniel Z. Freedman
Professor of Mathematics
Thesis Supervisor

Certified by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
Alan H. Guth
Professor of Physics
Thesis Supervisor

Accepted by . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
June L. Matthews
Senior Thesis Coordinator, Department of Physics

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## Abstract

Differential regularization is extended to superspace in an attempt to renormalize $N = 1$ supersymmetric gauge theories. The general formalism that underlies the structure of superspace is described, and the quantization of supersymmetric gauge theories in superspace is considered in detail. Superspace background field method is discussed and employed to study supersymmetric quantum field theories. To verify the consistency of differential renormalization in the superspace background field method, $\beta$-function of supersymmetric quantum electrodynamics is calculated to two-loop order and that of supersymmetric Yang-Mills theory to one-loop order.

Thesis Supervisor: Daniel Z. Freedman
Title: Professor of Mathematics

Thesis Supervisor: Alan H. Guth
Title: Professor of Physics

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# Chapter 1

# INTRODUCTION

Supersymmetry is an extension of the Poincaré and internal symmetries. It relates particles with different spins, and its algebra is the only graded Lie algebra that is consistent with the S-matrix symmetries in quantum field theories.[1] As a consequence, in addition to being mathematically elegant, supersymmetry might play an important role in physics. In this paper, we omit the standard discussion of the supersymmetry algebra. Therefore, the reader without any prior knowledge in supersymmetry might not benefit much from this paper. Nevertheless, instead of a detailed introduction to supersymmetry, we include below some interesting aspects of supersymmetry and hope that they would interest the newcomers to the subject.[1]

The Standard Model of particle physics based on the gauge group $SU(3)_C \times SU(2)_W \times U(1)_Y$ provides a very accurate description of the strong and electroweak interactions up to present accelerator energies. Its accuracy, however, is not expected to persist at arbitrarily high energies since new phenomena, such as the quantum effects of gravity, are expected to become important. Hence, the Standard Model

[1] To those who become excited enough by this presentation and wish to pursue studying supersymmetry, we recommend Refs.[1] $\sim$ [7] for the supersymmetry algebra and Refs.[8] $\sim$ [18] for more physical aspects of supersymmetry.

is likely to be an effective theory at low energy and is accompanied by problems such as the "hierarchy problem" which is the puzzle of why the ratio between the electroweak scale $\mathcal{O}(100\ GeV)$ and the Planck scale $\mathcal{O}(10^{19}\ GeV)$ is so large. There are currently two plausible solutions suggested in addressing these problems. In one approach, new fundamental fermions and forces[2] are introduced in lieu of the fundamental scalars.[19] The other solution involves a new symmetry that allows the exact cancellations of all quadratic divergences. The simplest model of the latter type is the Minimal Supersymmetric Standard Model (MSSM), in which the symmetry group is a direct product of the standard gauge group and the $N = 1$ supersymmetry. Supersymmetric theories predict that every boson (fermion) has a fermionic (bosonic) superpartner which has the same mass. Since this phenomenon is not observed in nature, supersymmetry must be spontaneously broken if it is an exact symmetry of the fundamental laws. Consequently, theoretical probing of supersymmetry breaking is an important topic of interest.

One of the prominent features of supersymmetry is that it contains much fewer independent parameters than a non-supersymmetric theory with the same particle content and gauge symmetry. Supersymmetry gives relations among observables; and if supersymmetry is softly broken, these relations get modified by radiative corrections. In particular, relations among the observables in the Higgs sector can receive large corrections from radiative effects. This distinctive feature of supersymmetry might solve the hierarchy problem to allow the electroweak breaking scale $\mathcal{O}(100\ GeV)$ to coexist with the unification scale $\mathcal{O}(10^{19}\ GeV)$.

In supersymmetric gauge field theories, certain quantum fluctuations of bosons and fermions cancel, and a certain set of gauge invariant Green's functions are related by supersymmetric Ward-Takahashi identities. Attributed to these features, supersymmetric gauge field theories, unlike their non-supersymmetric analogues, allow for the possibility of exactly calculating the vacuum expectation values of certain gauge invariant composite operators.[20] Since the properties of the vacuum are believed to be determined by the non-perturbative aspects, we then have a means of

[2] Known as technicolor.

studying the non-perturbative features of supersymmetric gauge field theories.

We now end our brief discussion of a few motivating concepts in supersymmetry and plunge into the core of this paper. Perturbative calculations in supersymmetric field theories can be greatly facilitated by using superfield Feynman rules in superspace. In addition to providing numerous simplifications, supergraph techniques maintain supersymmetry manifest throughout the calculations. These advantages are further enhanced in supersymmetric gauge theories if super background field method is utilized to maintain the explicit gauge invariance. However, these improvements are acquired at the cost of having to deal with some new problems. For example, additional infrared divergences[21, 22] hinder loop calculations in the superspace approach. Although this problem can be solved by using a non-local gauge-fixing term[23] with regularization by dimensional reduction[24], the method is rather complicated.

Renormalization of perturbative quantum field theories is a well-defined concept. Pauli-Villars, point splitting, and dimensional regularization are some examples of successful regularization procedures that are widely used. However, although some regularization procedures manifestly preserve the gauge symmetry, there exists no satisfactory procedure that explicitly maintains the significant symmetries of supersymmetric and chiral gauge theories. Differential renormalization is a procedure that has a potential to remedy this problem.

Differential renormalization[25] is a method that regularizes and renormalizes the coordinate space amplitudes that are too singular to have well defined Fourier transformations into the momentum space. No explicit cutoff or counterterms arise in this approach, and the renormalization procedure can be greatly simplified. Furthermore, infrared divergences generally do not appear, and this feature has naturally motivated us to use the technique in studying supersymmetric gauge theories. In this paper, we extend the differential renormalization technique to the aforementioned superspace formalism. Although differential renormalization becomes rather cumbersome to use in gauge theories with complicated tensor structures, we have observed that the superspace approach reduces some of this burden. In order to check the consistency of our work, we compute the $\beta$-functions of supersymmetric gauge theories and compare the results to previous calculations cited in the literature.

This paper is organized as follows: In Chapter 2, we introduce supersymmetric gauge theories in superspace, as well as some important formalisms that underlie the structure of superspace. In Chapter 3, we extend the ordinary background field method to superspace. We first present the basic mathematical development required for the discussion and, then, apply the method to both abelian and non-abelian supersymmetric gauge theories. In order to familiarize the reader with differential renormalization, we give a brief introduction to the method in Chapter 4. Chapter 5 contains materials directly pertinent to the title of this paper. The reader already familiar with superspace and differential renormalization may feel free to jump to this chapter. In Section 5.1 we differentially renormalize supersymmetric quantum electrodynamics and derive the $\beta$-function to two-loop order. We discuss the renormalization of supersymmetric Yang-Mills theory and its one-loop $\beta$-function in Section 5.2. Finally, we make some concluding remarks in Chapter 6.

Anyone who has ever tried to master supersymmetry would know that there are too many different conventions used in the literature. Some conventions are preferred over others for reasons of simplifying computations, saving time and paper, or perhaps personal preference. However, it is important to develop a consistent set of conventions in order to carry out a coherent communication, so we need to fix our conventions before we can embark on any serious investigation. This task is done in Appendix A. In Appendix B, we discuss the Gegenbauer technique of evaluating Feynman integrals.

# Chapter 2

# SUPERSYMMETRIC GAUGE THEORIES IN SUPERSPACE

In this chapter, we discuss the supersymmetric extensions of quantum electrodynamics and Yang-Mills theory. Since the latter is conceptually more difficult than the former, we discuss the latter first. With the exception of Chapter 4, we work in superspace formalism throughout this paper. Hence, we begin with a brief introduction to superspace in Section 2.1. We then discuss the theoretical framework of supersymmetric Yang-Mills theory in Section 2.2, and supersymmetric quantum electrodynamics in Section 2.3. In Section 2.4, we summarize the super Feynman rules for massless supersymmetric gauge theories.

## 2.1 A Modest Introduction to $N = 1$ Superspace

### 2.1.1 General Formalism

$N = 1$ superspace is an 8-dimensional manifold which can describe off-shell field representations of the supersymmetry algebra. It has 4 familiar spacetime coordinates $x^a$ and 4 Majorana spinors, $\theta^\alpha$ ($\alpha = 1, 2$) and $\overline{\theta}^{\dot{\alpha}}$ ($\dot{\alpha} = \dot{1}, \dot{2}$). While $x^a$ coordinates satisfy the usual condition

$$[x^a, x^b] \; = \; 0, \qquad (2.1.1)$$

$\theta^\alpha$ and $\overline{\theta}^{\dot{\beta}}$ form a Grassmann algebra

$$\{\theta^\alpha, \theta^\beta\} \; = \; \{\overline{\theta}^{\dot{\alpha}}, \overline{\theta}^{\dot{\beta}}\} \; = \; \{\theta^\alpha, \overline{\theta}^{\dot{\beta}}\} \; = \; 0. \qquad (2.1.2)$$

Just as the ordinary spacetime can be viewed as the coset space Poincaré/Lorentz, superspace can be considered as the coset space Super-Poincaré/Lorentz [17, 27, 12]. Rather than belaboring the mathematical development that underlies this structure, let us discuss some important implications of taking this view. First, define $\varepsilon^\alpha$ and $\overline{\varepsilon}^{\dot{\alpha}}$ to be anticommuting Majorana spinors that satisfy

$$[\varepsilon^\alpha, \text{anything}\} \; = \; 0 \qquad \text{and} \qquad [\overline{\varepsilon}^{\dot{\alpha}}, \text{anything}\} \; = \; 0, \qquad (2.1.3)$$

where $[\ ,\ \}$ is the graded Lie product, equivalent to a commutator when "anything" is bosonic, and an anticommutator when "anything" is fermionic. Then, the supersymmetry transformation on superspace is realized by the transformations[27]

$$x^a \;\longrightarrow\; x'^a \; = \; x^a - i\left(\varepsilon^\alpha \sigma^a_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} + \overline{\varepsilon}^{\dot{\alpha}} \sigma^a_{\alpha\dot{\alpha}} \theta^\alpha\right), \qquad (2.1.4)$$

$$\theta^\alpha \;\longrightarrow\; \theta'^\alpha \; = \; \theta^\alpha + \varepsilon^\alpha \qquad \text{and} \qquad \overline{\theta}^{\dot{\alpha}} \;\longrightarrow\; \overline{\theta}'^{\dot{\alpha}} \; = \; \overline{\theta}^{\dot{\alpha}} + \overline{\varepsilon}^{\dot{\alpha}}. \qquad (2.1.5)$$

We observe that in order for the transformations to preserve correct dimensions, $\theta, \overline{\theta}, \varepsilon$ and $\overline{\varepsilon}$ all must have mass dimension $-\frac{1}{2}$. The generators $Q_\alpha$ and $\overline{Q}_{\dot{\alpha}}$ of supersymmetry are represented by the following differential operators on the supermanifold[27]:

$$Q_\alpha \;=\; i\left(\frac{\partial}{\partial\theta^\alpha} \;-\; i\,\overline{\theta}^{\dot\beta}\sigma^a_{\alpha\dot\beta}\,\partial_a\right) \quad \text{and} \quad \overline{Q}_{\dot\alpha} \;=\; i\left(\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \;-\; i\,\theta^\beta\sigma^a_{\beta\dot\alpha}\,\partial_a\right) \qquad (2.1.6)$$

As expected from the supersymmetry algebra, these supercharges have mass dimension $+\frac{1}{2}$. It is trivial to check that they satisfy the correct anticommutation rules[1]

$$\{Q_\alpha, \overline{Q}_{\dot\alpha}\} \;=\; -2i\,\sigma^a_{\alpha\dot\alpha}\partial_a \qquad \text{and} \qquad \{Q_\alpha, Q_\beta\} \;=\; 0 \;=\; \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\}. \qquad (2.1.7)$$

The supercharges $Q_\alpha$ and $\overline{Q}_{\dot\alpha}$ generate coordinate transformations in superspace by mixing the $x^a$ coordinates with the $\theta^\alpha$ and $\overline{\theta}^{\dot\alpha}$ coordinates. We call this a supertranslation in contrast to the Poincaré transformations which do not mix the two types of coordinates. For example, the translation generators $P^a = i\partial_a$ of the Poincaré group transform the coordinates as

$$x^a \;\longrightarrow\; x'^a \;=\; x^a + c^a, \qquad (2.1.8)$$

$$\theta^\alpha \;\longrightarrow\; \theta'^\alpha \;=\; \theta^\alpha \qquad \text{and} \qquad \overline{\theta}^{\dot\alpha} \;\longrightarrow\; \overline{\theta}'^{\dot\alpha} \;=\; \overline{\theta}^{\dot\alpha}, \qquad (2.1.9)$$

where $c^a$ is a constant spacetime vector. In ordinary quantum field theories, the spacetime derivative $\partial_a$ is translation invariant, since

$$[\partial_a, P_b] \;=\; [\partial_a, i\partial_b] \;=\; 0. \qquad (2.1.10)$$

In supersymmetric quantum field theories, however, the supertranslation generators $Q_\alpha$ and $\overline{Q}_{\dot\alpha}$ are not invariant under supertranslations, because the anticommutator in (2.1.7) does not vanish. We need to find derivatives, say $D_\alpha$ and $\overline{D}_{\dot\alpha}$, that are invariant under super and ordinary translations; that is,

$$\{Q_\alpha, D_\beta\} \;=\; \{\overline{Q}_{\dot\alpha}, D_\beta\} \;=\; \{Q_\alpha, \overline{D}_{\dot\beta}\} \;=\; \{\overline{Q}_{\dot\alpha}, \overline{D}_{\dot\beta}\} \;=\; 0 \qquad (2.1.11)$$

[1] These rules correspond to the case with vanishing central charges.

and

$$[P_a, D_\beta] \;=\; [P_a, \overline{D}_{\dot\beta}] \;=\; 0. \tag{2.1.12}$$

We state without proof that the explicit representations of these derivatives are[27]

$$D_\alpha \;=\; \left(\frac{\partial}{\partial\theta^\alpha} \;+\; i\,\overline{\theta}^{\dot\beta}\sigma^a_{\alpha\dot\beta}\,\partial_a\right) \qquad \text{and} \qquad \overline{D}_{\dot\alpha} \;=\; \left(\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \;+\; i\,\theta^\beta\sigma^a_{\beta\dot\alpha}\,\partial_a\right). \tag{2.1.13}$$

These derivatives are covariant with respect to the Poincaré, chiral, and isospin transformations[27], and they are appropriately called the "covariant derivatives." Using (2.1.13), one can show that the covariant derivatives satisfy the following identities:

$$\{D_\alpha, \overline{D}_{\dot\alpha}\} \;=\; 2i\,\sigma^a_{\alpha\dot\alpha}\,\partial_a, \tag{2.1.14}$$

$$\left[D^\alpha, \overline{D}^2\right] = 4i\,(\sigma^a)^{\alpha\dot\alpha}\,\partial_a\overline{D}_{\dot\alpha}, \tag{2.1.15}$$

$$\{D_\alpha, D_\beta\} \;=\; 0 \;=\; \{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\}, \tag{2.1.16}$$

$$D^2\overline{D}^2D^2 \;=\; 16\Box D^2\,, \qquad \overline{D}^2D^2\overline{D}^2 \;=\; 16\Box\overline{D}^2, \tag{2.1.17}$$

$$D^n \;=\; 0 \;=\; \overline{D}^n\,, \qquad \forall\; n\geq 3,\; n\in\mathbb{Z}^+, \tag{2.1.18}$$

$$\begin{aligned} D_i\left[\left(D_1\cdots D_m\overline{D}_1\cdots\overline{D}_nF\right)G\right] \;&=\; \left(D_iD_1\cdots D_m\overline{D}_1\cdots\overline{D}_nF\right)G \\ &\quad + (-1)^{(m+n)}\left(D_1\cdots D_m\overline{D}_1\cdots\overline{D}_nF\right)D_iG. \end{aligned} \tag{2.1.19}$$

### 2.1.2 Superfields

Superfields $S(x,\theta,\overline{\theta})$ are multispinor funtions on superspace and give linear representations of the supersymmetry algebra [17, 18, 26]. Superfields transform as scalars under supersymmetry and as multispinors under the Lorentz symmetry. Component fields are obtained from a superfield by expanding the superfield in terms of $\theta$ and $\overline{\theta}$ as follows:

$$S(x,\theta,\overline{\theta}) \quad = \quad s(x) + \theta\eta(x) + \overline{\theta}\,\overline{\xi}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) + \theta\sigma^a\overline{\theta}A_a(x)$$

$$+ \theta\theta\overline{\theta}\,\overline{\lambda}(x) + \overline{\theta\theta}\theta\psi(x) + \theta\theta\overline{\theta\theta}d(x) \tag{2.1.20}$$

The expansion terminates at the $\theta\theta\overline{\theta\theta}$ level, since $\theta_\alpha$ and $\overline{\theta}_{\dot{\alpha}}$ are Grassmann variables. The component fields shown in (2.1.20) form a general multiplet, which is reducible. Hence, we need to impose covariant constraints to obtain irreducible representations on superfields. In particular, a superfield $\Phi$ characterized by the constraint[2]

$$\overline{D}^{\dot{\alpha}}\Phi = 0 \tag{2.1.21}$$

is called a **chiral** superfield. Similarly, a superfield $\overline{\Phi}$ satisfying the constraint[3]

$$D^\alpha\overline{\Phi} \;=\; 0 \tag{2.1.22}$$

is called an **antichiral** superfield. Sometimes, $\Phi$ and $\overline{\Phi}$ are collectively called scalar superfields. These superfields contain the matter fields as their components. Another type of superfield obeying the condition

$$V \;= \overline{V} \tag{2.1.23}$$

is called a **real** or a **vector** superfields. Vector superfields play the analogous role of gauge fields in ordinary quantum field theories; in fact, the ordinary gauge field resides in the vector super-multiplet. We will not attempt to write down the explicit component expansions of the constrained superfields. This omission should be forgivable since we will not make any reference to component fields in this paper. Also, we believe that this is one less convention by which the reader might get confused. However, one very important fact needs to be discussed about component fields. Supersymmetry transformation for a superfield is defined as[17]

$$\delta_\varepsilon S(x,\theta,\overline{\theta}) \;\equiv\; \left(\varepsilon Q + \overline{\varepsilon}\overline{Q}\right) S(x,\theta,\overline{\theta}), \tag{2.1.24}$$

---

[2] $\overline{D}^{\dot{\alpha}}$ is defined in (2.1.13)
[3] $D^\alpha$ is also defined in (2.1.13)

where $Q$ and $\epsilon$ were defined in the last section. We know that $Q$ transforms a component field into other component fields with mass dimensions less or greater than the original field by $\frac{1}{2}$. However, the component field with the highest dimension can only transform to fields whose dimensions are less by $\frac{1}{2}$. That is, in terms for the fields shown in (2.1.20), the supersymmetric variation of $d$ contains $\psi$ and $\overline{\lambda}$, but there is no field in the multiplet that has higher dimension than $d$. We also know that the dimension of $\varepsilon$ is $-\frac{1}{2}$. Therefore, since the supersymmetric variation of a field must have the same dimension as the field itself, we conclude that the variation $\delta_\varepsilon d$ contains terms proportional to a total spacetime derivative of $\psi$ and $\overline{\lambda}$. It is generally true that the component field with the highest mass dimension transforms into a spacetime divergence. Another important point that one should bare in mind is that all renormalizable supersymmetric theories can be constructed in terms of vector and scalar superfields.

### 2.1.3 Integration and Functional Differentiation in Superspace

Since superspace has anticommuting coordinates, we need to define the notion of integration over anticommuting variables. Integration over Grassmannian variables should be familiar to those who have worked with the path integral formalism in quantizing gauge theories. The Berezin integral[28] for a one-dimensional anticommuting variable $\eta$ is defined as

$$\int d\eta \;= 0 \qquad \text{and} \qquad \int d\eta\, \eta \;=\; 1. \tag{2.1.25}$$

If we adopt this definition into superspace, it is straightforward to show that

$$\int \mathrm{d}^2\theta\, \theta^2 \;=\; 1 \qquad \text{and} \qquad \int \mathrm{d}^2\overline{\theta}\, \overline{\theta}^2 \;=\; 1. \tag{2.1.26}$$

Hence, it appears that integration is equivalent to differentiation; that is, we can formally define

$$\int d^2\theta \;\equiv\; -\frac{1}{4}\partial^\alpha\partial_\alpha \qquad \text{and} \qquad \int d^2\overline{\theta} \;\equiv\; -\frac{1}{4}\overline{\partial}^{\dot{\alpha}}\overline{\partial}_{\dot{\alpha}}. \tag{2.1.27}$$

Under an integral over the spacetime measure $d^4x$ , the spinoral derivatives $\partial^\alpha$ and $\partial^{\dot{\alpha}}$ may be replaced by the covariant derivatives $D^\alpha$ and $\overline{D}^{\dot{\alpha}}$, respectively. For example,

$$\int d^8z \;\; S(x,\theta,\overline{\theta}) \;\equiv\; \int d^4x \; d^2\theta \; d^2\overline{\theta} \; S(x,\theta,\overline{\theta}) \;=\; \int d^4x \;\; \frac{D^2\overline{D}^2}{16} \; S(x,\theta,\overline{\theta}). \tag{2.1.28}$$

Notice that the covariant derivatives $D^2\overline{D}^2$ acting on the superfield $S(x,\theta,\overline{\theta})$ will pick out the component with the highest mass dimension. However, we argued in the last section that this component field transforms into a spacetime divergence under supersymmetry transformation. Thus, we conclude that the integral shown in (2.1.28) is invariant under supersymmetry transformations. As we will discuss in the forthcoming sections, this is how invariant actions for supersymmetric quantum field theories are constructed.

Having defined how we can integrate over Grassmannian variables, we proceed to consider a delta function in $\theta$-space. A very sensible definition of $\delta^4(\theta_1 - \theta_2) \equiv \delta_{12}$ is

$$\int d^2\theta_1 \; d^2\overline{\theta}_1 \;\; f(\theta_1,\overline{\theta}_1) \; \delta_{12} \;=\; f(\theta_2,\overline{\theta}_2), \tag{2.1.29}$$

where $f$ is an arbitrary function. Consider the case where $f = 1$. Then, (2.1.28) implies that

$$\frac{1}{16} \; D_1^2\overline{D}_1^2 \; \delta_{12} \;=\; 1 \tag{2.1.30}$$

under a spacetime integral. A specific representation that satisfies this condition is

$$\delta_{12} \;\equiv\; 4(\theta_1 - \theta_2)^2 \; (\overline{\theta}_1 - \overline{\theta}_2)^2, \tag{2.1.31}$$

and we will use this definition throughout this paper. An immediate consequence of adopting this definition is

$$D_i^2 \delta^2(\theta_{ij}) \;=\; D_i^2 \; 2\left(\theta_i - \theta_j\right)^2 \;=\; 4 \tag{2.1.32}$$

Since $\theta_\alpha$ and $\overline{\theta}_{\dot\alpha}$ are Grassmannian variables, we know that

$$(\theta_i^\alpha)^n \;=\; 0 \;= \left(\overline{\theta}_i^{\dot\alpha}\right)^n \;, \qquad \forall\; n \geq 2,\; n \in \mathbb{Z}^+, \tag{2.1.33}$$

and this implies that

$$\delta_{12}\; \theta_1^\alpha \;=\; \delta_{12}\; \overline{\theta}_1^{\dot\alpha} \;=\; \delta_{12}\; \theta_2^\alpha \;=\; \delta_{12}\; \overline{\theta}_2^{\dot\alpha} \;=\; 0. \tag{2.1.34}$$

Consequences of this property that will be of great importance to us in evaluating super Feynman graphs are

$$\begin{aligned}
\delta_{12}\delta_{12} \;&=\; \delta_{12} D_\alpha \delta_{12} \;=\; \delta_{12} \overline{D}_{\dot\alpha} \delta_{12} \;=\; \delta_{12} D^2 \delta_{12} \;=\; \delta_{12} \overline{D}^2 \delta_{12} \\
&=\; \delta_{12} D_\alpha \overline{D}_{\dot\alpha} \delta_{12} \;=\; \delta_{12} D^2 \overline{D}_{\dot\alpha} \delta_{12} \;=\; \delta_{12} D_\alpha \overline{D}^2 \delta_{12} \\
&=\; 0,
\end{aligned} \tag{2.1.35}$$

and

$$\delta_{12} D^2 \overline{D}^2 \delta_{12} \;=\; \delta_{12} \overline{D}^2 D^2 \delta_{12} \;=\; \delta_{12} D^\alpha \overline{D}^2 D_\alpha \delta_{12} \;=\; \delta_{12} \overline{D}^{\dot\alpha} D^2 \overline{D}_{\dot\alpha} \delta_{12} \;=\; 16\; \delta_{12}. \tag{2.1.36}$$

Another property that we will use repeatedly in Chapter 5 is the transfer rule defined as

$$D_\alpha(z_1) \left[\delta^{(4)}(\theta_1 - \theta_2) f(x_1 - x_2)\right] \;=\; -D_\alpha(z_2) \left[\delta^{(4)}(\theta_1 - \theta_2) f(x_1 - x_2)\right]. \tag{2.1.37}$$

We complete this section with a remark on functional differentiation in superspace.

Functional differentiation for vector superfields $V$ is defined as

$$\frac{\delta V(z_1)}{\delta V(z_2)} \;=\; \delta^8(z_{12}). \tag{2.1.38}$$

For chiral superfields, however, the defining condition $\overline{D}_{\dot\alpha}\Phi$ must be extended to

$$\overline{D}_{\dot\alpha}\frac{\delta\Phi(z_1)}{\delta\Phi(z_2)} \;=\; 0. \tag{2.1.39}$$

The same ideas apply to antichiral superfields as well. Hence, we judiciously define[29]

$$\frac{\delta\Phi(z_1)}{\delta\Phi(z_2)} \;=\; -\frac{1}{4}\overline{D}_1^2\delta^8(z_{12}) \qquad \text{and} \qquad \frac{\delta\overline{\Phi}(z_1)}{\delta\overline{\Phi}(z_2)} \;=\; -\frac{1}{4}D_1^2\delta^8(z_{12}) \tag{2.1.40}$$

## 2.2 Non-Abelian Theory (SUSY Yang-Mills)

The normalized generating functional[4] for an ordinary pure gauge field theory is given by

$$Z[J] \;=\; N\int(\mathcal{D}A)\; e^{[S_o(A)+S_s(J,A)]}, \tag{2.2.41}$$

where $S_o$ is the gauge invariant classical action for the gauge field $A$, $S_s$ the gauge breaking source term for source $J$, and $N$ the normalization constant. In supersymmetric gauge theories, the role of gauge field $A$ is taken up by a real superfield $V$, $J$ is also a real superfield, and the normalization constant $N$ is equal to unity [27]. For example, the super generating functional for supersymmetric Yang-Mills theory takes the form

$$Z[J] \;=\; \int(\mathcal{D}V)\, e^{[S_o(V)+S_s(J,V)]}, \tag{2.2.42}$$

where the gauge invariant classical action is

$$S_o \;=\; \frac{1}{64g^2}\mathrm{Tr}\int\; \mathrm{d}^4x\;\mathrm{d}^2\theta\; W^\alpha W_\alpha\;, \qquad W^\alpha \;=\; \overline{D}^2\left(e^{-gV}D^\alpha e^{gV}\right), \tag{2.2.43}$$

[4] We work in Euclidean $x$-space throughout this paper.

and the source term is

$$S_s \;=\; \mathrm{Tr} \int \mathrm{d}^8 z \;\; JV. \tag{2.2.44}$$

Since this is a non-abelian theory, the superfield $V$ is Lie-algebra-valued; that is, $V = V^a T_a$, where $T_a$ are the generators of the gauge group. Furthermore, the generators satisfy

$$\begin{aligned} [T_a, T_b] &= i f_{ab}{}^{c}\, T_c\ , & f_{acd} f_{bcd} &= C_2(G)\delta_{ab}, \\ \mathrm{Tr}(T_a T_b) &= T(R)\delta_{ab}\ , & T_a T_a &= C(R) I. \end{aligned} \tag{2.2.45}$$

The action shown in (2.2.43) is real, except for possible surface terms–for example, from instanton contributions [30]. This absence of the hermitian conjugate will greatly reduce the number of supergraphs we need to consider.

We proceed now to discuss the underlying gauge symmetry of the theory. It is trivial to show that (2.2.43) is invariant under the gauge transformations

$$e^{V'} \;=\; e^{i\overline{\Lambda}}\; e^{V}\; e^{-i\Lambda}, \tag{2.2.46}$$

where $\Lambda$ and $\overline{\Lambda}$ are Lie-algebra-valued chiral and anti-chiral superfields, respectively. In order to determine how $V$ transforms under (2.2.46), we can use the Baker-Campbell-Hausdorff formula[18]

$$\exp{(M)}\exp{(N)} \;=\; \exp\Big\{M + \pounds_{M/2}\cdot\Big[N + \Big(\coth \pounds_{M/2}\Big)\cdot N\Big] + \cdots\Big\}, \tag{2.2.47}$$

where $\pounds_{M/2}$ is the Lie derivative whose action is defined as $\pounds_{M/2}\cdot N = [M/2, N]$, and $\coth\Big(\pounds_{M/2}\Big)$ is given by its power series expansion with the definition

$$\Big(\pounds_{M/2}\Big)^k \cdot N \;=\; \underbrace{\Big[\frac{M}{2}, \Big[\frac{M}{2}, \Big[\cdots\Big[\frac{M}{2}}_{k}\,,\; N\Big]\cdots\Big]\Big]\Big]. \tag{2.2.48}$$

Upon using (2.2.47) and (2.2.48), we obtain the following transformation law for $V$

[27]:

$$\delta V \;=\; V' - V \;=\; i\pounds_{V/2}\left[-\left(\overline{\Lambda}+\Lambda\right)+\left(\coth \pounds_{V/2}\right)\left(\overline{\Lambda}-\Lambda\right)\right] \tag{2.2.49}$$

By using (2.1.27) and expanding the superfield strength $W^\alpha$ in terms of the vector superfield $V$, we can rewrite (2.2.43) as

$$\begin{aligned} S_o &= -\frac{1}{16g^2}\,\mathrm{Tr}\int \mathrm{d}^8 z \;\; \left(e^{-gV}D^\alpha\, e^{gV}\right)\overline{D}^2\left(e^{-gV}D_\alpha\, e^{gV}\right) \\ &= \frac{1}{16}\,\mathrm{Tr}\int \mathrm{d}^8 z\;\; \Big\{ VD^\alpha\overline{D}^2 D_\alpha V \;+\; \frac{1}{16}\,gV\left\{D^\alpha V, \overline{D}^2 D_\alpha V\right\} \\ &\qquad\qquad\qquad + \text{Higher order in } V\Big\}. \end{aligned} \tag{2.2.50}$$

The first term in (2.2.50) can be rewritten as

$$VD^\alpha\overline{D}^2 D_\alpha V \;=\; -8V\Box\Pi_{1/2}V, \tag{2.2.51}$$

where $\Pi_{1/2}$ is one of the superspin projection operators $\Pi_i = (\Pi_{0+}, \Pi_{1/2}, \Pi_{0-}), i \in \{1,2,3\}$, defined as

$$\Pi_{1/2} \;\equiv\; -\frac{D^\alpha\overline{D}^2 D_\alpha}{8\Box} \tag{2.2.52}$$

and

$$\Pi_{0+} \;\equiv\; \frac{\overline{D}^2 D^2}{16\Box}, \qquad \Pi_{0-} \;\equiv\; \frac{D^2\overline{D}^2}{16\Box}, \qquad \Pi_0 \;\equiv\; \Pi_{0+}+\Pi_{0-}. \tag{2.2.53}$$

As the usual projection operators do, the superspin projection operators satisfy

$$\Pi_{0+}+\Pi_{1/2}+\Pi_{0-} \;=\; \Pi_0+\Pi_{1/2} \;=\; 1 \qquad \text{and} \qquad \Pi_i\Pi_j \;=\delta_{ij}\Pi_i. \tag{2.2.54}$$

Given these facts, we note that the kinetic operator $\Box\Pi_{1/2}$ in (2.2.51) is not invertible because the superspin zero part $V_o \equiv \Pi_o V$ gets annihilated by $\Pi_{1/2}$. This failure should not be a big surprise since we have not fixed the gauge yet. However, we can infer from the failure that the gauge fixing term should take the form $V\Box\Pi_0 V$.

Indeed, as it will be discussed in Section 2.4, if we take the gauge fixing term to be

$$-\frac{1}{\alpha}\mathrm{Tr}\int \mathrm{d}^8 z\ \frac{1}{2}V\Box\Pi_0 V \ = \ -\frac{1}{16\alpha}\mathrm{Tr}\int \mathrm{d}^8 z\ \left(D^2 V\right)\left(\overline{D}^2 V\right), \tag{2.2.55}$$

then the part of the action quadratic in $V$ becomes invertible. Before we proceed with calculating the propagator, let us first consider how we could implement the gauge fixing term into the action while preserving unitarity. As in ordinary field theories[31, 32, 33], unitarity can be maintained by adding ghost superfields to the action. An appropriate gauge fixing function is

$$F \ = \ \overline{D}^2 V - f = 0 \qquad \left(\text{or } \ \overline{F} \ = \ D^2 V - \overline{f} = 0\right). \tag{2.2.56}$$

The Faddeev-Poppov determinant $\Delta_{FP}$ corresponding to this choice of gauge is

$$\Delta_{FP} \ = \ \int \left(\mathcal{D}\Lambda\mathcal{D}\overline{\Lambda}\right)\ \delta\left(D^2 V' - \overline{f}\right)\delta\left(\overline{D}^2 V' - f\right), \tag{2.2.57}$$

where $V'$, as originally defined in (2.2.49), is

$$V' \ = \ V + \xi(V)\Lambda + \overline{\xi}(V)\overline{\Lambda} + \cdots. \tag{2.2.58}$$

After inserting an identity **1** in the form of (2.2.57), (2.2.42) becomes

$$\begin{aligned} Z[J] \ &= \ \int (\mathcal{D}V)\ \Delta_{FP}^{-1}\int \left(\mathcal{D}\Lambda\mathcal{D}\overline{\Lambda}\right)\delta\left(D^2 V' - \overline{f}\right)\delta\left(\overline{D}^2 V' - f\right)e^{(S_o+S_s)} \\ &= \ \int (\mathcal{D}V)\ \Delta_{FP}^{-1}\int \left(\mathcal{D}\Lambda\mathcal{D}\overline{\Lambda}\right)\delta\left(D^2 V - \overline{f}\right)\delta\left(\overline{D}^2 V - f\right)e^{(S_o+S_s)}, \end{aligned} \tag{2.2.59}$$

where the last equation follows from a gauge transformation. Because of the delta functions, we only need to consider $\Delta_{FP}^{-1}$ for $D^2 V = \overline{f}$ and $\overline{D}^2 V = f$. Furthermore, since the main contribution to $\Delta_{FP}^{-1}$ comes from $\Lambda$, $\overline{\Lambda} \sim 0$, we may ignore terms with more than one $\Lambda$ or $\overline{\Lambda}$ [17]; so, we have

$$\Delta_{FP}| = \int \left(\mathcal{D}\Lambda\mathcal{D}\overline{\Lambda}\right) \delta\left(D^2\left(\xi(V)\Lambda + \overline{\xi}(V)\overline{\Lambda}\right)\right) \delta\left(\overline{D}^2\left(\xi(V)\Lambda + \overline{\xi}(V)\overline{\Lambda}\right)\right), \quad (2.2.60)$$

where the vertical line beside $\Delta_{FP}$ indicates that we are evaluating $\Delta_{FP}$ at $D^2V = \overline{f}, \overline{D}^2V = f$. We can now introduce ghost superfields to obtain the inverse functional determinant

$$\begin{aligned}
\Delta^{-1}_{FP}\Big| &\equiv \int \left(\mathcal{D}c\mathcal{D}c'\mathcal{D}\overline{c}\mathcal{D}\overline{c'}\right) \exp\left(S_{FP}\right) \\
&= \int \left(\mathcal{D}c\mathcal{D}c'\mathcal{D}\overline{c}\mathcal{D}\overline{c'}\right) \exp\left\{-\frac{i}{4}\mathrm{Tr} \int \mathrm{d}^4x \left[\int \mathrm{d}^2\theta\ c'\overline{D}^2\left(\xi(V)c + \overline{\xi}(V)\overline{c}\right)\right.\right. \\
&\qquad \left.\left. + \int \mathrm{d}^2\overline{\theta}\ \overline{c'}D^2\left(\xi(V)c + \overline{\xi}(V)\overline{c}\right)\right]\right\} \\
&= \int \left(\mathcal{D}c\mathcal{D}c'\mathcal{D}\overline{c}\mathcal{D}\overline{c'}\right) \exp\left\{\mathrm{Tr}\int \mathrm{d}^8z\ \times \right. \qquad (2.2.61) \\
&\qquad \left.\left\{\left(\overline{c'} + c'\right) \pounds_{V/2}\left[(\overline{c} + c) + \left(\coth \pounds_{V/2}\right)(c - \overline{c})\right]\right\}\right\}.
\end{aligned}$$

$c$ and $c'$ are chiral ghost superfields while $\overline{c}$ and $\overline{c'}$ are antichiral ghost superfields. We used their chirality and antichirality in the second equation of (2.2.61) to extract a total measure $\mathrm{d}^8z$ . Lastly, we average over $f$ and $\overline{f}$ as follows:

$$\int \left(\mathcal{D}f\mathcal{D}\overline{f}\right)\ \exp\left[-\frac{1}{16g^2\alpha}\int \mathrm{d}^8z\ \overline{f}f\right] \qquad (2.2.62)$$

Hence, the final super generating functional for supersymmetric pure Yang-Mills theory is

$$\begin{aligned}
Z[J] &\equiv \int \left(\mathcal{D}V\mathcal{D}c\mathcal{D}c'\mathcal{D}\overline{c}\mathcal{D}\overline{c'}\right)\ \exp(S_{\mathrm{SUSY\ YM}}) \\
&= \int \left(\mathcal{D}V\mathcal{D}c\mathcal{D}c'\mathcal{D}\overline{c}\mathcal{D}\overline{c'}\right)\ \exp\left\{\frac{1}{64g^2}\mathrm{Tr}\int\ \mathrm{d}^4x\ \mathrm{d}^2\theta\ W^\alpha W_\alpha\right. \\
&\quad - \frac{1}{16\alpha}\mathrm{Tr}\int \mathrm{d}^8z\ \left(D^2V\right)\left(\overline{D}^2V\right) + \mathrm{Tr}\int \mathrm{d}^8z\ \ JV \\
&\quad \left. + \mathrm{Tr}\int \mathrm{d}^8z\ \left(\overline{c'} + c'\right)\pounds_{V/2}\left[(\overline{c} + c) + \left(\coth \pounds_{V/2}\right)(c - \overline{c})\right]\right\}. \quad (2.2.63)
\end{aligned}$$

Interactions between matter and gauge superfields are given by the action

$$S_I \;=\; \int \mathrm{d}^8 z \;\; \overline{\Phi}\; e^{gV}\; \Phi. \tag{2.2.64}$$

$\Phi$ and $\overline{\Phi}$ are chiral and antichiral superfields, respectively, and they transform under gauge transformations as

$$\Phi \;\longrightarrow\; e^{ig\Lambda}\;\Phi \qquad \text{and} \qquad \overline{\Phi} \;\longrightarrow\; e^{-ig\overline{\Lambda}}\;\overline{\Phi}, \tag{2.2.65}$$

thus leaving (2.2.64) gauge invariant. In general, if we wish to consider the matter-gauge interaction, we simply need to change the integration measure, add new sources for matter, and include additional gauge invariant actions in (2.2.63).

## 2.3 Abelian Theory (SQED)

The supersymmetric extension of massless quantum electrodynamics (QED) is given by the following classical action [18]:

$$S_o = \frac{1}{64}\int \mathrm{d}^6 z\; W^\alpha W_\alpha \;+\; \int \mathrm{d}^8 z\left(\overline{\Phi}_+ e^{gV}\Phi_+ \;+\; \overline{\Phi}_- e^{-gV}\Phi_-\right) \tag{2.3.66}$$

$W_\alpha$ is the chiral superfield strength defined as $W_\alpha = \overline{D}^2 D_\alpha V$. Under infinitesimal local gauge transformations, the gauge and matter superfields transform as

$$\delta V = i(\Lambda - \overline{\Lambda}), \tag{2.3.67}$$

$$\delta\Phi_\pm = \mp ig\Lambda\Phi_\pm, \qquad \delta\overline{\Phi}_\pm = \pm ig\overline{\Lambda}\;\overline{\Phi}_\pm. \tag{2.3.68}$$

$\Lambda$ and $\overline{\Lambda}$ are chiral and anti-chiral superfields, respectively. As in the non-abelian case (c.f. (2.2.51)), the kinetic operator for the gauge superfield is not invertible, and we need to fix the gauge. Usual steps can be taken to obtain the same gauge fixing term as in (2.2.55); of course, the trace can be ignored since we are currently dealing with an abelian gauge symmetry. However, since (2.3.67) does not depend on $V$, the abelian analogue of the Faddeev-Poppov determinant in (2.2.57) can be dropped from the generating functional. In other words, we don't need to introduce ghost superfields in the abelian theory. Hence, the complete supersymmetric quantum electrodynamics (SQED) action including the source terms is given by

$$\begin{aligned} S_{SQED} \;=\; & \frac{1}{64}\int \mathrm{d}^6 z\; W^\alpha W_\alpha \;+\; \int \mathrm{d}^8 z \left(\overline{\Phi}_+ e^{gV}\Phi_+ \;+\; \overline{\Phi}_- e^{-gV}\Phi_-\right) \\ & - \frac{1}{16\alpha}\int \mathrm{d}^8 z (D^2 V)(\overline{D}^2 V) \qquad (2.3.69) \\ & + \int \mathrm{d}^6 z \,(j_-\Phi_+ \;+\; j_+\Phi_-) \;+\; \int \mathrm{d}^6 \overline{z} \left(\overline{j}_-\overline{\Phi}_+ \;+\; \overline{j}_+\overline{\Phi}_-\right) \;+\; \int \mathrm{d}^8 z\; JV, \end{aligned}$$

where $J$ and $V$ denote vector superfields; $\Phi_\pm$ and $j_\pm$ chiral superfields; and $\overline{j}_\pm$ and $\overline{\Phi}_\pm$ anti-chiral superfields. Lastly, we remark that the coupling constant $g$ is related to the usual electric charge $e$ in QED by $g = \sqrt{2}e$. This definition is necessary to have the correct coefficients for the component field strength $F_{\mu\nu}F^{\mu\nu}$ and for the gauge connection in the spacetime covariant derivative.

## 2.4 Super Feynman Rules

In this section, we discuss the super Feynman rules for SQED and SUSY Yang-Mills theory. We first derive the propagators for the gauge and matter superfields and, then, discuss the rules for vertices.

As can be seen from (2.2.50), (2.2.51) and (2.2.55), the part of the action quadratic in $V$ is

$$-\,\mathrm{Tr}\int \mathrm{d}^8 z\;\frac{1}{2}\left[V\Box\left(\Pi_{1/2} + \frac{1}{\alpha}\Pi_0\right)V\right]. \qquad (2.4.70)$$

This expression is easily invertible, and we obtain the following gauge superfield propagator without much effort:

$$\langle T\; V(z_1)V(z_2)\rangle \;=\; \left(\Pi_{1/2} + \alpha\Pi_0\right)\frac{1}{\Box}\,\delta^8(z_{12}). \qquad (2.4.71)$$

In Fermi-Feynman gauge ($\alpha = 1$), by the property of projection operators shown in

(2.2.54), the propagator takes the simple form

$$\langle T\ V(z_1)V(z_2)\rangle \;=\; \frac{1}{\Box}\,\delta^8(z_{12}) \;=\; -\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\,\delta_{12}. \tag{2.4.72}$$

Although we will not deal with the matter-present Yang-Mills theory in this paper, let us also derive the propagator for matter superfields for completeness. The free part of (2.2.64) and source terms, where matter superfields are coupled to the chiral source $j$ and antichiral source $\overline{j}$, are

$$S_{M_o} \;=\; \int \mathrm{d}^8z\ \ \overline{\Phi}\Phi \;+\; \int \mathrm{d}^6z\ \ j\Phi \;+\; \int \mathrm{d}^6\overline{z}\ \ \overline{j}\ \overline{\Phi}. \tag{2.4.73}$$

Before we can perform the super functional integral, we need to promote all integration measures to the full measure $\mathrm{d}^8z$ . Let us explicitly discuss how this can be done for the $j\Phi$ term. Since $\Phi$ is a chiral superfield, we know that $\overline{D}^2D^2\Phi = 16\Box\Phi$. Then, we can write

$$\begin{aligned}
\int \mathrm{d}^6z\ \ j\Phi \;&=\; \int \mathrm{d}^6z\ \ j\frac{1}{16}\Box^{-1}\overline{D}^2D^2\Phi \\
&=\; \int \mathrm{d}^6z\ \ \overline{D}^2\left(j\frac{1}{16}\Box^{-1}D^2\Phi\right) \\
&=\; -\frac{1}{4}\int \mathrm{d}^8z\ \ j\frac{D^2}{\Box}\Phi,
\end{aligned} \tag{2.4.74}$$

where chirality of $j$ was used in the second step, and (2.1.27) was used in the last step. Similar arguments can be applied to show that

$$\int \mathrm{d}^6\overline{z}\ \ \overline{j}\ \overline{\Phi} \;=\; -\frac{1}{4}\int \mathrm{d}^6\overline{z}\ \ \overline{j}\ \frac{\overline{D}^2}{\Box}\ \overline{\Phi}. \tag{2.4.75}$$

Hence, (2.4.73) is equivalent to

$$\begin{aligned}
S_{M_o} \;&=\; \int \mathrm{d}^8z\ \left[\frac{1}{2}\left(\Phi\ \overline{\Phi}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Phi \\ \overline{\Phi} \end{pmatrix} + \left(\Phi\ \overline{\Phi}\right)\begin{pmatrix} -\frac{1}{4}\frac{D^2}{\Box}j \\ -\frac{1}{4}\frac{\overline{D}^2}{\Box}\overline{j} \end{pmatrix}\right]. \\
&\equiv\; \int \mathrm{d}^8z\ \left(\frac{1}{2}\eta^T\Sigma\ \eta \;+\; \eta^T\Omega\right).
\end{aligned} \tag{2.4.76}$$

Now, the generating functional can be evaluated to give[34]

$$\begin{aligned} Z_{M_o}[j,\overline{j}] &= \int \left(\mathcal{D}\Phi\mathcal{D}\overline{\Phi}\right) e^{S_{M_o}} = \exp\left(-\frac{1}{2}\int \mathrm{d}^8 z \;\; \Omega^T \Sigma^{-1}\Omega\right) \\ &= \exp\left(-\int \mathrm{d}^8 z \;\; \overline{j}\Box^{-1} j\right), \end{aligned} \tag{2.4.77}$$

and we can read off the matter propagator

$$\left\langle T\ \overline{\Phi}(z_1)\Phi(z_2)\right\rangle = -\frac{1}{\Box}\,\delta^8(z_{12}) = \frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\,\delta_{12}. \tag{2.4.78}$$

If we have an interaction action $S_i(\Phi,\overline{\Phi})$, then we can find the vertices by using (2.1.40) on the generating functional[30, 34]

$$Z[j,\overline{j}] = \exp\left[S_i\left(\frac{\delta}{\delta j},\frac{\delta}{\delta \overline{j}}\right)\right] Z_{M_o}[j,\overline{j}]. \tag{2.4.79}$$

By virtue of (2.1.40), we must include a factor of $-\frac{1}{4}\overline{D}^2$ for each chiral superfield line, and a factor of $-\frac{1}{4}D^2$ for each antichiral superfield line leaving a vertex. However, there are exceptions to this rule. If the interaction action is of the form $\int \mathrm{d}^6 z \;\; \Phi^n$, then one factor of $-\frac{1}{4}\overline{D}^2$ must be used to convert the measure into $\mathrm{d}^8 z$ . Therefore, if the vertex is purely chiral, we must omit one factor of $-\frac{1}{4}\overline{D}^2$. Similarly, if the vertex if purely antichiral, we must omit one factor of $-\frac{1}{4}D^2$. The vertices for the vector and ghost superfields can be read off directly from (2.2.50) and (2.2.61).

We summarize below the $x$-space super Feynman rules for massless supersymmetric gauge theories:

(1) The gauge and matter propagators, respectively, are

$$\langle T\ V(z_1)V(z_2)\rangle = -\frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\delta_{12} \tag{2.4.80}$$

and

$$\left\langle T\ \overline{\Phi}(z_1)\Phi(z_2)\right\rangle = \frac{1}{4\pi^2}\frac{1}{(x_1-x_2)^2}\delta_{12}. \tag{2.4.81}$$

(2) Vertices are determined directly from the interaction Lagrangian as in ordinary

field theories. Include a factor of $\left(-\frac{1}{4}D^2\right)$ $-\frac{1}{4}\overline{D}^2$ for each (antichiral) chiral superfield line leaving a vertex. However, omit a factor of $\left(-\frac{1}{4}D^2\right)$ $-\frac{1}{4}\overline{D}^2$ for a purely (antichiral) chiral vertex.

(3) Integrate over internal total superspace coordinates $z_{int} = (x_{int}, \theta_{int})$ and external $\theta$ coordinates $\theta_{ext}$.

(4) Consider symmetry factors for each supergraph.

# Chapter 3

# BACKGROUND FIELD METHODS

In quantizing gauge field theories, gauge invariance manifest at the classical level is usually lost when a specific gauge is chosen. In the language of Lagrangian field theory, the gauge invariance of the classical Lagrangian is broken when the gauge-fixing and ghost terms are introduced. As a consequence, although physical quantities computed are gauge invariant and gauge-fixing independent, unphysical quantities such as counterterms are not gauge invariant. The background field method is a formalism which allows one to choose a gauge while maintaining the explicit gauge invariance of the original Lagrangian. In this method, counterterms are also gauge invariant [35, 36] and computations are greatly facilitated. Therefore, the background field method is a powerful technique for studying the renormalizability of gauge theories.

The conventional background field method is first discussed in section 1. In section 2, we describe the superspace background field method (SBFM). In Section 3.3, we apply the SBFM to SUSY Yang-Mills theory and in Section 3.4 to SQED.

## 3.1 Ordinary Background Field Method

The background field method was first developed by DeWitt[37] for one-loop computations, and it was later extended in references[38, 39] for higher orders of perturbation theory. In this section, we consider the version which is applicable to multi-loop computations. As a particular model, we consider the pure Yang-Mills theory, for which the generating functional is given by

$$\begin{aligned} Z[J] &= \int (\mathcal{D}A)\,(\mathcal{D}\eta)\,(\mathcal{D}\overline{\eta})\ \exp\left\{ i\ \mathrm{Tr} \int \mathrm{d}^4 x\ \left[ \mathcal{L}_{YM}(A) + \mathcal{L}_{GF}(A) \right.\right. \\ &\quad \left.\left. +\ \mathcal{L}_{FP}(A, \eta\,\overline{\eta}) + J^{\mu} A_{\mu} \right] \right\}, \\ \mathcal{L}_{YM}(A) &= -\frac{1}{4}\left(F^2_{\mu\nu}\right), \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]\,, \\ \mathcal{L}_{GF} &= -\frac{1}{2\alpha}\left(G^2\right). \end{aligned} \tag{3.1.1}$$

All fields are Lie algebra valued; that is, $A = A^a T_a$, etc., where $T^a$ are elements of the Lie algebra of the gauge group. A typical choice of $G$ is $G = \partial^\mu A_\mu$, and $\mathcal{L}_{FP}$ is the Faddeev-Popov ghost term corresponding to $\det\left[\frac{\delta G^a}{\delta \omega^b}\right]$, where $\omega$ is the gauge parameter. $\mathcal{L}_{YM}$ is invariant under an infinitesimal gauge transformation

$$\delta A_\mu = \partial_\mu \omega + ig\,[\omega, A_\mu]\,, \tag{3.1.2}$$

but $\mathcal{L}_{GF}$ and $\mathcal{L}_{FP}$ have residual terms.

The main idea of the background field method is to choose a gauge which maintains an explicit gauge invariance with respect to the background field. We begin the procedure by replacing the gauge field $A$ in the classical action $\mathcal{L}_{YM}$ by

$$A_\mu = Q_\mu + B_\mu, \tag{3.1.3}$$

where $Q_\mu$ denotes the quantum field and $B_\mu$ the background field. In the functional integral, only the $Q_\mu$ field couples to an external source. Also, we remark that we do not integrate over the $B$ field, but only over the $Q$ field. In terms of $Q$ and $B$, the gauge transformation in (3.1.2) is

$$\delta A_\mu \;=\; \delta\left(Q_\mu + B_\mu\right) \;=\; \partial_\mu \omega + ig\left[\omega, Q_\mu + B_\mu\right], \tag{3.1.4}$$

and it has the following two interpretations[35, 36]:

(a) Quantum transformation:

$$\delta B_\mu = 0, \qquad \delta Q_\mu = \underset{\sim}{\nabla}_\mu \omega + ig\left[\omega, Q_\mu\right],$$

$$\text{where } \; \underset{\sim}{\nabla}_\mu \equiv \partial \;+\; ig\left[\quad, B_\mu\right]. \tag{3.1.5}$$

(b) Background transformation:

$$\delta B_\mu = \underset{\sim}{\nabla}_\mu \omega, \qquad \delta Q_\mu = ig\left[\omega, Q_\mu\right]. \tag{3.1.6}$$

We proceed now to fix the quantum gauge. We carefully choose a gauge so that the gauge invariance with respect to the background gauge transformation in (3.1.6) is retained. For example, if we choose the gauge-fixing function

$$\underset{\sim}{G}(Q, B) \;= \underset{\sim}{\nabla}^\mu Q_\mu \;=\; \partial^\mu Q_\mu + ig\left[Q_\mu, B_\mu\right], \tag{3.1.7}$$

and an appropriate Faddeev-Popov ghost term, then the generating functional

$$\begin{aligned} \underset{\sim}{Z}[J, B] \quad &= \quad \int (\mathcal{D}A)\,(\mathcal{D}\eta)\,(\mathcal{D}\overline{\eta}) \exp\left\{ i \;\mathrm{Tr} \int \mathrm{d}^4x \;\left[ \mathcal{L}_{YM}(Q+B) - \frac{1}{2\alpha}\left(\underset{\sim}{G}^2\right)\right]\right. \\ &\qquad \left. + \;\underset{\sim}{\mathcal{L}}_{FP}(Q, B, \eta, \overline{\eta}) + J^\mu Q_\mu \right\} \end{aligned} \tag{3.1.8}$$

is invariant under (3.1.6).

We now wish to establish relationships between the ordinary generating functionals and the analogous ones in the background field method. Taking $Q \longrightarrow Q - B$ in (3.1.8) gives

$$\underset{\sim}{Z}[J, B] \;=\; Z[J, B] \;\exp\left\{ -i\mathrm{Tr} \int \mathrm{d}^4x \; J^\mu B_\mu \right\}, \tag{3.1.9}$$

where $Z[J,B]$ is the conventional generating functional with the background field dependent gauge-fixing and ghost terms. In our example, the gauge fixing function for $Z[J,B]$ is

$$G(Q,B) \ = \utilde{\nabla}^\mu Q_\mu - \partial^\mu B_\mu. \tag{3.1.10}$$

$Z$ ($\utilde{Z}$) is related to the generating functional $W$ ($\utilde{W}$) for the connected Feynman diagrams by $Z = e^{iW}$ ($\utilde{Z} = e^{i\utilde{W}}$). Using this definition and (3.1.9), we can write

$$\utilde{W}[J,B] \ = \ W[J,B] - \mathrm{Tr}\int \mathrm{d}^4x \ J^\mu B_\mu. \tag{3.1.11}$$

We proceed by making the Legendre transformations to get

$$\Gamma[\overline{Q},B] \ = \ W[J,B] - \mathrm{Tr}\int \mathrm{d}^4x \ J^\mu \overline{Q}_\mu,$$

$$\begin{aligned}
\utilde{\Gamma}[\widetilde{Q},B] \quad &= \quad \utilde{W}[J,B] - \mathrm{Tr}\int \mathrm{d}^4x \ J^\mu \widetilde{Q}_\mu \\
&= \quad W[J,B] - \mathrm{Tr}\int \mathrm{d}^4x \ J^\mu \left(\widetilde{Q}_\mu + B_\mu\right) \\
&= \quad \Gamma[\overline{Q},B]\Big|_{\overline{Q}=\widetilde{Q}+B} .
\end{aligned} \tag{3.1.12}$$

In particular, evaluating (3.1.12) for $\widetilde{Q} = 0$ gives

$$\utilde{\Gamma}[0,B] \ = \ \Gamma[\overline{Q},B]\Big|_{\overline{Q}=B} , \tag{3.1.13}$$

which suggests that $\utilde{\Gamma}[0,B]$ is equivalent to the usual effective action $\Gamma$ with an unusual gauge-fixing term given by (3.1.10). $\utilde{\Gamma}[0,B]$ is a gauge invariant functional of $B$; it can be used to generate the S-matrix, and physical quantities calculated will be equal to the results obtained by using the conventional $\Gamma$ [35].

## 3.2 Superspace Background Field Method (SBFM)

### 3.2.1 Representations of Supersymmetric Gauge Theories

Supersymmetric gauge theories can be formulated in either chiral or vector representation [27]. In the chiral representation, the abelian theory is first developed by investigating its off-shell representation using prepotentials which can be used to find covariant derivatives. In the vector representation, the approach is in the opposite order to the one taken in the chiral representation; that is, we postulate covariant derivatives *ab initio* and introduce covariant constraints which are solved in terms of prepotentials. For a more detailed discussion of the subject, see Refs. [27, 30, 40]. The presentation made in this section mainly comes from Ref.[27].

#### Chiral Representation

Since

$$\Phi \longrightarrow \Phi' = e^{ig\Lambda}\Phi, \qquad \text{where } \overline{D}_{\dot\alpha}\Lambda = 0, \tag{3.2.14}$$

covariant derivatives should have the characteristic

$$(\nabla_A\Phi) \longrightarrow (\nabla_A\Phi)' = e^{ig\Lambda}\left(\nabla_A\Phi\right). \tag{3.2.15}$$

In order words, we want to have

$$\nabla_A \longrightarrow \nabla'_A \;=\; e^{ig\Lambda}\;\nabla_A\;e^{-ig\Lambda}. \tag{3.2.16}$$

Since $\overline{D}_{\dot\alpha}\Lambda = 0$, $\overline{D}_{\dot\alpha}$ is automatically covariant with respect to the $\Lambda$ transformation; that is,

$$\nabla_{\dot\alpha} \equiv \overline{D}_{\dot\alpha} \longrightarrow \nabla'_{\dot\alpha} \;=\; e^{ig\Lambda}\;\overline{D}_{\dot\alpha}\;e^{-ig\Lambda} \;=\; \overline{D}_{\dot\alpha} \;=\; \nabla_{\dot\alpha}. \tag{3.2.17}$$

Similarly, $D_\alpha$ is covariant with respect to the $\overline{\Lambda}$ transformation. However, we can use the prepotential $V$ to make $D_\alpha$ covariant with respect to the $\Lambda$ transformation.

Define

$$\nabla_\alpha \equiv e^{-gV} D_\alpha e^{gV}. \tag{3.2.18}$$

Then,

$$\begin{aligned}
\nabla_\alpha \longrightarrow \nabla'_\alpha &= \left(e^{-gV}\right)' D_\alpha \left(e^{gV}\right)' \\
&= \left(e^{ig\Lambda} e^{-gV} e^{-ig\overline{\Lambda}}\right) D_\alpha \left(e^{ig\overline{\Lambda}} e^{gV} e^{-ig\Lambda}\right) \\
&= \left(e^{ig\Lambda} e^{-gV} e^{-ig\overline{\Lambda}}\right) \left[\left(D_\alpha e^{ig\overline{\Lambda}}\right) + e^{ig\overline{\Lambda}} D_\alpha\right] \left(e^{gV} e^{-ig\Lambda}\right) \\
&= e^{ig\Lambda} \left(e^{-gV} D_\alpha \, e^{gV}\right) e^{-ig\Lambda} \\
&= e^{ig\Lambda} \, \nabla_\alpha \, e^{-ig\Lambda}.
\end{aligned} \tag{3.2.19}$$

The vector part of $\nabla_A$ is defined by $\nabla_{\alpha\dot{\alpha}} \equiv -i\left\{\nabla_\alpha, \nabla_{\dot{\alpha}}\right\}$, and its covariance is guaranteed by those of $\nabla_\alpha$ and $\nabla_{\dot{\alpha}}$.

Hence, the complete chiral gauge covariant derivatives are defined by

$$\nabla_A = \left(\nabla_\alpha, \ \nabla_{\dot{\alpha}}, \ \nabla_{\alpha\dot{\alpha}}\right) = \left(e^{-gV} D_\alpha \, e^{gV}, \ \overline{D}_{\dot{\alpha}}, \ -i\left\{\nabla_\alpha, \nabla_{\dot{\alpha}}\right\}\right). \tag{3.2.20}$$

Similar arguments as the ones given above show that anti-chiral gauge covariant derivatives are

$$\overline{\nabla}_A = \left(\overline{\nabla}_\alpha, \ \overline{\nabla}_{\dot{\alpha}}, \ \overline{\nabla}_{\alpha\dot{\alpha}}\right) = \left(D_\alpha, \ e^{gV} \overline{D}_{\dot{\alpha}} e^{-gV}, \ -i\left\{\overline{\nabla}_\alpha, \overline{\nabla}_{\dot{\alpha}}\right\}\right), \tag{3.2.21}$$

with the transformation property

$$\overline{\nabla}_A \longrightarrow \overline{\nabla}'_A = e^{ig\overline{\Lambda}} \, \overline{\nabla}_A \, e^{-ig\overline{\Lambda}}. \tag{3.2.22}$$

### Vector Representation

In vector representation, we first postulate that the covariant derivatives transform as

$$\nabla_A \longrightarrow \nabla'_A \;=\; e^{igH}\nabla_A e^{-igH}, \tag{3.2.23}$$

where $H$ is a hermitian superfield ($H = \overline{H}$). Since $H$ is not chiral, gauge transformation will not preserve the (anti)chirality condition ($D_\alpha\overline{\Phi} = 0$)$\overline{D}_{\dot\alpha}\Phi = 0$. This problem can be remedied by defining covariantly chiral and antichiral superfields by

$$\overline{\nabla}_{\dot\alpha}\Phi_c = 0, \qquad \Phi_c \longrightarrow \Phi'_c \;=\; e^{igH}\Phi_c,$$

$$\nabla_\alpha\overline{\Phi}_c = 0, \qquad \overline{\Phi}_c \longrightarrow \overline{\Phi}'_c \;=\; \overline{\Phi}_c e^{-igH}. \tag{3.2.24}$$

(3.2.24) suggests that

$$\left\{\overline{\nabla}_{\dot\alpha}, \overline{\nabla}_{\dot\beta}\right\}\Phi \;=\; -iF_{\dot\alpha\dot\beta}\Phi \;=0. \tag{3.2.25}$$

This gives us the constraint

$$\{\nabla_\alpha, \nabla_\beta\} \;=\; 0 \;=\; \left\{\overline{\nabla}_{\dot\alpha}, \overline{\nabla}_{\dot\beta}\right\}, \tag{3.2.26}$$

whose solution takes the form

$$\nabla_\alpha \;=\; e^{-g\Omega}\, D_\alpha\, e^{g\Omega}\;, \qquad \overline{\nabla}_{\dot\alpha} \;=\; e^{g\overline{\Omega}}\, \overline{D}_{\dot\alpha}\, e^{-g\overline{\Omega}}\;,$$

$$\Omega \;=\Omega\;{}^iT_i \;\neq\; \overline{\Omega}, \tag{3.2.27}$$

where $\Omega$ is an arbitrary complex prepotential. In vector representation, the gauge covariant derivative multiplet is given by

$$\nabla_A \;=\; \left(\nabla_\alpha,\; \overline{\nabla}_{\dot\alpha},\; -i\left\{\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\right\}\right). \tag{3.2.28}$$

(3.2.27) implies that (3.2.23) can be achieved by

$$\left(e^{g\Omega}\right) \longrightarrow \left(e^{g\Omega}\right)' \;=\; e^{g\Omega}\; e^{-igH}. \tag{3.2.29}$$

Furthermore, if $D_\alpha \overline{\Lambda} = 0 = \overline{D}_{\dot\alpha}\Lambda$, (3.2.27) is invariant under the gauge transformations

$$\begin{aligned} \left(e^{g\Omega}\right) \longrightarrow \left(e^{g\Omega}\right)' &= e^{ig\overline{\Lambda}} e^{g\Omega} \\ \left(e^{g\overline{\Omega}}\right) \longrightarrow \left(e^{g\overline{\Omega}}\right)' &= e^{g\overline{\Omega}} e^{-ig\Lambda}. \end{aligned} \tag{3.2.30}$$

Therefore, the gauge group of $\Omega$ is actually larger than that of $\Gamma_A$ [27]. We can use the $H$ gauge freedom to set $\Omega = \overline{\Omega}$, and define

$$\left(e^{g\Omega}\right)' = e^{ig\overline{\Lambda}} e^{g\Omega} e^{-igH}. \tag{3.2.31}$$

We further define the hermitian part of $\Omega$ to be

$$e^{gV} = e^{g\Omega} e^{g\overline{\Omega}} \tag{3.2.32}$$

which has the correct transformation property

$$\left(e^{gV}\right)' = e^{ig\overline{\Lambda}} e^{gV} e^{-ig\Lambda}. \tag{3.2.33}$$

Finally, we note that the covariantly chiral superfield defined in (3.2.24) is given by

$$\Phi_c = e^{g\overline{\Omega}} \Phi \tag{3.2.34}$$

**Relationship Between Chiral and Vector Representations**

We can go from the vector to the chiral representation which transforms only under $\Lambda$ by evaluating $\nabla_A^{vector}$ between $e^{-g\overline{\Omega}}$ and $e^{g\overline{\Omega}}$. We have

$$\nabla_A^{chiral} = e^{-g\overline{\Omega}} \nabla_A^{vector} e^{g\overline{\Omega}} = \left(e^{-gV} D_\alpha e^{gV},\ \overline{D}_{\dot\alpha},\ -i\left\{\nabla_\alpha^{chrial}, \overline{\nabla}_{\dot\alpha}^{chiral}\right\}\right), \tag{3.2.35}$$

which agrees with (3.2.20). As expected, no $\Omega$ or $H$ appears in the chiral representation.

### 3.2.2 Quantum-Background Splitting

#### Gauge Field Splitting

In analogy with (3.2.27), we define the background covariant derivative as

$$\underset{\sim}{\nabla}_\alpha \;=\; e^{-g\underset{\sim}{\Omega}}\, D_\alpha\, e^{g\underset{\sim}{\Omega}}, \qquad \overline{\underset{\sim}{\nabla}}_{\dot\alpha} \;=\; e^{g\overline{\underset{\sim}{\Omega}}}\, \overline{D}_{\dot\alpha}\, e^{-g\overline{\underset{\sim}{\Omega}}}, \tag{3.2.36}$$

where $\underset{\sim}{\Omega}$ is the background [1] prepotential. We then define the quantum-background splitting as

$$\nabla_\alpha \;=\; e^{-gV_Q}\, \underset{\sim}{\nabla}_\alpha\, e^{gV_Q}, \qquad \overline{\nabla}_{\dot\alpha} \;=\; \overline{\underset{\sim}{\nabla}}_{\dot\alpha}, \qquad \nabla_{\alpha\dot\alpha} \;=\; -i\left\{\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\right\}. \tag{3.2.37}$$

We can make a transformation to a background chiral representation by evaluating (3.2.37) in between $e^{-g\overline{\underset{\sim}{\Omega}}}$ and $e^{g\overline{\underset{\sim}{\Omega}}}$. We have

$$\underset{\sim}{\nabla}^{chiral}_\alpha \;=\; e^{-g\overline{\underset{\sim}{\Omega}}}\, e^{-gV_Q}\, \underset{\sim}{\nabla}_\alpha\, e^{gV_Q}\, e^{g\overline{\underset{\sim}{\Omega}}} \;=\; \left(e^{-g\overline{\underset{\sim}{\Omega}}}\, e^{-gV_Q}\, e^{-g\underset{\sim}{\Omega}}\right) D_\alpha \left(e^{g\underset{\sim}{\Omega}}\, e^{gV_Q}\, e^{g\overline{\underset{\sim}{\Omega}}}\right),$$

$$\underset{\sim}{\nabla}^{chiral}_{\dot\alpha} \;=\; e^{-g\overline{\underset{\sim}{\Omega}}}\, \overline{\underset{\sim}{\nabla}}_{\dot\alpha}\, e^{g\overline{\underset{\sim}{\Omega}}} \;=\; \left(e^{-g\overline{\underset{\sim}{\Omega}}}\, e^{g\overline{\underset{\sim}{\Omega}}}\right) \overline{D}_{\dot\alpha} \left(e^{-g\overline{\underset{\sim}{\Omega}}}\, e^{g\overline{\underset{\sim}{\Omega}}}\right) \;=\; \overline{D}_{\dot\alpha} \tag{3.2.38}$$

Comparing this to (3.2.20), we observe that the splitting is equivalent to

$$e^{gV_T} \;\longrightarrow\; e^{g\underset{\sim}{\Omega}}\, e^{gV_Q}\, e^{g\overline{\underset{\sim}{\Omega}}}. \tag{3.2.39}$$

We know that the unsplit gauge field $V_T$ transforms as

$$e^{gV_T} \;\longrightarrow\; \left(e^{gV_T}\right)' \;=\; e^{ig\overline{\Lambda}}\, e^{gV_T}\, e^{-ig\Lambda}, \tag{3.2.40}$$

where $D_a\overline{\Lambda} = 0 = \overline{D}_{\dot\alpha}\Lambda$. In terms of (3.2.39),

$$e^{gV_T} \;\longrightarrow\; \left(e^{g\underset{\sim}{\Omega}}\, e^{gV_Q}\, e^{g\overline{\underset{\sim}{\Omega}}}\right)' \;=\; e^{ig\overline{\Lambda}} \left(e^{g\underset{\sim}{\Omega}}\, e^{gV_Q}\, e^{g\overline{\underset{\sim}{\Omega}}}\right) e^{-ig\Lambda}. \tag{3.2.41}$$

[1] Henceforth, background superfields and derivatives will be denoted by the "∼" sign underneath the variables.

This transformation can be interpreted in the following two ways:

(a) Quantum Chiral Transformation:

$$\left(e^{g\underset{\sim}{\Omega}}\; e^{gV_Q}\; e^{g\underset{\sim}{\overline{\Omega}}}\right)' \;=\; e^{g\underset{\sim}{\Omega}}\left[\left(e^{-g\underset{\sim}{\Omega}} e^{ig\overline{\Lambda}} e^{g\underset{\sim}{\Omega}}\right)\; e^{gV_Q}\; \left(e^{g\underset{\sim}{\overline{\Omega}}} e^{-ig\Lambda} e^{-g\underset{\sim}{\overline{\Omega}}}\right)\right] e^{g\underset{\sim}{\overline{\Omega}}}, \quad (3.2.42)$$

which implies that

$$e^{g\underset{\sim}{\Omega}} \longrightarrow \left(e^{g\underset{\sim}{\Omega}}\right)' \;=\; e^{g\underset{\sim}{\Omega}} \quad\Longrightarrow\quad \underset{\sim}{\nabla}_\alpha \longrightarrow \left(\underset{\sim}{\nabla}_\alpha\right)' \;=\; \underset{\sim}{\nabla}_\alpha \qquad (3.2.43)$$

$$e^{gV_Q} \longrightarrow e^{ig\underset{\sim}{\overline{\Lambda}}} e^{gV_Q} e^{-ig\underset{\sim}{\Lambda}}, \qquad \text{where } \underset{\sim}{\Lambda} \;=\; e^{g\underset{\sim}{\overline{\Omega}}}\Lambda e^{-g\underset{\sim}{\overline{\Omega}}} \quad \text{and} \quad \underset{\sim}{\overline{\nabla}}_{\dot\alpha}\underset{\sim}{\Lambda} = 0. \qquad (3.2.44)$$

(3.2.37), (3.2.43) and (3.2.44) together suggest that

$$\nabla_A \longrightarrow (\nabla_A)' \;=\; e^{ig\underset{\sim}{\Lambda}}\; \nabla_A\; e^{-ig\underset{\sim}{\Lambda}}. \qquad (3.2.45)$$

(b) Background Vector Transformation:

$$\left(e^{g\underset{\sim}{\Omega}}\; e^{gV_Q}\; e^{g\underset{\sim}{\overline{\Omega}}}\right)' \;=\; \left(e^{ig\overline{\Lambda}} e^{g\underset{\sim}{\Omega}} e^{-igH}\right)\left(e^{igH} e^{gV_Q} e^{-igH}\right)\left(e^{igH} e^{g\underset{\sim}{\overline{\Omega}}} e^{-ig\Lambda}\right) \quad (3.2.46)$$

This means that the background field $\underset{\sim}{\Omega}$ transforms as

$$e^{g\underset{\sim}{\Omega}} \longrightarrow \left(e^{g\underset{\sim}{\Omega}}\right)' \;=\; e^{ig\overline{\Lambda}}\; e^{g\underset{\sim}{\Omega}}\; e^{-igH} \Longrightarrow$$

$$\begin{aligned}\underset{\sim}{\nabla}_\alpha \longrightarrow \left(\underset{\sim}{\nabla}_\alpha\right)' \;&=\; \left(e^{igH} e^{g\underset{\sim}{\Omega}} e^{-ig\overline{\Lambda}}\right) D_\alpha \left(e^{ig\overline{\Lambda}} e^{g\underset{\sim}{\Omega}} e^{-igH}\right)\\ &=\; e^{igH} e^{g\underset{\sim}{\Omega}} D_\alpha e^{g\underset{\sim}{\Omega}} e^{-igH} \;=\; e^{igH}\; \underset{\sim}{\nabla}_\alpha\; e^{-igH}, \end{aligned} \quad (3.2.47)$$

and the quantum field $V_Q$ as

$$e^{gV_Q} \;\longrightarrow\; \left(e^{gV_Q}\right)' \;=\; e^{igH}\; e^{gV_Q}\; e^{-igH} \Longrightarrow V_Q \longrightarrow e^{igH} V_Q e^{-igH}. \qquad (3.2.48)$$

(3.2.37), (3.2.47) and (3.2.48) imply that

$$\nabla_A \longrightarrow (\nabla_A)' \;=\; e^{igH}\; \nabla_A \; e^{-igH} \tag{3.2.49}$$

**Matter Field Splitting**

Consider the following original Lagrangian:

$$\mathcal{L} \;=\; \overline{\Phi}\, e^{gV_T}\Phi, \qquad \text{where} \;\; D_\alpha\overline{\Phi} = 0 = \overline{D}_{\dot\alpha}\Phi. \tag{3.2.50}$$

After the quantum-background splitting, (3.2.50) transforms as

$$\mathcal{L} \longrightarrow \mathcal{L}' \;=\; \overline{\Phi}\; e^{g\underset{\sim}{\Omega}} e^{gV_Q} e^{g\underset{\sim}{\overline{\Omega}}}\; \Phi \;\equiv\; \underset{\sim}{\overline{\Phi}}\; e^{gV_Q}\; \underset{\sim}{\Phi}, \tag{3.2.51}$$

where $\underset{\sim}{\Phi} \equiv e^{g\underset{\sim}{\overline{\Omega}}}\Phi$ is background chiral; that is, $\underset{\sim}{\nabla}_{\dot\alpha}\underset{\sim}{\Phi} = 0$. $\underset{\sim}{\Phi}$ splits linearly into $\underset{\sim}{\Phi} = \underset{\sim}{\Phi}^Q + \underset{\sim}{\Phi}^B$ and has the following transformation properties:

(a) Quantum Transformation:

$$\underset{\sim}{\Phi} \longrightarrow e^{ig\underset{\sim}{\Lambda}}\underset{\sim}{\Phi} \qquad , \qquad \underset{\sim}{\overline{\Phi}} \longrightarrow \underset{\sim}{\overline{\Phi}} e^{-ig\underset{\sim}{\overline{\Lambda}}}. \tag{3.2.52}$$

(b) Background Transformation:

$$\underset{\sim}{\Phi} \longrightarrow e^{igH}\underset{\sim}{\Phi} \qquad , \qquad \underset{\sim}{\overline{\Phi}} \longrightarrow \underset{\sim}{\overline{\Phi}} e^{-igH}. \tag{3.2.53}$$

## 3.3 SBFM in SUSY Yang-Mills Theory

We will choose the quantum gauge fixing term to be background covariant in order to obtain gauge invariant generating functionals. Before we proceed any further, however, we need to consider a subtle issue that arises in gauge-fixing. For a gauge theory with field $V$ and gauge condition $G(V) = h$, the gauge-fixing term in the Lagrangian is obtained by [30]

$$\begin{aligned} \delta\left[G(V)-h\right] \quad &\longrightarrow \quad \int (\mathcal{D}h)\,(\mathcal{D}\eta)\,\delta\left[G(V)-h\right]\exp\left\{-\int \mathrm{d}^8z\,(hMh+\eta M\eta)\right\} \\ &= \quad \int (\mathcal{D}\eta)\exp\left\{-\int \mathrm{d}^8z\,(GMG+\eta M\eta)\right\}, \end{aligned} \tag{3.3.54}$$

where $\eta$ is the Nielson-Kallosh ghost with opposite statistics to $h$. This new ghost is needed to implement the correct normalization

$$\int (\mathcal{D}h)\,(\mathcal{D}\eta)\exp\left\{-\int \mathrm{d}^8z\,(hMh+\eta M\eta)\right\} \;=\; 1. \tag{3.3.55}$$

Normally $M$ is independent of the fields and the additional ghost contribution can be ignored. However, in the background field method, $M$ does depend on the background field, and $\eta M\eta$ contribution cannot be decoupled. In the case of supersymmetric Yang-Mills theory, the quantum gauge is fixed by

$$\int \mathcal{D}\underset{\sim}{h}\mathcal{D}\overline{\underset{\sim}{h}}\mathcal{D}\underset{\sim}{\eta}\mathcal{D}\overline{\underset{\sim}{\eta}}\;\delta\left(\overline{\underset{\sim}{\nabla}}^2V_Q-\underset{\sim}{h}\right)\delta\left(\underset{\sim}{\nabla}^2V_Q-\overline{\underset{\sim}{h}}\right)\exp\left\{-\int \mathrm{d}^8z\left(\overline{\underset{\sim}{h}}\underset{\sim}{h}+\overline{\underset{\sim}{\eta}}\underset{\sim}{\eta}\right)\right\}, \tag{3.3.56}$$

which gives the gauge-fixing term

$$S_{GF} \;=\; -\frac{1}{16}\mathrm{Tr}\int \mathrm{d}^8z\left(\underset{\sim}{\nabla}^2V_Q\right)\left(\overline{\underset{\sim}{\nabla}}^2V_Q\right). \tag{3.3.57}$$

Corresponding ghost terms are obtained in the same fashion as described in Section 2.2. The complete[2] background covariant action is

[2] Actually, without the source terms.

$$\begin{aligned} S &= \int \mathrm{d}^8 z\ \overline{\underaccent{\sim}{\Phi}} e^{gV_Q} \underaccent{\sim}{\Phi} \ - \ \frac{1}{16g^2} \mathrm{Tr} \int \mathrm{d}^8 z \left( e^{-gV_Q} \underaccent{\sim}{\nabla}^\alpha e^{gV_Q} \right) \overline{\underaccent{\sim}{\nabla}}^2 \left( e^{-gV_Q} \underaccent{\sim}{\nabla}_\alpha e^{gV_Q} \right) \\ & \quad - \frac{1}{16\alpha} \mathrm{Tr} \int \mathrm{d}^8 z \left( \underaccent{\sim}{\nabla}^2 V_Q \right) \left( \overline{\underaccent{\sim}{\nabla}}^2 V_Q \right) \ + \ \mathrm{Tr} \int \mathrm{d}^8 z\ \overline{\underaccent{\sim}{\eta}}\ \underaccent{\sim}{\eta} \\ & \quad + \mathrm{Tr} \int \mathrm{d}^8 z \left( \overline{\underaccent{\sim}{c}}' + \underaccent{\sim}{c}' \right) L_{V_Q/2} \left[ \left( \overline{\underaccent{\sim}{c}} + \underaccent{\sim}{c} \right) + \left( \coth L_{V_Q/2} \right) \left( \underaccent{\sim}{c} - \overline{\underaccent{\sim}{c}} \right) \right], \end{aligned} \tag{3.3.58}$$

where $\underaccent{\sim}{c}, \underaccent{\sim}{c}', \underaccent{\sim}{\eta}$ and $\underaccent{\sim}{\Phi}$ are all background covariant.

We note that the Nielson-Kallosh ghost $\eta$ does not couple to any other quantum fields and, therefore, makes contributions only at the one-loop level. However, $\eta$ interacts with background fields through[41]

$$\overline{\underaccent{\sim}{\eta}}\ \underaccent{\sim}{\eta} \ = \ \overline{\eta}\ e^{gB}\ \eta, \tag{3.3.59}$$

where $D_\alpha \overline{\eta} = 0 = \overline{D}_{\dot{\alpha}} \eta$ and $B$ is the background gauge field. Similar splitting is also applied to $\overline{\underaccent{\sim}{c}}\ \underaccent{\sim}{c}$ and $\overline{\underaccent{\sim}{c}}'\ \underaccent{\sim}{c}'$. Hence, the part of the action that will contribute to one-loop calculation is

$$\mathrm{Tr} \int \mathrm{d}^8 z\ \left[ -\frac{1}{2}\ V_Q \left( \underaccent{\sim}{\nabla}^a \underaccent{\sim}{\nabla}_a - \underaccent{\sim}{W}^\alpha \underaccent{\sim}{\nabla}_\alpha + \overline{\underaccent{\sim}{W}}^{\dot{\alpha}} \overline{\underaccent{\sim}{\nabla}}_{\dot{\alpha}} \right) V_Q + \overline{\eta} B \eta + \overline{c}' B c + c' B \overline{c} \right]. \tag{3.3.60}$$

We end this section with the remark that the identities shown in (2.1.35) and (2.1.36) would still hold true if the covariant derivatives are replaced by background covariant derivatives. In particular, we would have

$$\delta_{ij}\ \underaccent{\sim}{\nabla}^2 \overline{\underaccent{\sim}{\nabla}}^2\ \delta_{ij} \ = \ 16\ \delta_{ij} \qquad \text{and} \qquad \delta_{ij}\ \underaccent{\sim}{\nabla}_\alpha \delta_{ij} \ = \ 0 \ = \ \delta ij \overline{\underaccent{\sim}{\nabla}}_{\dot{\alpha}} \delta ij \tag{3.3.61}$$

The aforementioned advantage of using the superspace background field method will be exemplified in Chapter 5.

## 3.4 SBFM in SQED

For SQED, splitting in the gauge superfield is linear; that is, the gauge superfield $V_T$ splits as

$$V_T \;\longrightarrow\; V_Q + B. \tag{3.4.62}$$

In fact, since all superfields commute in the abelian case, (3.4.62) follows directly from (3.2.39) and the definitions

$$e^V \;=\; e^{\Omega} e^{\overline{\Omega}} \qquad \text{and} \qquad e^B \;=\; e^{\underset{\sim}{\Omega}} e^{\underset{\sim}{\overline{\Omega}}}. \tag{3.4.63}$$

As in the non-abelian case, the total gauge transformation

$$V_T \;\longrightarrow\; V_T' \;=\; V + i\left(\Lambda - \overline{\Lambda}\right) \tag{3.4.64}$$

has two interpretations. They are as follows:

(a) Quantum Transformation:

$$V_Q \;\longrightarrow\; V_Q' \;=\; V + i\left(\Lambda - \overline{\Lambda}\right) \tag{3.4.65}$$

$$B \;\longrightarrow\; B' \;=\; B \tag{3.4.66}$$

(b) Background Transformation:

$$V_Q \;\longrightarrow\; V_Q' \;=\; V_Q \tag{3.4.67}$$

$$B \;\longrightarrow\; B' \;=\; V + i\left(\Lambda - \overline{\Lambda}\right) \tag{3.4.68}$$

Thus, we see that, under the quantum transformation, the quantum field $V_Q$ transformations like the total gauge field $V_T$, and the background field $B$ is inert. Under the background transformation, however, the opposite holds true. This is a remarkable simplification compared to the corresponding transformations in the non-abelian theory. In addition to this simplification, we wish to note an oddness in this method.

Since the gauge superfield splitting is linear and the transformation in (3.4.68) is the same as (3.4.64), it appears that nothing really changes in the perturbation theory, except that now only the background superfields are allowed as external superfields. In the background field method, one can calculate the $\beta$-function by considering only the 1-PI $B$–$B$ two-point function. In the non-background field method, however, this is not possible. It seems strange, to us at least, that although the same diagrams are considered, one case allows us to compute the $\beta$-function while the other does not.

In SQED, the matter field transformations are also very simple. In comparison to the distinct transformations shown in (3.2.52) and (3.2.53), the matter superfields transform as

$$\delta\Phi_\pm = \mp ig\Lambda\Phi_\pm \qquad , \qquad \delta\overline{\Phi}_\pm = \pm ig\overline{\Lambda}\,\overline{\Phi}_\pm \tag{3.4.69}$$

under both quantum and background transformations.

# Chapter 4

# A PRELUDE TO DIFFERENTIAL RENORMALIZATION

In this section, we summarize some important results of Ref.[25] pertinent to our work. We sometimes make claims without due explanations and give examples that already have been considered in the reference. In contrast to the above statement by Pascal, presentation made in this chapter is very concise. Those who want to master differential renormalization are strongly recommended to consult Ref.[25], as well as Refs.[42] $\sim$ [50].

## 4.1 General Idea

Differential regularization is a procedure defined in coordinate space ($x$-space) that handles ultraviolet divergences in perturbative quantum field theories. The method simultaneously regularizes and renormalizes singular amplitudes, which, after being renormalized, should satisfy the Callan-Symanzik equations. In this regularization procedure, the $x$-space bare amplitude too singular to have a well-defined Fourier transform into the momentum space is re-written as derivatives of a less singular expression. Then, the derivatives can be integrated by parts, and the Fourier transform can be performed on the resulting expression.

Consider field theories in Euclidean space, where $x^{2n} \equiv (x_\mu x_\mu)^n$. Here, terms of the form $1/x^4$ appear very often in bare amplitudes. For example, the expression for the one-loop 4-point function in massless $\phi^4$ theory is proportional to $1/x^4$, and massless supersymmetric gauge theories which we will consider in Section 5 have bare amplitudes that contain $1/x^4$. However, because of its singularity at $x \sim 0$, $1/x^4$ has no well-defined Fourier transform. Let us proceed to regularize this expression by finding an Euclidean invariant function $F(x^2)$ such that

$$\frac{1}{x^4} \;=\; \Box F(x^2), \tag{4.1.1}$$

where $\Box \equiv \partial_\mu \partial_\mu$. In terms of the new variable $s \equiv x^2$, (4.1.1) can be rewritten as

$$\frac{1}{s^2} \;=\; \frac{4}{s}\frac{d}{ds}\left(s^2 \frac{dF}{ds}\right), \tag{4.1.2}$$

whose general solution is given by

$$F(s) \;=\; -\frac{1}{4}\frac{\ln(sM^2)}{s} + \alpha. \tag{4.1.3}$$

$\alpha$ and $M$ are dimensionful constants arising from integration. $\alpha$ can safely be dropped, but $M^2$ is needed for the correct dimension in the logarithm. In fact, $M$ has a more physical meaning; it will be discussed later that $M$ actually is the renormalization group scale in Callan-Symanzik equations. Therefore, we have the identity

$$\frac{1}{x^4} \;=\; -\frac{1}{4}\Box \frac{\ln(x^2 M^2)}{x^2}. \tag{4.1.4}$$

The left and right hand sides of this equation are identical for $x \neq 0$. However, $F(x^2)$ is less singular than $1/x^4$, and this fact plays a pivotal role in differential renormalization. Consider a test function $T(x)$ which is regular as $x \to 0$. In differential regularization, partial integration prescription is given by

$$\begin{aligned}\int \mathrm{d}^4x\ \frac{T(x)}{x^4} &= -\frac{1}{4}\int \mathrm{d}^4x\ \ T(x)\ \Box\frac{\ln(x^2M^2)}{x^2} = \frac{1}{4}\int \mathrm{d}^4x\ \ [\partial_\mu T(x)]\ \partial_\mu\frac{\ln(x^2M^2)}{x^2} \\ &= -\frac{1}{4}\int \mathrm{d}^4x\ \ [\Box T(x)]\ \frac{\ln(x^2M^2)}{x^2},\end{aligned} \tag{4.1.5}$$

where surface terms have been naively dropped. In principle, the surface terms can be calculated by introducing a proper cutoff; in Ref.[25], an infinitesimal ball of radius $\varepsilon$ at the origin is excluded from the integration space. As $\varepsilon \to 0$, singularity arises from the surface term, but it is shown to be canceled if one adds an appropriate counterterm in the action. Hence, ultraviolet counterterms are implicitly present in differential regularization, and (4.1.5) is justified[1]. As a concrete example, let us take the test function to be $e^{ik\cdot x}$ so that the integration corresponds to the Fourier transform. According to (4.1.5), we have the well-defined Fourier transform

$$\begin{aligned}\int \mathrm{d}^4x\ \ e^{ik\cdot x}\frac{1}{x^4} &= -\frac{1}{4}\int \mathrm{d}^4x\ \ e^{ik\cdot x}\ \Box\frac{\ln(x^2M^2)}{x^2} = \frac{k^2}{4}\int \mathrm{d}^4x\ \ e^{ik\cdot x}\frac{\ln(x^2M^2)}{x^2} \\ &= -\pi^2\ln\left(\frac{k^2\gamma_E^2}{4M^2}\right),\end{aligned} \tag{4.1.6}$$

where $\gamma_E$ is the Euler-Mascheroni constant. We note that because differential renormalization does not involve explicit divergent subtractions, the mass parameter $M$ plays the role of the renormalization group scale that is necessary for verifying the consistency of the procedure.

As discussed above, the general idea of differential regularization goes as follows. For each singular amplitude, we first find an alternate expression which is identical to the original one at non-coincident points. The alternate amplitude involves derivative(s) of a less singular expression, which, when used in conjunction with (4.1.5), gives a well-defined Fourier transformation.

[1]An alert reader might be also concerned with the other end of scale; that is, infrared divergence. However, it has been shown that surface terms at large distances are generally well-damped, and therefore, (4.1.5) still holds.

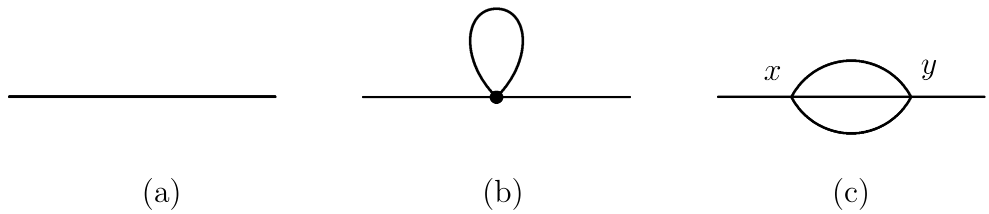


Figure 4-1: Feynman diagrams contributing to the 1-PI two-point function. (a) Tree. (b) Tadpole. (c) Two-loop.

We proceed now to find more identities similar to (4.1.4) that will be used in Section 5. Rather than simply writing down expressions that are equivalent, let us consider identities in the context of a field theory. In particular, we will work with massive $\phi^4$ theory[51] whose Euclidean action is given by

$$S \;=\; \int \mathrm{d}^4x \;\; \left[\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial_\mu\phi\right) \;+\; \frac{1}{2}m^2\phi^2 \;+\; \frac{1}{4!}\lambda\phi^4\right]. \tag{4.1.7}$$

The bare propagator for this theory is

$$\Delta(x-y) \;=\; \frac{1}{4\pi^2}\sqrt{\frac{m^2}{(x-y)^2}}\; K_1\left(\sqrt{m^2(x-y)^2}\right), \tag{4.1.8}$$

where $K_1$ is a modified Bessel function. Near the coincident point $x \sim y$, $K_1$ can be expanded to give

$$\Delta(z \equiv x-y) \;=\; \frac{1}{4\pi^2}\left\{\frac{1}{z^2} + \frac{1}{4}m^2\ln\left(m^2z^2\right) + \frac{m^2}{4}\left[1-2\psi(2)\right] + g(mz)\right\}, \tag{4.1.9}$$

where $g(mz) \to mz$ as $x \to y$. However, the propagator $\Delta(x-y)$ is singular at $x = y$. After regularization, we have $\Delta(0) = \mu^2$, where $\mu$ is a dimensionful parameter fixed by a mass renormalization scheme [25].

We now use (4.1.9) to compute the 1-PI two-point function $\Gamma^{(2)}(x-y)$. At the tree level (Figure 4-1a), we simply read it off from the action given in (4.1.7); that is, $\Gamma^{(2)}(x-y) \;=\; (-\Box + m^2)\,\delta^4(x-y)$ at the tree level. At the one-loop level (Figure 4-1b), there is a tadpole diagram which is proportional to $\Delta(0) = \mu^2$. Hence, the first non-trivial calculation arises at the two-loop level shown in Figure 4-1c. The bare amplitude corresponding to the two-loop self-energy correction is

$$\Sigma_o(x-y) \;=\; -\frac{1}{6}\lambda^2\left[\Delta(x-y)\right]^3, \tag{4.1.10}$$

which has the following expansion near $x \sim y$:

$$\Sigma_o(z \equiv x-y) \;=\; -\frac{\lambda^2}{384\pi^6}\left[\frac{1}{z^6}+\frac{3m^2}{4}\frac{\ln\left(m^2z^2\right)}{z^4}-\frac{3m^2}{4}\frac{1-\ln\gamma_E^2}{z^4}+G(z)\right] \tag{4.1.11}$$

The last term in (4.1.11) goes as $mz/z^4$ and has a finite Fourier transform. Furthermore, the third term can be regularized by using (4.2.20). Hence, let us concentrate on regularizing the first two expressions in (4.1.11). We simply follow the aforementioned procedure of regularization and try to find Euclidean invariant functions $A(z^2)$ and $B(z^2)$ such that

$$\frac{1}{z^6} \;=\; \Box\Box A(z^2), \tag{4.1.12}$$

$$\frac{\ln(z^2m^2)}{z^4} \;=\; \Box B(z^2). \tag{4.1.13}$$

The general solutions for $x \neq y$ are[25]

$$A(z^2) \;=\; -\frac{1}{32}\,\frac{\ln(z^2M_1^2)}{z^2} \;+\; \beta_1^2\,\ln(z^2\beta_2^2) \;+\; \beta_3 z^2, \tag{4.1.14}$$

$$B(z^2) \;=\; -\frac{1}{8}\,\frac{\left[\ln(z^2m^2)\right]^2+2\ln(z^2M_2^2)}{z^2} \;+\; \kappa_1\frac{1}{z^2} \;+\; \kappa_2. \tag{4.1.15}$$

$M_1, M_2, \beta_1, \beta_2, \beta_3, \kappa_1$, and $\kappa_2$ are arbitrary parameters, while $m$ is the Lagrangian mass. Substituting (4.1.14) and (4.1.15) into (4.1.12) and (4.1.13), respectively, gives

$$\frac{1}{z^6} \;=\; -\frac{1}{32}\,\Box\Box\,\frac{\ln(z^2M_1)}{z^2} \;-\; 16\pi^2\beta_1^2\,\delta^4(z), \tag{4.1.16}$$

$$\frac{\ln(z^2m^2)}{z^4} \;=\; -\frac{1}{8}\,\Box\,\frac{\left[\ln(z^2m^2)\right]^2+2\ln(z^2M_2^2)}{z^2} \;-\; 4\pi^2\kappa_1\,\delta^4(z). \tag{4.1.17}$$

Using these identities, the regularized version of the self-energy in (4.1.11) can be written as

$$\Sigma_r(z \equiv x - y) \quad = \quad \frac{\lambda^2}{384\pi^6}\left\{\frac{1}{32}\,\Box\Box\,\frac{\ln(z^2M_1^2)}{z^2} \;+\; \frac{3m^2}{32}\,\Box\,\frac{\left[\ln(z^2m^2)\right]^2 + 2\ln(z^2M_2^2)}{z^2}\right.$$
$$\left. -\frac{3m^2}{16}\left(1-\gamma_E^2\right)\Box\frac{\ln(z^2M_3^2)}{z^2} \;+\; \xi^2\delta^4(z) \;-\; G(z)\right\}. \qquad (4.1.18)$$

Terms proportional to $\delta^4(z)$ have finite Fourier transforms, so they have been combined into a single term $\xi^2\delta^4(z)$.

## 4.2 Renomalization Conditions

As in other renormalization procedures, there are certain ambiguities in differential renormalization. For example, in our example of the 1-PI two-point function, the self-energy correction contains terms of the form

$$\Sigma(x-y) \;\supset\; C_1\,\Box\delta^4(x-y) \;+\; C_2\delta^4(x-y). \qquad (4.2.19)$$

We need to specify the renormalization conditions that would fix these ambiguous local terms, and they are as follows:

(1) <u>Wave function and coupling constant renormalization conditions</u>.

In $\phi^4$ theory, all mass parameters that arise in logarithms are taken to be the same. In our example, this would correspond to setting $M_1 = M_2 = M_3$. In gauge theories, however, there is no *a priori* reason to set all scale parameters to be equal. Instead, Ward-Takahashi or BRST identities must be imposed to relate various parameters. This is done for SQED in Section 5.1.2 and Section 5.1.3 of this work.

(2) <u>Mass renormalization condition</u>.

Parameters that arise as coefficients of $\delta^4(z)$ are set to zero. In particular, $\beta_1$ in (4.1.14) and $\kappa_1$ in (4.1.15) would become zero. Hence, the only explicit term in the 1-PI two-point function that is proportional to $\delta^4(z)$ comes from the tree approximation $(-\Box + m^2)\delta^4(z)$.

After ambiguities are removed by these conditions, the regularized amplitudes satisfy the renormalization group equations, in which the $M$'s play the role of scale parameters. Hence, differential regularization automatically gives renormalized amplitudes, and its simple nature is easily appreciated if one tries to perform the same computations in the momentum space using Feynman parameter integrals.

We end this chapter with a list of some formulas that we will find useful in Chapter 5.

$$\frac{1}{x_{ij}^4} = -\frac{1}{4}\Box\frac{\ln(x_{ij}^2 M^2)}{x_{ij}^2} \tag{4.2.20}$$

$$\frac{1}{x_{ij}^6} = -\frac{1}{32}\Box\Box\frac{\ln(x_{ij}^2 M^2)}{x_{ij}^2} \tag{4.2.21}$$

$$\Box\frac{1}{x_{ij}} = -4\pi^2\delta^4(x_{ij}) \tag{4.2.22}$$

# Chapter 5

# DIFFERENTIAL RENORMALIZATION OF SUSY GAUGE THEORIES

In this chapter, we justify the title of this paper. We use the method described in the previous chapter to renormalize supersymmetric gauge theories in superspace. At the risk of being pedagogical, we try to make the computations as explicit as possible for clarity. We begin with SQED in Section 5.1 and proceed to SUSY Yang-Mills theory in Section 5.2.

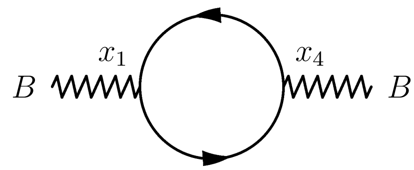


Figure 5-1: One-loop contribution to the SQED $\beta$-function. $B$ denotes the background field. Internal chiral fields could be either $\Phi_+$ or $\Phi_-$.

## 5.1 SQED

### 5.1.1 One-Loop Level

#### Renormalization

One-loop contribution from $\Phi_-$ to the background superfield vacuum polarization shown in Figure 5-1 is

$$\Gamma_-^{(1)} = \frac{1}{2}\frac{(-g)^2}{(4\pi^2)^2}\left(-\frac{1}{4}\right)^4 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\left(\overline{D}_4^2 D_4^2 \frac{\delta_{41}}{x_{41}^2}\right)\left(D_4^2 \overline{D}_4^2 \frac{\delta_{41}}{x_{41}^2}\right). \quad (5.1.1)$$

Notice that this expression contains a factor of $\frac{1}{4\pi^2}$ associated with each propagator, and $-\frac{1}{4}$ with each $D_\alpha$ or $\overline{D}_{\dot\alpha}$. Let us now consider a useful "D-Algebra"[27] that we will use repeatedly in this chapter. Let $F$ be a general function in superspace that depends on $z_j \equiv (x_j, \theta_j)$. Further assume that the expression we are about to consider is inside an integration over $z_j$. In order to simply notations, propagators will be denoted by $P_{ij}$. Then,

$$\left[D_i^2 \overline{D}_i^2 P_{ij}\right]\left[\overline{D}_i^2 D_i^2 P_{ik}\right] F(z_i) = \left[\overline{D}_i^2 P_{ij}\right]\ D_i^2 \left\{\left[\overline{D}_i^2 D_i^2 P_{ik}\right] F(z_i)\right\}$$

$$\begin{aligned}
&= \left[\overline{D}_i^2 P_{ij}\right]\left[\left(D_i^2 \overline{D}_i^2 D_i^2 P_{ik}\right) + 2\left(D_i^\alpha \overline{D}_i^2 D_i^2 P_{ik}\right) D_{i\alpha} + \left(\overline{D}_i^2 D_i^2 P_{ik}\right) D_i^2\right] F(z_i)\\
&= P_{ij}\overline{D}_i^2 \left[\left(16\Box D_i^2 P_{ik}\right) + 2\left(D_i^\alpha \overline{D}_i^2 D_i^2 P_{ik}\right) D_{i\alpha} + \left(\overline{D}_i^2 D_i^2 P_{ik}\right) D_i^2\right] F(z_i)\\
&= P_{ij}\left[\left(16\Box \overline{D}_i^2 D_i^2 P_{ik}\right) - 4\left(\overline{D}_i^{\dot\beta} D_i^\alpha \overline{D}_i^2 D_i^2 P_{ik}\right)\overline{D}_{i\dot\beta} D_{i\alpha} + \left(\overline{D}_i^2 D_i^2 P_{ik}\right)\overline{D}_i^2 D_i^2\right] F(z_i)\\
&\quad + \text{additional terms that would vanish if } \delta_{ij} = \delta_{ik}
\end{aligned} \quad (5.1.2)$$

where identities (2.1.17), (2.1.18), and (2.1.19) have been used. The last statement in (5.1.2) needs a bit of clarification. If a term has less than 2 $D$'s or 2 $\overline{D}$'s between two $\delta_{ij}$'s, then the term vanishes. If a term has more than 2 $D$'s or 2 $\overline{D}$'s, then (2.1.18) and (2.1.14) can be used to reduce the number of operators. Following this line of argument, one can show that the "additional terms" in (5.1.2) effectively contains less than 2 $D$'s or 2 $\overline{D}$'s. Upon using (2.1.14), we can rewrite (5.1.2) as

$$P_{ij}\left[\left(16\Box\overline{D}_i^2 D_i^2 P_{ik}\right) - 8i\,(\sigma^a)^{\alpha\dot\beta}\left(\partial_{ia}\overline{D}_i^2 D_i^2 P_{ik}\right)\overline{D}_{i\dot\beta}D_{i\alpha} + \left(\overline{D}_i^2 D_i^2 P_{ik}\right)\overline{D}_i^2 D_i^2\right] F(z_i)$$
$$+ \text{ additional terms that would vanish if } \delta_{ij} = \delta_{ik}. \tag{5.1.3}$$

Armed with this useful identity, we can proceed now to renormalize (5.1.1). We simply substitute $i = 4, j = k = 1$ in (5.1.3) and use (4.2.22) and (2.1.36) to obtain

$$\begin{aligned}\Gamma_-^{(1)} &= \frac{1}{2}\frac{g^2}{(4\pi^2)^2}\left(-\frac{1}{4}\right)^4 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)\frac{16\delta_{41}}{x_{41}^2}\Big[-4\pi^2(16)\delta^4(x_{41}) \\ &\quad - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{4a}\frac{1}{x_{41}^2}\right)\overline{D}_{4\dot\alpha}D_{4\alpha} + \frac{1}{x_{41}^2}\overline{D}_4^2 D_4^2\Big] B(z_4).\end{aligned} \tag{5.1.4}$$

However, we observe that

$$\frac{1}{x^2}\partial_a\frac{1}{x^2} = -\frac{2x_a}{x^6} = \frac{1}{2}\partial_a\frac{1}{x^4}, \tag{5.1.5}$$

so (5.1.4) becomes

$$\begin{aligned}\Gamma_-^{(1)} &= -\frac{1}{2}\frac{g^2}{4\pi^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x_{41}^2} \\ &\quad + \frac{1}{2}\frac{g^2}{(4\pi^2)^2}\frac{1}{16}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)\frac{\delta_{41}}{x_{41}^4}\left[4i\,(\sigma^a)^{\alpha\dot\alpha}\,\partial_{4a}\overline{D}_{4\dot\alpha}D_{4\alpha} + \overline{D}_4^2 D_4^2\right] B(z_4),\end{aligned} \tag{5.1.6}$$

where $\partial_{4a}$ has been integrated by parts to act on $B(z_4)$. Apart from the first term, rest of the expression can be combined into a gauge invariant form. We know from

(2.1.15) that

$$\overline{D}^2 D^2 = \overline{D}^2 D^\alpha D_\alpha = \left[\overline{D}^2, D^\alpha\right] D_\alpha + D^\alpha \overline{D}^2 D_\alpha = -4i\,(\sigma^a)^{\alpha\dot{\alpha}}\,\partial_a \overline{D}_{\dot{\alpha}} D_\alpha + D^\alpha \overline{D}^2 D_\alpha. \tag{5.1.7}$$

Hence, using (5.1.7) and (4.2.20) gives the fully regularized contribution

$$\begin{aligned} \Gamma_-^{(1)} &= -\frac{1}{2}\frac{g^2}{4\pi^2}\int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_4 \;\; B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x_{41}^2} \\ &\quad -\frac{1}{2}\frac{g^2}{(4\pi^2)^2}\frac{1}{64}\int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_4 \;\; \left[B(z_1) D^\alpha \overline{D}^2 D_\alpha B(z_4)\right]\delta_{41}\Box\frac{\ln(x_{41}^2 M^2)}{x_{41}^2}. \end{aligned} \tag{5.1.8}$$

The first term is purely local and finite. $B(z_1)D^\alpha\overline{D}^2 D_\alpha B(z_4)$ in the second term is manifestly gauge invariant since it appears as the kinetic term for $B$ in the SQED Lagrangian.

If we carry out analogous calculations for the contribution coming from $\Phi_+$, we will observe that it is exactly equal to $\Gamma_-^{(1)}$.

### One-Loop $\beta$-Function

In calculating the $\beta$-function, it is advantageous to redefine the field as

$$B \longrightarrow B' = gB. \tag{5.1.9}$$

The key point is that in background field method, the renormalization constant for the $B$ field is related to that of the coupling constant $g$ by $Z_g\sqrt{Z_B} = 1$. This suggests that

$$B' = g_o B_o = Z_g\sqrt{Z_B} g_r B_r = g_r B_r, \tag{5.1.10}$$

where subscripts “o” and “r” denote bare and renormalized quantities, respectively. Therefore, there is no anomalous dimension for the $B'$ field, and the renormalization group equation takes the following simple form:

$$\left[\sum_i M_i \frac{\partial}{\partial M_i} + \beta(e)\frac{\partial}{\partial e}\right] G(x_4 - x_1) = 0 \tag{5.1.11}$$

$$\beta(e) = \beta_1 e^3 + \beta_2 e^5 + \cdots,$$

where $e \equiv \frac{g}{\sqrt{2}}$ is the conventional electric charge as discussed in Section 2.3. Here, we define

$$G(x_4 - x_1) \equiv \sum_n \left[ G_+^{(n)}(x_4 - x_1) + G_-^{(n)}(x_4 - x_1) \right]$$
$$\text{and} \quad \Gamma_\pm^{(n)} \equiv \int \mathrm{d}^8 z_1 \, \mathrm{d}^8 z_4 \, \left[ B'(z_1) D^\alpha \overline{D}^2 D_\alpha B'(z_4) \right] G_\pm^{(n)}(x_4 - x_1), \tag{5.1.12}$$

where "$n$" denotes the $n^{th}$ loop contributions. $\Gamma^0$ is just the kinetic term in the original action; more specifically,

$$\begin{aligned}
\Gamma^0 &= \frac{1}{64g^4} \int \mathrm{d}^6 z \, W^\alpha W_\alpha = -\frac{1}{16g^2} \int \mathrm{d}^8 z \, (D^\alpha B'(z)) \left( \overline{D}^2 D_\alpha B'(z) \right) \\
&= \frac{1}{32e^2} \int \mathrm{d}^8 z_1 \, \mathrm{d}^8 z_4 \, \left[ B'(z_1) D^\alpha \overline{D}^2 D_\alpha B'(z_4) \right] \, \delta^8(z_{41}).
\end{aligned} \tag{5.1.13}$$

Combining (5.1.13) and (5.1.8) gives

$$G(z_4 - z_1) = \frac{1}{32e^2} \, \delta^8(z_{41}) + 2 \left[ -\frac{1}{128} \frac{1}{(4\pi^2)^2} \, \delta_{41} \, \Box \frac{\ln(x_{41}^2 M^2)}{x_{41}^2} \right] + \cdots. \tag{5.1.14}$$

Substituting (5.1.14) into (5.1.12) and using (4.2.22), we readily derive the following one-loop contribution to the $\beta$-function:

$$\boxed{\beta_1 = \frac{1}{8\pi^2}} \tag{5.1.15}$$

According to Shifman *et al.*, the exact $\beta$-function for SQED is given by[52, 53, 54]

$$\beta(\alpha) = \frac{\alpha^2}{\pi} \left[ 1 - \gamma(\alpha) \right], \tag{5.1.16}$$

where $\alpha$ has the usual definition $e^2/4\pi$, and $\gamma(\alpha)$ is the anomalous dimension of the matter superfield. Since $\beta(e) = \frac{e}{2\pi}\beta(\alpha)$, this suggests that to the lowest order in $e$,

$$\beta(e) \;=\; \frac{e^3}{8\pi^2}, \tag{5.1.17}$$

which agrees with our result.

### 5.1.2 Ward-Takahashi Identities in SQED

Before we proceed to compute two-loop contributions to the $\beta$-function, we need to construct a method to relate mass parameters that will arise from subdivergences. As in differential renormalization of ordinary quantum field theories [25, 43, 45], this is done by considering the underlying symmetry expressed in terms of Ward-Takahashi Identities (WTI). As discussed in Section 2.3, the complete action of SQED consists of the free, gauge fixing, and source terms; that is,

$$S = S_0 + S_{GF} + S_{Source}, \tag{5.1.18}$$

where

$$\begin{aligned} S_0 &= \frac{1}{64}\int \mathrm{d}^6 z\; W^\alpha W_\alpha \;+\; \int \mathrm{d}^8 z\left(\overline{\Phi}_+ e^{gV}\Phi_+ + \overline{\Phi}_- e^{-gV}\Phi_-\right) \\ S_{GF} &= -\frac{1}{16\alpha}\int \mathrm{d}^8 z (D^2 V)(\overline{D}^2 V) \\ S_{Source} &= \int \mathrm{d}^6 z\,(j_-\Phi_+ + j_+\Phi_-) \;+\; \int \mathrm{d}^6\overline{z}\left(\overline{j}_-\overline{\Phi}_+ + \overline{j}_+\overline{\Phi}_-\right) \;+\; \int \mathrm{d}^8 z\; JV. \end{aligned} \tag{5.1.19}$$

$W_\alpha$ is the chiral field strength given by $W_\alpha = \overline{D}^2 D_\alpha V$. Recall that $J$ and $V$ denote vector superfields, $\Phi_\pm$ and $j_\pm$ chiral superfields, and $\overline{j}_\pm$ and $\overline{\Phi}_\pm$ anti-chiral superfields. Under infinitesimal local gauge transformations, we have

$$\delta\Phi_\pm = \mp ig\Lambda\Phi_\pm, \qquad \delta\overline{\Phi}_\pm = \pm ig\overline{\Lambda}\;\overline{\Phi}_\pm,$$

$$\delta V = i(\Lambda - \overline{\Lambda}). \tag{5.1.20}$$

$\Lambda$ and $\overline{\Lambda}$ are chiral and anti-chiral superfields, respectively; that is, $\overline{D}_{\dot{\alpha}}\Lambda = 0$ and $D_\alpha \overline{\Lambda} = 0$. The normalized generating functional for Green's functions is

$$Z[J, j_\pm, \overline{j}_\pm] \; = \; \frac{1}{N} \int (\mathcal{D}F) \exp\{i(S_0 + S_{GF} + S_{Source})\} \tag{5.1.21}$$

where the integration measure $(\mathcal{D}F)$ is defined as

$$(\mathcal{D}F) \; \equiv \; (\mathcal{D}V)(\mathcal{D}\overline{\Phi}_+)(\mathcal{D}\Phi_+)(\mathcal{D}\overline{\Phi}_-)(\mathcal{D}\Phi_-), \tag{5.1.22}$$

and $N$ is the normalization factor which we will omit in our further consideration of generating functionals. $S_0$ is invariant under the gauge transformations defined in (5.1.20). However, $S_{GF}$ and $S_{Source}$ are not gauge invariant, and the generating functional $Z$ is thus not gauge invariant. Upon performing the infinitesimal gauge transformations on $Z$, we obtain

$$\begin{aligned} Z \longrightarrow Z' \; &= \; \int (\mathcal{D}F) e^{iS} \Big\{ 1 - \frac{i}{16\alpha} \int \mathrm{d}^8 z \left[ D^2 i(\Lambda - \overline{\Lambda}) \overline{D}^2 V + D^2 V \overline{D}^2 i(\Lambda - \overline{\Lambda}) \right] \\ &\quad + ig \int \mathrm{d}^6 z \left[ j_-(-i\Lambda\Phi_+) + j_+(i\Lambda\Phi_-) \right] \\ &\quad + ig \int \mathrm{d}^6 \overline{z} \left[ \overline{j}_-(i\overline{\Lambda}\,\overline{\Phi}_+) + \overline{j}_+(-i\overline{\Lambda}\,\overline{\Phi}_-) \right] + i \int \mathrm{d}^8 z \; J i(\Lambda - \overline{\Lambda}) \Big\} \\ &= \; \int (\mathcal{D}F) e^{iS} \Big\{ 1 + \frac{1}{16\alpha} \int \mathrm{d}^8 z \left[ \Lambda D^2 \overline{D}^2 V - \overline{\Lambda}\, \overline{D}^2 D^2 V \right] \\ &\quad + g \int \mathrm{d}^6 z \; \Lambda \left( j_- \Phi_+ - j_+ \Phi_- \right) - g \int \mathrm{d}^6 \overline{z} \; \overline{\Lambda} \left( \overline{j}_- \overline{\Phi}_+ - \overline{j}_+ \overline{\Phi}_- \right) \\ &\quad - \int \mathrm{d}^8 z \; J(\Lambda - \overline{\Lambda}) \Big\}. \end{aligned} \tag{5.1.23}$$

Upon using (2.1.40) to evaluate $\left.\frac{\delta Z'}{\delta \Lambda}\right|_{\Lambda = \overline{\Lambda} = 0} = 0$, we derive the following relation:

$$\int (\mathcal{D}F) e^{iS} \left[ -\frac{1}{4} \frac{1}{16\alpha} \overline{D}^2 D^2 \overline{D}^2 V + \frac{1}{4} \overline{D}^2 J + g(j_- \Phi_+ - j_+ \Phi_-) \right] = 0 \tag{5.1.24}$$

In terms of $Z$ and its functional derivatives with respect to sources, (5.1.24) can be rewritten as

$$\left[\frac{1}{\alpha}\Box\overline{D}^2\frac{1}{i}\frac{\delta}{\delta J}-\overline{D}^2J-4g\left(j_-\frac{1}{i}\frac{\delta}{\delta j_-}-j_+\frac{1}{i}\frac{\delta}{\delta j_+}\right)\right]Z=0, \tag{5.1.25}$$

where we made use of the identity (2.1.17). If we use the definition $Z=e^{iW_c}$, where $W_c$ is the generating functional for the connected Feynman supergraphs, (5.1.25) becomes

$$-\overline{D}^2J+\left[\frac{1}{\alpha}\Box\overline{D}^2\frac{\delta}{\delta J}-4g\left(j_-\frac{\delta}{\delta j_-}-j_+\frac{\delta}{\delta j_+}\right)\right]W_c=0. \tag{5.1.26}$$

We are now ready to perform the Legendre transformation to obtain the generating functional $\Gamma$ for 1-particle irreducible (1PI) Green's functions. We first define the Legendre transformation by

$$\begin{aligned}\Gamma[V,\Phi_\pm,\overline{\Phi}_\pm] &= W_c[J,j_\pm,\overline{j}_\pm]-\int \mathrm{d}^8z\; JV\\ &\quad -\int \mathrm{d}^6z\,(j_-\Phi_++j_+\Phi_-)-\int \mathrm{d}^6\overline{z}\left(\overline{j}_-\overline{\Phi}_++\overline{j}_+\overline{\Phi}_-\right),\end{aligned} \tag{5.1.27}$$

which implies the relation

$$\begin{gathered}\frac{\delta W_c}{\delta J}=V,\qquad \frac{\delta W_c}{\delta j_\pm}=\Phi_\mp,\qquad \frac{\delta W_c}{\delta \overline{j}_\pm}=\overline{\Phi}_\mp,\\ J=-\frac{\delta\Gamma}{\delta V},\qquad j_\pm=-\frac{\delta\Gamma}{\delta\Phi_\mp},\qquad \overline{j}_\pm=-\frac{\delta\Gamma}{\delta\overline{\Phi}_\mp}.\end{gathered} \tag{5.1.28}$$

After implementing all substitutions, we arrive at the following Ward-Takahashi(WT) identity:

$$\frac{1}{\alpha}\Box\overline{D}^2V+\left[\overline{D}^2\frac{\delta}{\delta V}-4g\left(\Phi_-\frac{\delta}{\delta\Phi_-}-\Phi_+\frac{\delta}{\delta\Phi_+}\right)\right]\Gamma=0. \tag{5.1.29}$$

If we take $\left.\frac{\delta Z'}{\delta\overline{\Lambda}}\right|_{\Lambda=\overline{\Lambda}=0}=0$ in (5.1.23) and perform analogous operations used to find (5.1.29), we can obtain another WT identity; in fact, it will be the complex conjugate of (5.1.29), namely

$$\frac{1}{\alpha}\Box D^2 V + \left[D^2\frac{\delta}{\delta V} - 4g\left(\overline{\Phi}_-\frac{\delta}{\delta\overline{\Phi}_-} - \overline{\Phi}_+\frac{\delta}{\delta\overline{\Phi}_+}\right)\right]\Gamma = 0. \tag{5.1.30}$$

We now wish to derive the WT identities relating the 3-point 1PI Green's function $\left\langle TV\Phi_\pm\overline{\Phi}_\pm\right\rangle_{\text{1PI}}$ and the 2-point 1PI Green's function $\left\langle T\Phi_\pm\overline{\Phi}_\pm\right\rangle_{1PI}$. This can be done by operating on (5.1.29) and (5.1.30) with $\frac{\delta}{\delta\Phi_-(z_2)}\frac{\delta}{\delta\overline{\Phi}_-(z_3)}\Big|_{V=\Phi_\pm=\overline{\Phi}_\pm=0}$, and the results are

$$\overline{D}^2(z_1)\left\langle T\ V(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\right\rangle_{\text{1PI}} = -g\left\langle T\ \Phi_-(z_1)\overline{\Phi}_-(z_3)\right\rangle_{\text{1PI}}\overline{D}^2(z_1)\delta^8(z_1-z_2), \tag{5.1.31}$$

$$D^2(z_1)\left\langle T\ V(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\right\rangle_{\text{1PI}} = -g\left\langle T\ \Phi_-(z_2)\overline{\Phi}_-(z_1)\right\rangle_{\text{1PI}}D^2(z_1)\delta^8(z_1-z_3). \tag{5.1.32}$$

Similarly, for the matter fields with plus charges, we derive

$$\overline{D}^2(z_1)\left\langle T\ V(z_1)\Phi_+(z_2)\overline{\Phi}_+(z_3)\right\rangle_{\text{1PI}} = g\left\langle T\ \Phi_+(z_1)\overline{\Phi}_+(z_3)\right\rangle_{\text{1PI}}\overline{D}^2(z_1)\delta^8(z_1-z_2), \tag{5.1.33}$$

$$D^2(z_1)\left\langle T\ V(z_1)\Phi_+(z_2)\overline{\Phi}_+(z_3)\right\rangle_{\text{1PI}} = g\left\langle T\ \Phi_+(z_2)\overline{\Phi}_+(z_1)\right\rangle_{\text{1PI}}D^2(z_1)\delta^8(z_1-z_3). \tag{5.1.34}$$

We remark that the vector superfield $V$ shown above is a quantum superfield. In the superspace background field method, we need to consider three-point structures involving external background superfields $B$. However, this does not suggest that what we have shown in this section is useless. From our discussion in Section 3.4, we know that under quantum transformation, the quantum superfield transforms as in (5.1.20) and the background superfield is inert. In contrast, under background transformation, the quantum superfield is inert and the background superfield transforms as in (5.1.20). What this implies is that if we try to derive the super WTI involving the background superfield, we will arrive at the same equations as (5.1.31) $\sim$ (5.1.34) with $V$'s replaced by $B$'s.[1]

[1]This has been privately verified.

### 5.1.3 Perturbative Calculations of Super WTI

In this section, we perform explicit one-loop computations of the $B - \Phi - \overline{\Phi}$ vertex function and $\Phi\overline{\Phi}$ self-energy correction. In particular, we will consider supergraphs shown in Figure 5-2.

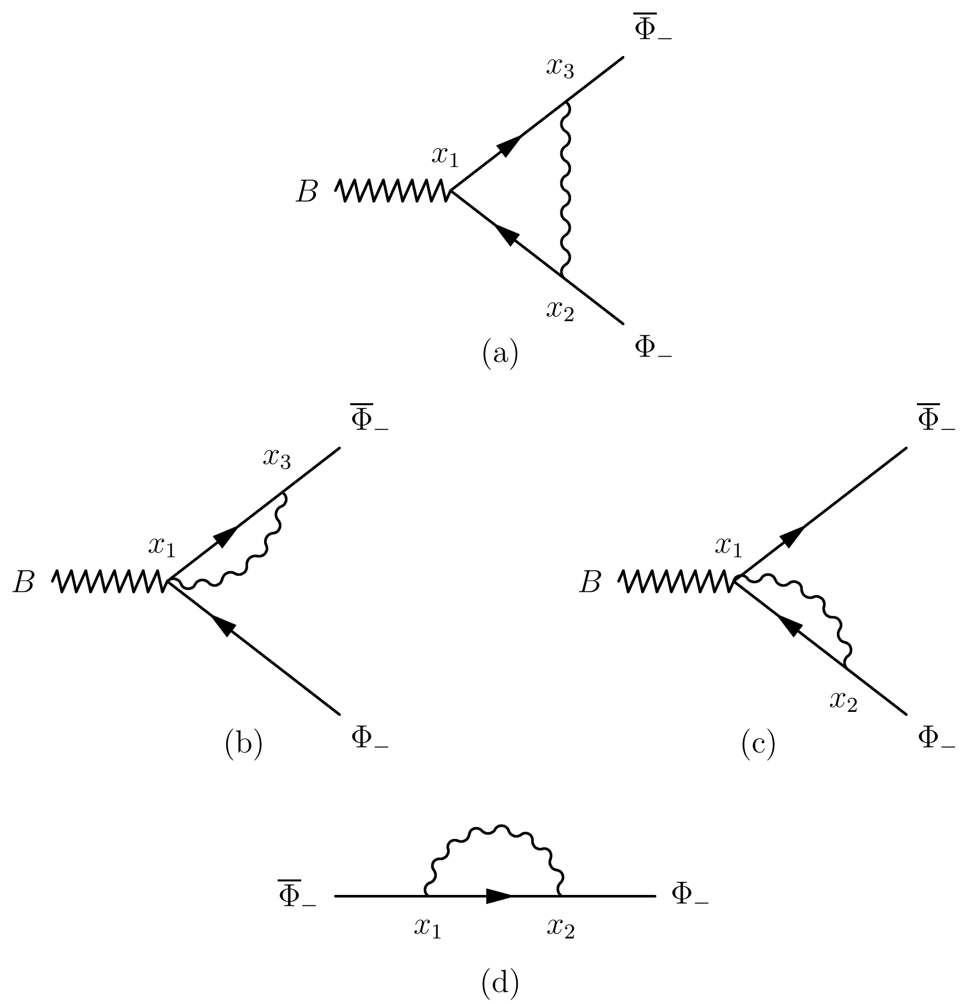


Figure 5-2: One Loop Contributions to super Ward-Takahashi Identity in SQED. (a)-(c) Vertex functions. (d) Self-energy correction.

The 1-PI Green's functions in Figure 5-2 are related by (5.1.32) which will be used to obtain a relation among the mass parameters. However, rather than working directly with Green's functions, we found it advantageous to calculate contributions to the effective action which contains the Green's functions we want.

Let us begin with the vertex supergraphs. Contribution to the effective action from Figure 5-2a is given by

$$
\begin{aligned}
\Gamma_{Va} &= \frac{(-g)^3}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_3\ \left(-\frac{1}{4}\right)^4 D_1^2\overline{D}_1^2\frac{\delta_{13}}{x_{13}^2}\left(-\frac{\delta_{23}}{x_{23}^2}\right)\overline{D}_1^2 D_1^2\frac{\delta_{12}}{x_{12}^2}B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3) \\
&= \frac{1}{4^4}\frac{g^3}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_3\ \Phi_-(z_2)\overline{\Phi}_-(z_3)\frac{\delta_{13}}{x_{13}^2}\frac{\delta_{23}}{x_{23}^2}\Big[-4\pi^2(16)\left(\overline{D}_1^2D_1^2\delta^8(z_{12})\right) \\
&\quad - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{1a}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha} + \left(\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\overline{D}_1^2D_1^2\Big]B(z_1), \qquad (5.1.35)
\end{aligned}
$$

where (5.1.3), (4.2.22) and (2.1.36) have been used. Note that the "additional terms" in (5.1.3) were dropped since $\delta_{23}$ effectively sets $\theta_2 = \theta_3$. For the first term in (5.1.35), we first integrate over $\mathrm{d}^4\theta_3$ and use (2.1.36) to eliminate $\overline{D}_1^2D_1^2$. This allows us to perform an integral over the full measure $\mathrm{d}^8z_2$. For the second and the third terms in (5.1.35), we remove all the $D$'s and $\overline{D}$'s from the background superfield $B$ onto propagators. The final result is

$$
\begin{aligned}
\Gamma_{V_a} &= -\frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8z_1\ \mathrm{d}^4x_3\ \ B(z_1)\Phi_-(z_1)\overline{\Phi}_-(x_3,\theta_1)\ \frac{1}{x_{13}^4} \\
&\quad + \ \frac{1}{4^4}\frac{g^3}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\ \frac{\delta_{23}}{x_{23}^2}\ \times \\
&\qquad\qquad \times\left[8i\,(\sigma^a)^{\alpha\dot\alpha}\,D_{1\alpha}\overline{D}_{1\dot\alpha}\left(\frac{\delta_{13}}{x_{13}^2}\partial_{1a}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right) + D_1^2\overline{D}_1^2\left(\frac{\delta_{13}}{x_{13}^2}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\right] \\
&= \int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\Bigg\{-\frac{g^3}{(4\pi^2)^2}\delta^8(z_{21})\frac{\delta_{13}}{x_{13}^4}\ + \\
&\quad + \frac{1}{4^4}\frac{g^3}{(4\pi^2)^3}\frac{\delta_{23}}{x_{23}^2}\left[8i\,(\sigma^a)^{\alpha\dot\alpha}\,D_{1\alpha}\overline{D}_{1\dot\alpha}\left(\frac{\delta_{13}}{x_{13}^2}\partial_{1a}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right) + D_1^2\overline{D}_1^2\left(\frac{\delta_{13}}{x_{13}^2}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\right]\Bigg\}. \\
&= \int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\ \ \Bigg\{\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\delta^8(z_{21})\delta_{13}\Box\frac{\ln(x_{13}^2M_{V_a}^2)}{x_{13}^2}\ + \\
&\quad + \frac{1}{4^4}\frac{g^3}{(4\pi^2)^3}\frac{\delta_{23}}{x_{23}^2}\left[8i\,(\sigma^a)^{\alpha\dot\alpha}\,D_{1\alpha}\overline{D}_{1\dot\alpha}\left(\frac{\delta_{13}}{x_{13}^2}\partial_{1a}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right) + D_1^2\overline{D}_1^2\left(\frac{\delta_{13}}{x_{13}^2}\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\right]\Bigg\}.
\end{aligned}
$$
$$(5.1.36)$$

Similarly, contributions from Figure 5-2b and Figure 5-2c can be straightforwardly shown to be

$$\begin{aligned}
\Gamma_{V_b} &= \frac{g^2(-g)}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_3\ \ B(z_1)\Phi_-(z_1)\overline{\Phi}_-(z_3)\left(-\frac{1}{4}\right)^2 D_1^2\overline{D}_1^2\frac{\delta_{13}}{x_{13}^2}\left(-\frac{\delta_{13}}{x_{13}^2}\right)\\
&= \frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\ \frac{\delta_{13}}{x_{13}^4}\delta^8(z_{12}) \qquad (5.1.37)\\
&= -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\ \delta_{13}\delta^8(z_{12})\Box\frac{\ln(x_{13}^2 M_{V_b}^2)}{x_{13}^2}
\end{aligned}$$

and

$$\Gamma_{V_c} = -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\ \delta_{12}\delta^8(z_{13})\Box\frac{\ln(x_{12}^2 M_{V_c}^2)}{x_{12}^2},$$

respectively. Finally, regularization of the self-energy diagram shown in Figure 5-2d gives

$$\begin{aligned}
\Gamma_\Sigma &= \frac{(-g)^2}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \ \Phi_-(z_2)\overline{\Phi}_-(z_1)\left(-\frac{1}{4}\right)^2\overline{D}_1^2 D_1^2\frac{\delta_{12}}{x_{12}^2}\left(-\frac{\delta_{12}}{x_{12}^2}\right)\\
&= \frac{1}{4}\frac{g^2}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \Phi_-(z_2)\overline{\Phi}_-(z_1)\delta_{12}\Box\frac{\ln(x_{12}^2 M_{V_\Sigma}^2)}{x_{12}^2}. \qquad (5.1.38)
\end{aligned}$$

Hence, there are four mass parameters in the scheme, and we need to use (5.1.31) or (5.1.32) in order to establish a relation among these parameters. A judicious choice is to use (5.1.32) since (2.1.18) tells us that the last two terms in (5.1.36) vanish if they are acted on by $D_1^2$. Extracting 1-PI Green's functions from (5.1.36), (5.1.38), and (5.1.38) gives

$$\begin{aligned}
\mathrm{LHS} \equiv\ & D^2(z_1)\left\langle T\ B(z_1)\Phi_-(z_2)\overline{\Phi}_-(z_3)\right\rangle_{1PI} = -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}D_1^2\left[-\delta_{13}\delta^8(z_{21})\Box\frac{\ln(x_{13}^2 M_{V_a}^2)}{x_{13}^2}\right.\\
&\left.+\ \delta_{13}\delta^8(z_{12})\Box\frac{\ln(x_{13}^2 M_{V_b}^2)}{x_{13}^2}+\delta_{12}\delta^8(z_{13})\Box\frac{\ln(x_{12}^2 M_{V_c}^2)}{x_{12}^2}\right]\\
=\ & -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}D_1^2\left[\delta_{13}\delta^8(z_{12})\Box\frac{\ln(M_{V_b}^2/M_{V_a}^2)}{x_{13}^2}+\delta_{12}\delta^8(z_{13})\Box\frac{\ln(x_{12}^2 M_{V_c}^2)}{x_{12}^2}\right]. \qquad (5.1.39)
\end{aligned}$$

Similarly, using the 1-PI Green's function from (5.1.38), we get

$$\begin{aligned} \text{RHS} &\equiv -g\left\langle T\ \Phi_-(z_2)\overline{\Phi}_-(z_1)\right\rangle_{1PI} D^2(z_1)\delta^8(z_1-z_3) \\ &= -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\delta_{12}\Box\frac{\ln(x_{12}^2 M_{V_\Sigma}^2)}{x_{12}^2}D_1^2\delta^8(z_{13}). \end{aligned} \tag{5.1.40}$$

Since we know that LHS = RHS from (5.1.32), it is certainly true that the equality still holds if we multiply both sides by $\delta^2(\theta_{13})$ and then integrate over the $z_1$ coordinate; that is,

$$\int \mathrm{d}^8 z_1\ \delta^2(\theta_{13})\ \text{LHS}\ =\ \int \mathrm{d}^8 z_1\ \delta^2(\theta_{13})\ \text{RHS}. \tag{5.1.41}$$

The reason for doing this somewhat odd manipulation is as follows: Observe that both of LHS and RHS already contain $\delta^2(\theta_{13})$. Therefore, (2.1.33) tells us that when acted upon by $\delta^2(\theta_{13})$, all terms in LHS and RHS vanish, except for those terms with $D_1^2\delta^2(\theta_{13})$. Hence, using (2.1.32) in an intermediate step, we obtain

$$\begin{aligned} \int \mathrm{d}^8 z_1\ \delta^2(\theta_{13})\ \text{LHS} &= -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \left[\delta^2(\theta_{13})D_1^2\delta^2(\theta_{13})\right]\times \\ &\quad \left\{\delta^2(\overline{\theta}_{13})\delta^8(z_{12})\Box\frac{\ln(M_{V_c}^2/M_{V_a}^2)}{x_{13}^2}+\delta_{12}\delta^6(\overline{z}_{13})\Box\frac{\ln(x_{12}^2 M_{V_b}^2)}{x_{12}^2}\right\} \\ &= -\frac{g^3}{(4\pi^2)^2}\delta_{23}\left[\Box\frac{\ln(M_{V_c}^2/M_{V_a}^2)}{x_{23}^2}+\Box\frac{\ln(x_{23}^2 M_{V_b}^2)}{x_{23}^2}\right] \end{aligned} \tag{5.1.42}$$

and

$$\begin{aligned} \int \mathrm{d}^8 z_1\ \delta^2(\theta_{13})\ \text{RHS} &= -\frac{1}{4}\frac{g^3}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \left[\delta^2(\theta_{13})D_1^2\delta^2(\theta_{13})\right]\delta_{12}\delta^6(\overline{z}_{13})\Box\frac{\ln(x_{12}^2 M_{V_\Sigma}^2)}{x_{12}^2} \\ &= -\frac{g^3}{(4\pi^2)^2}\delta_{23}\Box\frac{\ln(x_{23}^2 M_{V_\Sigma}^2)}{x_{23}^2}. \end{aligned} \tag{5.1.43}$$

Equating (5.1.42) with (5.1.43) gives the result

$$\boxed{M_{V_a}^2 M_{V_\Sigma}^2\ =\ M_{V_b}^2 M_{V_c}^2}\ . \tag{5.1.44}$$

Although we will not attempt to show it explicitly here, it has been verified that if we had applied the Ward-Takahashi identity (5.1.31), then the same mass relation as (5.1.44) would have followed.

### 5.1.4 Two-Loop Level

**Renormalization**

Supergraphs contributing to the two-loop $\beta$-function are shown in Figure 5-3. The amplitude for the supergraph in Figure 5-3a1 is given by

$$\begin{aligned}\Gamma^{(2)}_{a1} &= \frac{1}{2}\frac{(-g)^2}{(4\pi^2)^3}\left(-\frac{1}{4}\right)^6 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\times \\ &\left[\left(\overline{D}^2_1 D^2_1 \frac{\delta_{12}}{x^2_{12}}\right)\ \Sigma(x_{23})\ \left(\overline{D}^2_3 D^2_3 \frac{\delta_{34}}{x^2_{34}}\right)\ \left(D^2_1 \overline{D}^2_1 \frac{\delta_{14}}{x^2_{14}}\right)\right], \quad (5.1.45)\end{aligned}$$

where $\Sigma(x_{23})$ is the matter field self-energy insertion which we choose to regularize first. Since $\Sigma(x_{23})$ is given by

$$\Sigma(x_{23}) = \frac{(-g)^2}{(4\pi^2)^2}\left(-\frac{1}{4}\right)^2 \left(\overline{D}^2_2 D^2_2 \frac{\delta_{23}}{x^2_{23}}\right)\left(-\frac{\delta_{23}}{x^2_{23}}\right) = \frac{g^2}{(4\pi^2)^2}\ \delta_{23}\frac{1}{4}\Box\frac{\ln x^2_{23}M^2_{V_\Sigma}}{x^2_{23}}, \quad (5.1.46)$$

we can integrate $\Box$ by parts onto the propagator $P_{12}$ and use (4.2.22) to obtain

$$\begin{aligned}\Gamma^{(2)}_{a1} &= -\frac{1}{2}\frac{g^4}{(4\pi^2)^4}\left(\frac{1}{4}\right)^7 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\times \\ &\left\{\left[D^2_2\overline{D}^2_2\delta^8(z_{21})\right]\delta_{23}\frac{\ln x^2_{23}M^2_{V_\Sigma}}{x^2_{23}}\ \left(\overline{D}^2_3 D^2_3\frac{\delta_{34}}{x^2_{34}}\right)\ \left(D^2_1\overline{D}^2_1\frac{\delta_{14}}{x^2_{14}}\right)\right\}. \quad (5.1.47)\end{aligned}$$

Further standard manipulations gives

$$\begin{aligned}\Gamma^{(2)}_{a1} &= -\frac{1}{2}\frac{g^4}{(4\pi^2)^4}\left(\frac{1}{4}\right)^7 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\times \\ &\qquad \left\{\delta^8(z_{21})\delta_{23}\frac{\ln x^2_{23}M^2_{V_\Sigma}}{x^2_{23}}\ \left(16\Box\overline{D}^2_3 D^2_3\frac{\delta_{34}}{x^2_{34}}\right)\left(D^2_1\overline{D}^2_1\frac{\delta_{14}}{x^2_{14}}\right)\right\} \\ &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_2\ \mathrm{d}^8 z_3\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\times \\ &\qquad \left\{\delta^8(z_{21})\delta_{23}\frac{\ln x^2_{23}M^2_{V_\Sigma}}{x^2_{23}}\ \left[D^2_4\overline{D}^2_4\delta^8(z_{43})\right]\left(\overline{D}^2_4 D^2_4\frac{\delta_{41}}{x^2_{41}}\right)\right\} \\ &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5 \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)\left\{\frac{\ln x^2_{14}M^2_{V_\Sigma}}{x^2_{14}}\ 16\delta_{14}\ \times\right.\end{aligned}$$

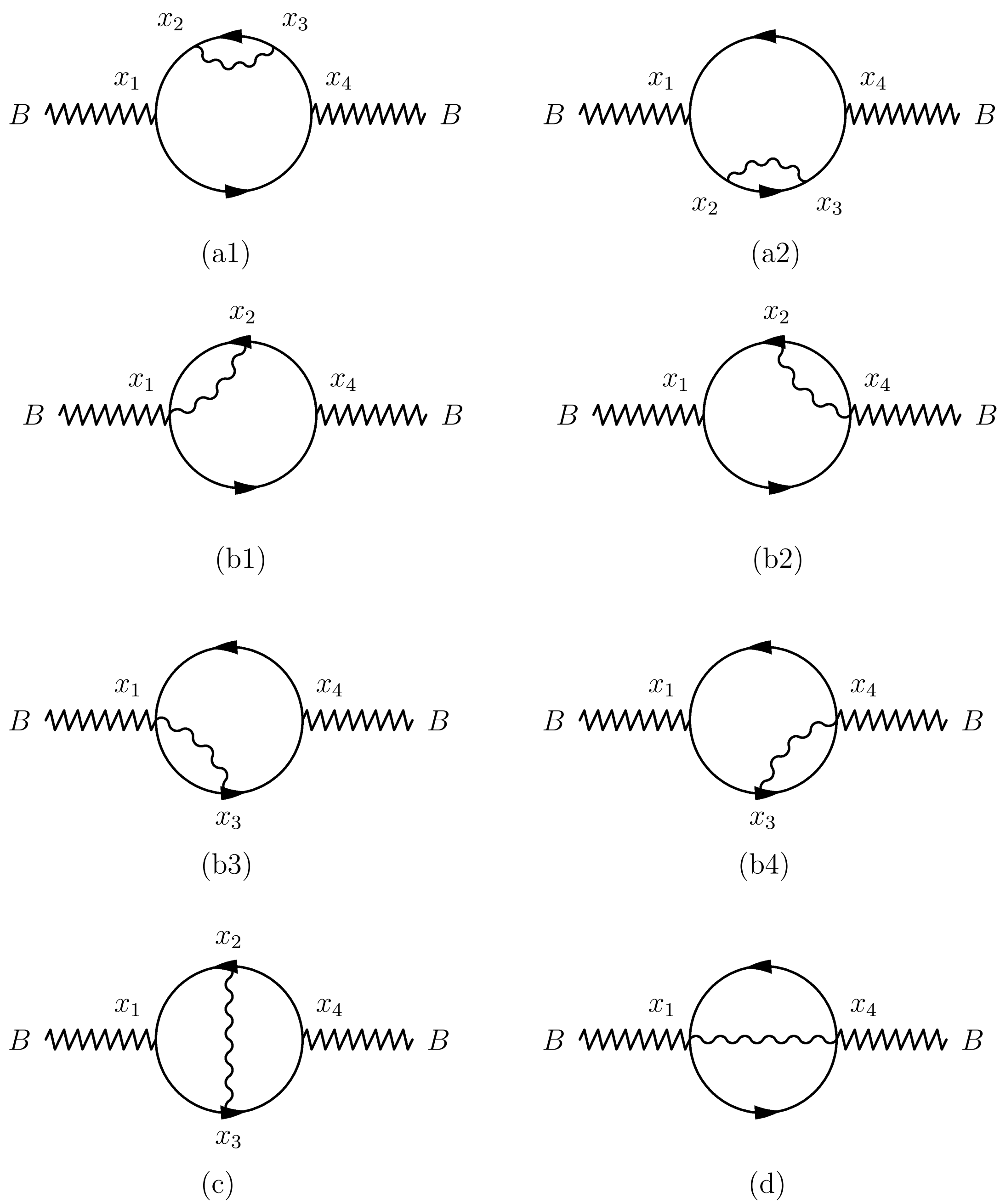


Figure 5-3: Two-loop supergraphs contributing to the $\beta$ function.

$$\left[-4\pi^2(16)\delta^4(x_{41}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{4a}\frac{1}{x_{41}^2}\right)\overline{D}_{4\dot\alpha}D_{4\alpha} + \left(\frac{1}{x_{41}^2}\right)\overline{D}_4^2 D_4^2\right] B(z_4)\Bigg\}. \tag{5.1.48}$$

Same computations presented above can be applied directly to evaluate the supergraph shown in Figure 5-3a2, and the result is

$$\begin{aligned}\Gamma^{(2)}_{a2} &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \ B(z_4)\left\{\frac{\ln x_{14}^2 M_{V_\Sigma}^2}{x_{14}^2}\ 16\delta_{14}\ \times\right.\\ &\left.\left[-4\pi^2(16)\delta^4(x_{14}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{1a}\frac{1}{x_{14}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha} + \left(\frac{1}{x_{14}^2}\right)\overline{D}_1^2 D_1^2\right] B(z_1)\right\}.\end{aligned} \tag{5.1.49}$$

As expected, this is exactly (5.1.48) with 1 and 4 coordinates exchanged. Since supergraphs shown in Figure 5-3b1∼4 are very similar in nature, we will outline the computation for only one of them. Let us consider Figure 5-3b1 first. This is given by

$$\begin{aligned}\Gamma^{(2)}_{b1} &= \frac{1}{2}\frac{g^2(-g)^2}{(4\pi^2)^4}\left(-\frac{1}{4}\right)^6\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_4\ \ B(z_1)B(z_4)\ \times\\ &\quad\left[\left(\overline{D}_1^2D_1^2\frac{\delta_{12}}{x_{12}^2}\right)\left(-\frac{\delta_{12}}{x_{12}^2}\right)\left(D_4^2\overline{D}_4^2\frac{\delta_{42}}{x_{42}^2}\right)\left(\overline{D}_4^2D_4^2\frac{\delta_{41}}{x_{41}^2}\right)\right]\\ &= \frac{1}{2}\frac{g^4}{(4\pi^2)^4}\left(\frac{1}{4}\right)^5\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_4\ \ B(z_1)B(z_4)\ \times\\ &\quad\left[\delta_{12}\left(\Box\frac{\ln x_{21}^2M_{V_c}^2}{x_{21}^2}\right)\left(D_4^2\overline{D}_4^2\frac{\delta_{42}}{x_{42}^2}\right)\left(\overline{D}_4^2D_4^2\frac{\delta_{41}}{x_{41}^2}\right)\right]\\ &= -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5\int \mathrm{d}^8z_1\ \mathrm{d}^8z_2\ \mathrm{d}^8z_4\ \ B(z_1)B(z_4)\ \times\\ &\quad\left[\delta_{12}\frac{\ln x_{21}^2M_{V_c}^2}{x_{21}^2}\left(D_4^2\overline{D}_4^2\delta^8(z_{42})\right)\left(\overline{D}_4^2D_4^2\frac{\delta_{41}}{x_{41}^2}\right)\right]\\ &= -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \ B(z_1)\left\{\frac{\ln x_{14}^2M_{V_c}^2}{x_{14}^2}\ 16\delta_{14}\times\right.\\ &\left.\left[-4\pi^2(16)\delta^4(x_{41}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{4a}\frac{1}{x_{41}^2}\right)\overline{D}_{4\dot\alpha}D_{4\alpha} + \left(\frac{1}{x_{41}^2}\right)\overline{D}_4^2D_4^2\right] B(z_4)\right\}.\end{aligned} \tag{5.1.50}$$

Apart from the sign and the mass parameter, this expression is exactly the same as that shown in (5.1.48). In evaluating the supergraph shown in Figure 5-3b2, the same steps taken to obtain (5.1.50) can be applied, and the only difference is in mass parameters arising from sub-divergences. Furthermore, Figure 5-3b3 is related to Figure 5-3b1 under the exchange of coordinates $1 \leftrightarrow 4$, and there exists a similar relation between Figure 5-3b3 and Figure 5-3b4 as that between Figure 5-3b1 and Figure 5-3b2. Hence, we arrive at the following expressions:

$$\Gamma^{(2)}_{b2} = -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5 \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_4 \;\; B(z_1)\left\{\frac{\ln x_{14}^2 M_{V_b}^2}{x_{14}^2} 16\delta_{14}\times \right. \tag{5.1.51}$$
$$\left.\left[-4\pi^2(16)\delta^4(x_{41}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{4a}\frac{1}{x_{41}^2}\right)\overline{D}_{4\dot\alpha}D_{4\alpha} + \left(\frac{1}{x_{41}^2}\right)\overline{D}_4^2 D_4^2\right] B(z_4)\right\}.$$

$$\Gamma^{(2)}_{b3} = -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5 \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_4 \;\; B(z_4)\left\{\frac{\ln x_{14}^2 M_{V_b}^2}{x_{14}^2} 16\delta_{14}\times \right. \tag{5.1.52}$$
$$\left.\left[-4\pi^2(16)\delta^4(x_{14}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{1a}\frac{1}{x_{14}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha} + \left(\frac{1}{x_{14}^2}\right)\overline{D}_1^2 D_1^2\right] B(z_1)\right\}.$$

$$\Gamma^{(2)}_{b4} = -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\left(\frac{1}{4}\right)^5 \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_4 \;\; B(z_4)\left\{\frac{\ln x_{14}^2 M_{V_c}^2}{x_{14}^2} 16\delta_{14}\times \right. \tag{5.1.53}$$
$$\left.\left[-4\pi^2(16)\delta^4(x_{14}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{1a}\frac{1}{x_{14}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha} + \left(\frac{1}{x_{14}^2}\right)\overline{D}_1^2 D_1^2\right] B(z_1)\right\}.$$

Simply writing down the amplitude for the diagram shown in Figure 5-3c should be clear. Straightforward manipulations show

$$\begin{aligned}
\Gamma^{(2)}_{c} &= \frac{1}{2}\frac{(-g)^4}{(4\pi^2)^5}\left(-\frac{1}{4}\right)^8 \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_2\, \mathrm{d}^8 z_3\, \mathrm{d}^8 z_4 \;\; B(z_1)B(z_4)\times \\
&\quad \left\{\left[\left(\overline{D}_1^2 D_1^2 \frac{\delta_{12}}{x_{12}^2}\right)\left(D_1^2\overline{D}_1^2\frac{\delta_{23}}{x_{23}^2}\right)\right]\left(-\frac{\delta_{23}}{x_{23}^2}\right)\left[\left(\overline{D}_4^2 D_4^2\frac{\delta_{43}}{x_{43}^2}\right)\left(D_4^2\overline{D}_4^2\frac{\delta_{42}}{x_{42}^2}\right)\right]\right\} \\
&= -\frac{1}{2}\frac{g^4}{(4\pi^2)^5}\left(\frac{1}{4}\right)^8 \int \mathrm{d}^8 z_1\, \mathrm{d}^8 z_2\, \mathrm{d}^8 z_3\, \mathrm{d}^8 z_4 \;\; \frac{16\delta_{12}}{x_{13}^2}\frac{1}{x_{23}^2}\frac{16\delta_{42}}{x_{42}^2}\times \\
&\quad \left\{\left[-4\pi^2(16)\delta^4(x_{12}) - 8i\,(\sigma^a)^{\alpha\dot\alpha}\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha} + \left(\frac{1}{x_{12}^2}\right)\overline{D}_1^2 D_1^2\right] B(z_1)\right\}\times
\end{aligned}$$

$$\left\{\left[-4\pi^2(16)\delta^4(x_{43}) - 8i\left(\sigma^b\right)^{\alpha\dot{\alpha}}\left(\partial_{4b}\frac{1}{x_{43}^2}\right)\overline{D}_{4\dot{\beta}}D_{4\beta} + \left(\frac{1}{x_{43}^2}\right)\overline{D}_4^2 D_4^2\right]B(z_4)\right\}. \tag{5.1.54}$$

In order to make computations more lucid, we will break up the above expression into the following four components:

$$\begin{aligned}
\Gamma_{c1}^{(2)} &\equiv \frac{1}{2}\frac{g^4}{(4\pi^2)^4}\left(\frac{1}{4}\right)^2\int \mathrm{d}^8z_1\ \mathrm{d}^4x_3\ \mathrm{d}^8z_4\ \ B(z_1)\left\{\frac{1}{x_{13}^4}\frac{\delta_{41}}{x_{41}^2}\times\right. &(5.1.55)\\
&\qquad \left.\left[-8i\left(\sigma^b\right)^{\beta\dot{\beta}}\left(\partial_{4b}\frac{1}{x_{43}^2}\right)\overline{D}_{4\dot{\beta}}D_{4\beta} + \left(\frac{1}{x_{43}^2}\right)\overline{D}_4^2 D_4^2\right]B(z_4)\right\}\\
\Gamma_{c2}^{(2)} &\equiv \frac{1}{2}\frac{g^4}{(4\pi^2)^4}\left(\frac{1}{4}\right)^2\int \mathrm{d}^8z_1\ \mathrm{d}^4x_2\ \mathrm{d}^8z_4\ \ B(z_4)\left\{\frac{1}{x_{42}^4}\frac{\delta_{14}}{x_{14}^2}\times\right. &(5.1.56)\\
&\qquad \left.\left[-8i\left(\sigma^a\right)^{\alpha\dot{\alpha}}\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\overline{D}_{1\dot{\alpha}}D_{1\alpha} + \left(\frac{1}{x_{12}^2}\right)\overline{D}_1^2 D_1^2\right]B(z_1)\right\}\\
\Gamma_{c3}^{(2)} &\equiv -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \ B(z_1)B(z_4)\ \frac{\delta_{41}}{x_{41}^6} &(5.1.57)\\
\Gamma_{c4}^{(2)} &\equiv -\frac{1}{2}\frac{g^4}{(4\pi^2)^5}\left(\frac{1}{4}\right)^4\int \mathrm{d}^8z_1\ \mathrm{d}^4x_2\ \mathrm{d}^4x_3\ \mathrm{d}^8z_4\ \ \delta_{14}\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\frac{1}{x_{42}^2}\times &(5.1.58)\\
&\qquad \left\{\left[-8i\left(\sigma^a\right)^{\alpha\dot{\alpha}}\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\overline{D}_{1\dot{\alpha}}D_{1\alpha} + \left(\frac{1}{x_{12}^2}\right)\overline{D}_1^2 D_1^2\right]B(z_1)\right\}\times\\
&\qquad \left\{\left[-8i\left(\sigma^b\right)^{\beta\dot{\beta}}\left(\partial_{4b}\frac{1}{x_{43}^2}\right)\overline{D}_{4\dot{\beta}}D_{4\beta} + \left(\frac{1}{x_{43}^2}\right)\overline{D}_4^2 D_4^2\right]B(z_4)\right\}
\end{aligned}$$

In evaluating $\Gamma_{c1}^{(2)}$, we first use the identity (4.2.20) for $1/x_{13}^4$. Then, since

$$\Box_1 f(x_{13}) \;=\; \Box_3 f(x_{13}), \tag{5.1.59}$$

we can integrate $\Box_3$ by parts to make it act on $1/x_{43}^4$. As a result, we obtain

$$\begin{aligned}
\Gamma_{c1}^{(2)} &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{1}{64}\int \mathrm{d}^8z_1\ \mathrm{d}^4x_3\ \mathrm{d}^8z_4\ \ \left\{B(z_1)\frac{\ln x_{13}^2 M_{V_a}^2}{x_{13}^2}\frac{\delta_{41}}{x_{41}^2}\right.\\
&\qquad \left.\left[-8i\left(\sigma^b\right)^{\beta\dot{\beta}}\left(\partial_{4b}\delta^4(x_{43})\right)\overline{D}_{4\dot{\beta}}D_{4\beta} + \left(\delta^4(x_{43})\right)\overline{D}_4^2 D_4^2\right]B(z_4)\right\}\\
&= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{1}{64}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \ \left\{B(z_1)\frac{\delta_{41}}{x_{41}^2}\times\right.
\end{aligned}$$

$$\left[-8i\left(\sigma^b\right)^{\beta\dot\beta}\left(\partial_{4b}\frac{\ln x_{41}^2 M_{V_a}^2}{x_{41}^2}\right)\overline{D}_{4\dot\beta}D_{4\beta}+\left(\frac{\ln x_{41}^2 M_{V_a}^2}{x_{41}^2}\right)\overline{D}_4^2 D_4^2\right]B(z_4)\Big\}$$

$$\begin{aligned} &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{1}{64}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\quad \Big\{B(z_1)\delta_{41}\left(\frac{\ln x_{41}^2 M_{V_a}^2}{x_{41}^2}\right)\times \\ &\qquad \left[-8i\left(\sigma^b\right)^{\beta\dot\beta}\left(\partial_{4b}\frac{1}{x_{41}^2}\right)\overline{D}_{4\dot\beta}D_{4\beta}+\left(\frac{1}{x_{41}^2}\right)\overline{D}_4^2 D_4^2\right]B(z_4)\Big\} \qquad (5.1.60) \\ &\quad -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^b\right)^{\beta\dot\beta}}{8}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \left[B(z_1)\overline{D}_{4\dot\beta}D_{4\beta}B(z_4)\right]\frac{\delta_{41}}{x_{41}^4}\partial_{4b}\left(\ln x_{41}^2 M_{V_a}^2\right)\end{aligned}$$

Similarly,

$$\begin{aligned}\Gamma_{c2}^{(2)} &= \frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{1}{64}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\quad \Big\{B(z_4)\delta_{41}\left(\frac{\ln x_{41}^2 M_{V_a}^2}{x_{41}^2}\right)\times \\ &\qquad \left[-8i\left(\sigma^a\right)^{\alpha\dot\alpha}\left(\partial_{1a}\frac{1}{x_{14}^2}\right)\overline{D}_{1\dot\alpha}D_{1\alpha}+\left(\frac{1}{x_{14}^2}\right)\overline{D}_1^2 D_1^2\right]B(z_1)\Big\} \qquad (5.1.61) \\ &\quad -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^a\right)^{\alpha\dot\alpha}}{8}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \left[B(z_4)\overline{D}_{1\dot\alpha}D_{1\alpha}B(z_1)\right]\frac{\delta_{14}}{x_{14}^4}\left(\partial_{1a}\ln x_{14}^2 M_{V_a}^2\right)\end{aligned}$$

Before we proceed with $\Gamma_{c4}^{(2)}$, let us first consider the contribution from the supergraph shown in Figure 5-3d. The computation of this amplitude shows

$$\begin{aligned}\Gamma_d^{(2)} &= \frac{1}{2}\frac{(g^2)^2}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ B(z_1)B(z_4)\ \left(-\frac{1}{4}\right)^4\left(\overline{D}_1^2D_1^2\frac{\delta_{14}}{x_{14}^2}\right)\left(-\frac{\delta_{14}}{x_{14}^2}\right)\left(D_1^2\overline{D}_1^2\frac{\delta_{14}}{x_{14}^2}\right) \\ &= -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ B(z_1)B(z_4)\ \frac{\delta_{14}}{x_{14}^6}. \qquad (5.1.62)\end{aligned}$$

All these computations are very complicated, and we fear that the reader might be either bored or discouraged. Therefore, we now take a moment to simplify what we have done so far. Let us sum the expressions for $\Gamma_{a1}^{(2)}, \Gamma_{b1}^{(2)}, \Gamma_{b2}^{(2)}$ and $\Gamma_{c1}^{(2)}$ shown in (5.1.48), (5.1.50), (5.1.52) and (5.1.60), respectively. Then, we observe that we get terms proportional to

$$\ln\left(\frac{M_{V_\Sigma}^2 M_{V_a}^2}{M_{V_b}^2 M_{V_c}^2}\right). \qquad (5.1.63)$$

However, this becomes zero when we impose the Ward-Takahashi identity which relates the mass parameters as shown in (5.1.44). Hence, we obtain the following clean result:

$$
\begin{aligned}
I \quad &\equiv \quad \Gamma^{(2)}_{a1} + \Gamma^{(2)}_{b1} + \Gamma^{(2)}_{b2} + \Gamma^{(2)}_{c1} \\
&= \quad -\frac{1}{8}\frac{g^4}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x^2_{41}} \ln\left(\frac{1}{x^2_{41}}\frac{M^2_{V_\Sigma}}{M^2_{V_b}M^2_{V_c}}\right) \qquad (5.1.64) \\
&\quad\ -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^b\right)^{\beta\dot\beta}}{8}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad \left[B(z_1)\overline{D}_{4\dot\beta}D_{4\beta}B(z_4)\right]\frac{\delta_{41}}{x^4_{41}}\left(\partial_{4b}\ln x^2_{41}M^2_{V_a}\right).
\end{aligned}
$$

Similarly, if we sum $\Gamma^{(2)}_{a2}, \Gamma^{(2)}_{b3}, \Gamma^{(2)}_{b4}$ and $\Gamma^{(2)}_{c2}$ shown in (5.1.49), (5.1.53), (5.1.54) and (5.1.61), respectively, we obtain

$$
\begin{aligned}
II \quad &\equiv \quad \Gamma^{(2)}_{a2} + \Gamma^{(2)}_{b3} + \Gamma^{(2)}_{b4} + \Gamma^{(2)}_{c2} \\
&= \quad -\frac{1}{8}\frac{g^4}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x^2_{41}} \ln\left(\frac{1}{x^2_{41}}\frac{M^2_{V_\Sigma}}{M^2_{V_b}M^2_{V_c}}\right) \\
&\quad\ -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^a\right)^{\alpha\dot\alpha}}{8}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad \left[B(z_4)\overline{D}_{1\dot\alpha}D_{1\alpha}B(z_1)\right]\frac{\delta_{14}}{x^4_{14}}\left(\partial_{1a}\ln x^2_{14}M^2_{V_a}\right) \\
&= \quad -\frac{1}{8}\frac{g^4}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x^2_{41}} \ln\left(\frac{1}{x^2_{41}}\frac{M^2_{V_\Sigma}}{M^2_{V_b}M^2_{V_c}}\right) \qquad (5.1.65) \\
&\quad\ -\frac{1}{2}\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^a\right)^{\alpha\dot\alpha}}{8}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad \left[B(z_1)\overline{D}_{4\dot\alpha}D_{4\alpha}B(z_4)\right]\frac{\delta_{41}}{x^4_{41}}\left(\partial_{4a}\ln x^2_{41}M^2_{V_a}\right)
\end{aligned}
$$

Collecting $\Gamma^{(2)}_{c3}$ from (5.1.57) and $\Gamma^{(2)}_{d}$ from (5.1.62) gives

$$
III \ \equiv\ \Gamma^{(2)}_{c3} + \Gamma^{(2)}_{d} \ =\ -\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad B(z_1)B(z_4)\ \frac{\delta_{14}}{x^6_{14}}. \qquad (5.1.66)
$$

Hence,

$$
\begin{aligned}
\sum_{A\neq 3d}\Gamma^{(2)}_A \quad &= \quad I + II + III \\
&= \quad -\frac{1}{4}\frac{g^4}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x^2_{41}} \ln\left(\frac{1}{x^2_{41}}\frac{M^2_{V_\Sigma}}{M^2_{V_b}M^2_{V_c}}\right) \\
&\quad\ -\frac{g^4}{(4\pi^2)^3}\frac{i\left(\sigma^a\right)^{\alpha\dot\alpha}}{8}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4 \quad \left[B(z_1)\overline{D}_{4\dot\alpha}D_{4\alpha}B(z_4)\right]\frac{\delta_{41}}{x^4_{41}}\left(\partial_{4a}\ln x^2_{41}M^2_{V_a}\right)
\end{aligned}
$$

$$- \frac{g^4}{(4\pi^2)^3} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad B(z_1) B(z_4) \; \frac{\delta_{14}}{x_{14}^6}. \tag{5.1.67}$$

Let us consider the second term in the above equation. We observe that

$$\frac{1}{x_{41}^4} \partial_{4a} \left( \ln x_{41}^2 M_{V_a} \right) \;=\; \frac{1}{x_{41}^4} \frac{2\,(x_{41})_a}{x_{41}^2} \;= -\frac{1}{2} \partial_{4a} \left( \frac{1}{x_{41}^4} \right). \tag{5.1.68}$$

Therefore, upon using the identities (4.2.20) and (4.2.21), (5.1.67) becomes

$$\begin{aligned}
\sum_{A \neq 3d} \Gamma_A^{(2)} \;=\; & -\frac{1}{4} \frac{g^4}{(4\pi^2)^2} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad B(z_1) B(z_4) \frac{\delta^8(z_{41})}{x_{41}^2} \ln \left( \frac{1}{x_{41}^2} \frac{M_{V_\Sigma}^2}{M_{V_b}^2 M_{V_c}^2} \right) \\
& - \frac{g^4}{(4\pi^2)^3} \frac{i\,(\sigma^a)^{\alpha\dot\alpha}}{8} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad \left[ B(z_1) \overline{D}_{4\dot\alpha} D_{4\alpha} B(z_4) \right] \delta_{41} \left( \frac{1}{8} \right) \partial_{4a} \Box \frac{\ln x_{41}^2 M'^2}{x_{41}^2} \\
& - \frac{g^4}{(4\pi^2)^3} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad B(z_1) B(z_4) \; \delta_{14} \left( -\frac{1}{32} \right) \Box\Box \frac{\ln x_{41}^2 M'^2}{x_{41}^2}.
\end{aligned} \tag{5.1.69}$$

In the second term of (5.1.69), $\partial_{4a}$ can be integrated by parts onto the background field. Then, by using (2.1.15), we see that

$$\begin{aligned}
B(z_1) \left[ D^\alpha, \overline{D}^2 \right] D_\alpha B(z_4) \;&=\; B(z_1) D^\alpha \overline{D}^2 D_\alpha B(z_4) - B(z_1) \overline{D}^2 D^2 B(z_4) \\
&=\; B(z_1) D^\alpha \overline{D}^2 D_\alpha B(z_4) - \frac{1}{2} B(z_1) \left( \overline{D}^2 D^2 + D^2 \overline{D}^2 \right) B(z_4) \\
&=\; B(z_1) D^\alpha \overline{D}^2 D_\alpha B(z_4) - 8 B(z_1) \Box \Pi_0 B(z_4),
\end{aligned} \tag{5.1.70}$$

where $\Pi_0$ was defined in (2.2.53). (5.1.69) now becomes

$$\begin{aligned}
\sum_{A \neq 3d} \Gamma_A^{(2)} \;=\; & -\frac{1}{4} \frac{g^4}{(4\pi^2)^2} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad B(z_1) B(z_4) \frac{\delta^8(z_{41})}{x_{41}^2} \ln \left( \frac{1}{x_{41}^2} \frac{M_{V_\Sigma}^2}{M_{V_b}^2 M_{V_c}^2} \right) \\
& + \frac{1}{4} \, \frac{1}{64} \frac{g^4}{(4\pi^2)^3} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad \left[ B(z_1) D_4^\alpha \overline{D}_4^2 D_{4\alpha} B(z_4) \right] \delta_{41} \Box \frac{\ln x_{41}^2 M'^2}{x_{41}^2} \\
& + \frac{1}{32} \frac{g^4}{(4\pi^2)^3} \int \mathrm{d}^8 z_1 \; \mathrm{d}^8 z_4 \quad \left[ B(z_1) \left( \Box - \Box \Pi_0 \right) B(z_4) \right] \; \delta_{14} \Box \frac{\ln x_{41}^2 M'^2}{x_{41}^2}.
\end{aligned} \tag{5.1.71}$$

However, since

$$\Box - \Box\Pi_0 \;=\; \Box\Pi_{1/2} \;= -\frac{D^\alpha \overline{D}^2 D_\alpha}{8}, \tag{5.1.72}$$

the last two terms in (5.1.71) cancel each other. In summary,

$$\sum_{A\neq 3d} \Gamma_A^{(2)} \;= -\frac{1}{4}\frac{g^4}{(4\pi^2)^2}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x_{41}^2}\ln\left(\frac{1}{x_{41}^2}\frac{M_{V_\Sigma}^2}{M_{V_b}^2 M_{V_c}^2}\right). \tag{5.1.73}$$

We observe that this is purely local[2], and therefore, they cannot contribute to $\beta$-function. Thus, we conclude that $\Gamma_{c4}^{(2)}$ gives the sole contribution to $\beta$-function. Let us now evaluate $\Gamma_{c4}^{(2)}$. At first sight, this task seems rather formidable. There are four separate terms in (5.1.58), but we can use $D$-algebra to argue that we need to consider only one of them. The argument goes as follows. After some standard manipulations, three of the four terms in (5.1.58) will be in the form of either

$$\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ \overline{D}_1^2 D_1^2\ B(z_1)\ G_1(z_1 - z_4)\ \overline{D}_4^2 D_4^2 B(z_4), \tag{5.1.74}$$

$$\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ \overline{D}_{1\dot\alpha} D_{1\alpha} B(z_1)\ G_2(z_1 - z_4)\ \overline{D}_4^2 D_4^2 B(z_4), \tag{5.1.75}$$

or

$$\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ \overline{D}_1^2 D_1^2 B(z_1)\ G_3(z_1 - z_4)\ \overline{D}_{4\dot\beta} D_{4\beta} B(z_4). \tag{5.1.76}$$

However, we can readily show that all of these vanish by integrating $\overline{D}$'s by parts and using (2.1.18). Hence, the only amplitude that we need to compute is

$$\begin{aligned}\Gamma_{c4}^{(2)} \;\;=\;\; & -\frac{1}{2}\frac{g^4}{(4\pi^2)^5}\ \frac{i^2}{4}\,(\sigma^a)^{\alpha\dot\alpha}\left(\sigma^b\right)^{\beta\dot\beta}\int \mathrm{d}^8 z_1\ \mathrm{d}^4 x_2\ \mathrm{d}^4 x_3\ \mathrm{d}^8 z_4\ \times \\ & \left\{\overline{D}_{1\dot\alpha} D_{1\alpha} B(z_1)\overline{D}_{4\dot\beta} D_{4\beta} B(z_4)\ \delta_{14}\left[\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\frac{1}{x_{42}^2}\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\left(\partial_{4b}\frac{1}{x_{43}^2}\right)\right]\right\}.\end{aligned} \tag{5.1.77}$$

In order to evaluate this integral, we integrate $\partial_{4b}$ by parts to obtain

$$\Gamma_{c4}^{(2)} \;\;=\;\; -\frac{1}{2}\frac{g^4}{(4\pi^2)^5}\ \frac{i^2}{4}\,(\sigma^a)^{\alpha\dot\alpha}\left(\sigma^b\right)^{\beta\dot\beta}\int \mathrm{d}^8 z_1\ \mathrm{d}^4 x_2\ \mathrm{d}^4 x_3\ \mathrm{d}^8 z_4\ \times \tag{5.1.78}$$

[2] This term also can be canceled, if we consider sea gull diagrams.

$$\Big\{\overline{D}_{1\dot\alpha}D_{1\alpha}B(z_1)\overline{D}_{4\dot\beta}D_{4\beta}B(z_4)\ \delta_{14}\left[\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\left(-\partial_{4b}\frac{1}{x_{42}^2}\right)\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\frac{1}{x_{43}^2}\right]$$
$$+\ \overline{D}_{1\dot\alpha}D_{1\alpha}B(z_1)\partial_{4b}\overline{D}_{4\dot\beta}D_{4\beta}B(z_4)\ \delta_{14}\left[\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\frac{1}{x_{42}^2}\left(\partial_{1a}\frac{1}{x_{12}^2}\right)\frac{1}{x_{43}^2}\right]\Big\}.$$

At this point, we note that the second term in (5.1.78) is finite by power counting. Therefore, this term will not contribute to the two-loop $\beta$-function and may be ignored for our purpose. In evaluating the first term in (5.1.78), we break it up into a traceless part and a trace part as follows:

$$\begin{aligned}\Gamma_{c4}^{(2)} \;=\;& -\frac{1}{8}\frac{g^4}{(4\pi^2)^5}\ (\sigma^a)^{\alpha\dot\alpha}\left(\sigma^b\right)^{\beta\dot\beta}\int \mathrm{d}^8z_1\ \mathrm{d}^4x_2\ \mathrm{d}^4x_3\ \mathrm{d}^8z_4\ \overline{D}_{1\dot\alpha}D_{1\alpha}B(z_1)\overline{D}_{4\dot\beta}D_{4\beta}B(z_4)\times\\ &\left\{\delta_{14}\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\frac{1}{x_{43}^2}\left[\left(\frac{\partial}{\partial x_4^a}\frac{\partial}{\partial x_1^b}-\frac{\delta_{ab}}{4}\frac{\partial}{\partial x_4}\cdot\frac{\partial}{\partial x_1}\right)+\frac{\delta_{ab}}{4}\frac{\partial}{\partial x_4}\cdot\frac{\partial}{\partial x_1}\right]\left(\frac{1}{x_{42}^2}\frac{1}{x_{12}^2}\right)\right\}\\ &+\text{Finite Terms}\end{aligned} \tag{5.1.79}$$

The traceless part in the above expression is convergent[45]; in fact, it can be shown[55] to be purely local in spacetime. Therefore, we conclude that the traceless part also may be ignored in calculating the two-loop $\beta$-function. Furthermore, the trace part can be computed by using the Gegenbauer technique discussed in Appendix B, and the result is

$$\int \mathrm{d}^4x_2\ \mathrm{d}^4x_3\ \left[\frac{1}{x_{13}^2}\frac{1}{x_{23}^2}\frac{1}{x_{43}^2}\ \frac{\partial}{\partial x_4}\cdot\frac{\partial}{\partial x_1}\left(\frac{1}{x_{42}^2}\frac{1}{x_{12}^2}\right)\right] \;=\; 4\pi^4\frac{1}{x_{41}^4} \;=\; -\pi^4\Box\frac{\ln x_{41}^2M'^2}{x_{41}^2}. \tag{5.1.80}$$

Upon using the identity

$$(\sigma^a)^{\alpha\dot\alpha}(\sigma_a)^{\beta\dot\beta} \;=\; 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} \tag{5.1.81}$$

and transferring all the $D$'s and $\overline{D}$'s to one of the background superfield, (5.1.79) becomes

$$\begin{aligned}\Gamma_{c4}^{(2)} \;=\;& -\frac{1}{8}\frac{1}{32}\frac{g^4}{(4\pi^2)^3}\int \mathrm{d}^8z_1\ \mathrm{d}^8z_4\ \left[B(z_1)\ D_4^\alpha\overline{D}_4^2D_{4\alpha}\ B(z_4)\right]\delta_{41}\Box\frac{\ln x_{14}^2M'^2}{x_{14}^2}\\ &+\text{Finite Terms.}\end{aligned} \tag{5.1.82}$$

**Two-Loop $\beta$-Function**

As in Section 5.1.1, we redefine the background field as

$$B \longrightarrow B' = gB. \tag{5.1.83}$$

Then, according to the definition given in (5.1.12), the Green's function up to two-loop order is

$$\begin{aligned} G(z_4 - z_1) \quad &= \quad \frac{1}{32e^2}\,\delta^8(z_{41}) + 2\left[-\frac{1}{128}\frac{1}{(4\pi^2)^2}\;\delta_{41}\;\Box\frac{\ln(x_{41}^2 M^2)}{x_{41}^2}\right] \\ &\qquad +2\left[-\frac{1}{8}\frac{1}{32}\frac{(\sqrt{2}e)^2}{(4\pi^2)^3}\;\delta_{41}\;\Box\frac{\ln x_{41}^2 M'^2}{x_{41}^2}\right] + \cdots \end{aligned} \tag{5.1.84}$$

From using (4.2.22), we see that

$$M'\frac{\partial G}{\partial M'} \;=\; -\frac{1}{64}\frac{e^2}{(4\pi^2)^3}2\left(-4\pi^2\right)\delta^8(z_{41}) \;=\; \frac{1}{32}\frac{e^2}{(4\pi^2)^2}\delta^8(z_{41}), \tag{5.1.85}$$

and to second order in $e$,

$$\mathcal{O}(e^2): \qquad \beta(e)\,\frac{\partial G}{\partial e} \;= \beta_2\; e^5\left(-\frac{2}{32e^3}\right)\delta^8(z_{41}) \;=\; -\frac{e^2}{16}\;\beta_2\;\delta^8(z_{41}). \tag{5.1.86}$$

Finally, the renormalization group equation in (5.1.12) gives the following two-loop contribution to the $\beta$-function:

$$\boxed{\beta_2 \;=\; \frac{1}{32\pi^4}} \tag{5.1.87}$$

Recall the exact $\beta$-function for SQED given in (5.1.16). Since the anomalous dimension of the matter superfield is [52, 53, 54]

$$\gamma(\alpha) \;=\; -\frac{\mathrm{d}\ln[Z(\mu)]}{\mathrm{d}\ln\mu} \;=\; -\frac{\alpha}{\pi} + \cdots, \tag{5.1.88}$$

the $\beta$-function is equal to

$$\beta(\alpha) \;=\; \frac{\alpha^2}{\pi} + \frac{\alpha^3}{\pi^2} + \cdots. \tag{5.1.89}$$

In terms of the coupling constant $e$, this is

$$\beta(e) \;=\; \frac{e^3}{8\pi^2} + \frac{e^5}{32\pi^4} + \cdots. \tag{5.1.90}$$

This agrees with (5.1.87), and the consistency of differential renormalization is verified.

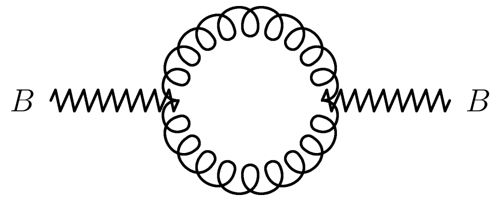


Figure 5-4: $V_Q$-loop contribution to the vacuum polarization of background field.

## 5.2 SUSY Yang-Mills

In this section, we present our computation of the $\beta$-function for supersymmetric Yang-Mills theory to one-loop order. If ordinary superspace approach is taken for this calculation, one necessarily has to deal with numerous complications. For example, the one-loop vacuum polarization of the vector superfield $V$ with an internal $V$-loop contains 36 terms. Although symmetries can be used to reduce the number of terms that need to be calculated, it is clear that this approach is cumbersome. We already saw in Section 5.1 how background field method can simplify computations. However, the method's power and usefulness become more apparent in non-abelian gauge theories.

We begin with the supergraph shown in Figure 5-4. This supergraph is the aforementioned troublesome graph with 36 terms. Let us see if there is any improvement. The part of the action that contributes to one-loop computations are shown in (3.3.60). Consider the interaction terms involving two quantum vector fields $V_Q$ and a background superfield strength $\underset{\sim}{W}$ or $\overline{\underset{\sim}{W}}$. Further observe that such terms contain either only one $\underset{\sim}{\nabla}$ or one $\overline{\underset{\sim}{\nabla}}$. Then, (3.3.61) implies that the one-loop contribution from the $V_Q$-loop vanishes simply because there are not enough covariant derivatives in the interaction. This finding is very striking. The most challenging computation in the ordinary superspace approach has become a very trivial one in the background field method.

As an immediate consequence of the above discussion, we observe that in supersymmetric pure Yang-Mills theory the only non-vanishing contributions to the one-loop two-point function of the background superfield come from ghost super-

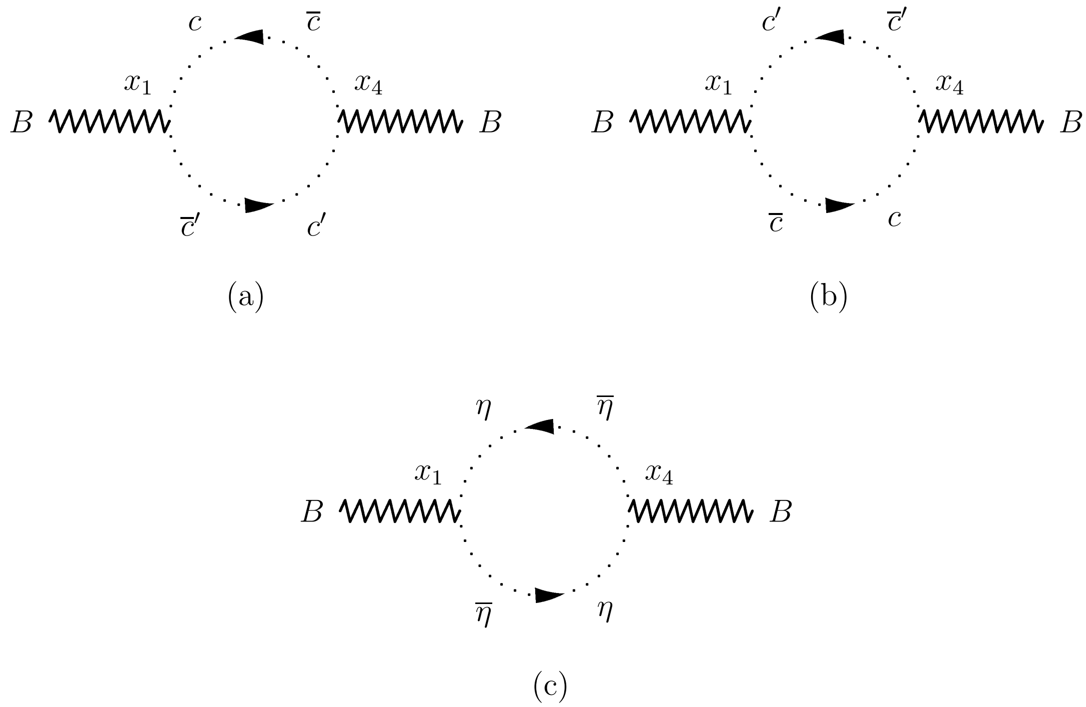


Figure 5-5: Ghost loop contributions to $\beta$-function. (a) & (b) Faddeev-Poppov ghost loops (c) Nielson-Kallosh ghost-loop.

fields. Recall that there are three ghosts in the theory–two Faddeev-Poppov and one Nielson-Kallosh. The super Feynman rule for interactions involving one background field and two ghosts is given by

$$if^{abc}\Big(\overline{c}'^a B^b c^c \;+\; c'^a B^b \overline{c}^c \;+\; \overline{\eta}^a B^b \eta^c\Big)\,. \tag{5.2.91}$$

Accordingly, supergraphs that we need to consider are shown in Figure 5-5. Computations of these three supergraphs are identical, so we will consider only one of them and multiply the result by 3 at the end of the day.

Further simplification can be made by noting that ghost superfields are chiral superfields, and that the computation we are considering here resembles the one from Section 5.1.1 very closely. Apart from some minor details, the general idea is identical to the method presented in Section 5.1.1. Hence, in comparison to (5.1.8), the total regularized contribution from ghosts is

$$\begin{aligned}\Gamma^{(1)}_{ghosts} &= 3\ C_2(G)\frac{g^2}{4\pi^2}\mathrm{Tr}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)B(z_4)\frac{\delta^8(z_{41})}{x_{41}^2} \\ &+ \frac{3\ C_2(G)}{64}\frac{g^2}{(4\pi^2)^2}\mathrm{Tr}\int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \ B(z_1)D^\alpha\overline{D}^2 D_\alpha B(z_4)\delta_{41}\Box\frac{\ln(x_{41}^2 M^2)}{x_{41}^2},\end{aligned} \tag{5.2.92}$$

where $C_2(G)$ was defined in (2.2.45). As in Section 5.1.1, we redefine the background field as

$$B \longrightarrow B' = gB \tag{5.2.93}$$

so that the anomalous dimension of the background field vanishes. Therefore, the Callan-Symanzik equation for SUSY Yang-Mills theory in background field method is

$$\begin{gathered}\left[M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g}\right] G(x_4 - x_1) = 0 \\ \beta(g) = \beta_1 g^3 + \beta_2 g^5 + \cdots,\end{gathered} \tag{5.2.94}$$

where we define the Green's function $G(x_4 - x_1)$ as

$$\Gamma \equiv \sum_n \Gamma^{(n)} \ = \ \int \mathrm{d}^8 z_1\ \mathrm{d}^8 z_4\ \left[B'(z_1)D^\alpha\overline{D}^2 D_\alpha B'(z_4)\right] G^{(n)}(x_4 - x_1). \tag{5.2.95}$$

As before, $n$ refers to the $n^{th}$ loop. Up to one-loop order, the Greens function is

$$G(z_4 - z_1) \ = \ \frac{1}{16g^2}\ \delta^8(z_{41}) + \frac{3}{64}\frac{C_2(G)}{(4\pi^2)^2}\ \delta_{41}\ \Box\frac{\ln(x_{41}^2 M^2)}{x_{41}^2}. \tag{5.2.96}$$

From substituting this into (5.2.94), we obtain the result

$$\boxed{\beta_1 = -\frac{3\ C_2(G)}{16\pi^2}.} \tag{5.2.97}$$

Supersymmetric Yang-Mills theory has been widely studied, and various aspects of the theory are well-known. In particular, Refs.[56, 57, 58, 59] discuss the $\beta$-function which is known to be

$$\beta(g) \;=\; -\frac{3\,C_2(G)}{16\pi^2}\,g^3 \;-\; 6\left[\frac{C_2(G)}{16\pi^2}\right]^2 g^5 \;+\; \mathcal{O}(g^7). \qquad (5.2.98)$$

Hence, our result (5.2.97) agrees with the standard value.

# Chapter 6

# CONCLUDING REMARKS

In this paper, we have discussed the renormalization of supersymmetric gauge theories using supergraph techniques and differential regularization. We have discussed the super background field method in considerable detail and have shown that it leads to computational simplifications. Ward-Takahashi identity for the abelian gauge theory has been derived, and a relation among the mass scale parameters has been obtained. We have successfully calculated the $\beta$-function of supersymmetric quantum electrodynamics to two-loop order and that of supersymmetric Yang-Mills theory to one-loop order, thus verifying the consistency of using differential regularization to renormalize supersymmetric gauge theories. Dimensional reduction contains intrinsic ambiguities concerning the dimension, and other renormalization schemes are rather cumbersome to implement. Differential renormalization, however, is a dimension-specific procedure that is considerably easier to use than such schemes as the supersymmetric version of Pauli-Villars regularization. We hope that further examination of differential renormalization will show that it is a clearly advantageous and completely consistent renormalization procedure for supersymmetric quantum field theories. We end this paper with the following remarks on possible future investigations.

We regret that we could not present higher-loop computations for the supersymmetric Yang-Mills theory. There were too many supergraphs to consider, the amount of available time did not permit us to pursue this study. Although super background field method leads to great calculational simplifications, further improvements in perturbation theory can be made if one uses the covariant supergraph techniques[60, 61]. We hope that using these techniques in the context of differential regularization will lead to a simpler and more powerful renormalization scheme. Furthermore, we hope that using differential renormalization to study anomalies will be able to shed some new light on the subject.

# Appendix A

# Conventions

The spacetime and spinor indices[1] are collectively denoted by a super index[2] $A \equiv \{a, \alpha, \dot{\alpha}\}$, where $a \in \{1,2,3,4\}$, $\alpha \in \{1,2\}$ and $\dot{\alpha} \in \{\dot{1}, \dot{2}\}$. For example, a superspace coordinate is defined as $z^A \equiv (x^a, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})$. Likewise, partial derivatives are collectively denoted by $\partial_A \equiv (\partial_a, \partial_\alpha, \overline{\partial}_{\dot{\alpha}})$. Our Minkowski spacetime metric is $\eta_{ab} = \mathrm{diag}(-1,+1,+1,+1)$. Under Wick-rotation of the time coordinate, the spacetime measure $\mathrm{d}^4x$ becomes $-i\mathrm{d}^4x$ ; and, consequently, $e^{iS}$ becomes $e^S$. Contraction of spacetime indices using $\eta_{ab}$ is understood in the same sense as in ordinary quantum field theory. Two component spinors $\lambda^\alpha$ belong to the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while $\overline{\lambda}_{\dot{\alpha}}$ belong to the $(0, \frac{1}{2})$ representation. Spinor indices may be raised and lowered with the invariant Levi-Civita tensors. These tensors satisfy

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \quad , \quad \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\beta}\dot{\alpha}}, \tag{A.1}$$

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^\alpha_\gamma \quad , \quad \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}} = -\delta^{\dot{\alpha}}_{\dot{\gamma}}. \tag{A.2}$$

We define the super metric as $\xi_{AB} = (\eta_{ab}, \epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$. Using the above definitions, our conventions are as follows:

$$z_A = z^B \xi_{BA} \quad , \quad z^A = \xi^{AB} z_B \ , \tag{A.3}$$

[1] We use lowercase Latin letters to denote the spacetime indices and lowercase Greek letters to denote the spinor indices.

[2] Super indices are written in uppercase Latin letters.

$$\partial_A z_B \;=\; \xi_{AB} \qquad , \qquad \partial_A z^B \;=\; \delta_A^{\ B} \;, \tag{A.4}$$

$$z_1 \cdot z_2 \;=\; z_1{}^A z_2{}^B \xi_{BA} \;=\; x_1{}^a x_{2a} + \theta_1{}^\alpha \theta_{2\alpha} + \overline{\theta}_1{}^{\dot\alpha} \overline{\theta}_{2\dot\alpha} \;=\; x_{1a} x_2{}^a - \theta_{1\alpha} \theta_2{}^\alpha - \overline{\theta}_{1\dot\alpha} \overline{\theta}_2{}^{\dot\alpha} \;. \tag{A.5}$$

Furthermore, if $\chi_\alpha$ is a spinor variable, then

$$\chi^2 \;=\; \chi^\alpha \chi_\alpha \;=\; \chi^\alpha \chi^\beta \epsilon_{\beta\alpha} \qquad , \qquad \overline{\chi}^2 \;=\; \overline{\chi}^{\dot\alpha} \overline{\chi}_{\dot\alpha} \;=\; \overline{\chi}^{\dot\alpha} \overline{\chi}^{\dot\beta} \epsilon_{\dot\beta\dot\alpha} \;, \tag{A.6}$$

$$\chi_\alpha \chi_\beta \;=\; -\frac{1}{2} \epsilon_{\alpha\beta} \chi^2 \qquad , \qquad \overline{\chi}_{\dot\alpha} \overline{\chi}_{\dot\beta} \;=\; -\frac{1}{2} \epsilon_{\dot\alpha\dot\beta} \overline{\chi}^2 \;, \tag{A.7}$$

$$\chi^\alpha \chi^\beta \;=\; -\frac{1}{2} \epsilon^{\alpha\beta} \chi^2 \qquad , \qquad \overline{\chi}^{\dot\alpha} \overline{\chi}^{\dot\beta} \;=\; -\frac{1}{2} \epsilon^{\dot\alpha\dot\beta} \overline{\chi}^2 \;, \tag{A.8}$$

We work in Euclidean $x$-space, where $\eta_{ab} \;=\; \eta^{ab} \;=\; \delta_{ab}$. The $\sigma$ matrices satisfy

$$\sigma^a_{\alpha\dot\alpha} \sigma_b^{\alpha\dot\alpha} \;=\; 2\delta^a_b \qquad , \qquad \sigma^a_{\alpha\dot\alpha} \sigma_{a\,\beta\dot\beta} \;=\; 2\epsilon_{\alpha\beta} \epsilon_{\dot\alpha\dot\beta} \;. \tag{A.9}$$

Coordinate differences are denoted by

$$z_{ij} \;\equiv z_i - z_j \quad , \quad \theta_{ij} \;\equiv\; \theta_i - \theta_j \quad , \quad \text{etc.} \;. \tag{A.10}$$

Delta functions and integration measures are defined by

$$\delta_{ij} \;\equiv\; \delta^4(\theta_i - \theta_j) \quad , \quad \delta^8(z_{ij}) \;\equiv\; \delta^4(x_{ij})\, \delta_{ij} \;, \tag{A.11}$$

$$\delta^2(\theta_{ij}) \;\equiv\; 2(\theta_i - \theta_j)^2 \quad , \quad \delta^2(\overline{\theta}_{ij}) \;\equiv\; 2(\overline{\theta}_i - \overline{\theta}_j)^2 \;, \tag{A.12}$$

$$\delta^6(z_{ij}) \;\equiv\; \delta^4(x_{ij}) \delta^2(\theta_{ij}) \quad , \quad \delta^6(\overline{z}_{ij}) \;\equiv\; \delta^4(x_{ij}) \delta^2(\overline{\theta}_{ij}) \;, \tag{A.13}$$

$$\mathrm{d}^8 z \;\;\equiv\; \mathrm{d}^4 x\; \mathrm{d}^2\theta\; \mathrm{d}^2\overline{\theta} \quad , \quad \mathrm{d}^6 z \;\;\equiv\; \mathrm{d}^4 x\; \mathrm{d}^2\theta \quad , \quad \mathrm{d}^6 \overline{z} \;\;\equiv\; \mathrm{d}^4 x\; \mathrm{d}^2\overline{\theta} \;. \tag{A.14}$$

# Appendix B

# Gegenbauer Technique

In this section, we adopt the notations in Ref.[25]. Let $y_\mu$ and $z_\mu$ be Euclidean 4-vectors. Then, Gegenbauer polynomials $C_n(\hat{z}\cdot\hat{y})$ are defined as

$$\frac{1}{(z-y)^2} \;\equiv\; \frac{1}{w_>^2}\sum_{n=0}^{\infty}\left(\frac{w_<}{w_>}\right)^n C_n(\hat{z}\cdot\hat{y}), \tag{B.1}$$

where

$$w_> \;=\; \begin{cases} z & \text{if } |z|>|y| \\ y & \text{if } |z|<|y| \end{cases} \quad , \quad w_< \;=\; \begin{cases} z & \text{if } |z|<|y| \\ y & \text{if } |z|>|y| \end{cases} \tag{B.2}$$

(In other words, $w_>$ is one of the two variables with a larger norm, and $w_<$ with a smaller norm). By using $\hat{z}\cdot\hat{y} = \cos\theta$, an explicit representation of Gegenbauer polynomials is given by

$$C_n(\cos\theta) \;=\; \frac{\sin\left[(n+1)\theta\right]}{\sin\theta}. \tag{B.3}$$

Furthermore, Gegenbauer polynomials satisfy the orthogonal condition

$$\int \frac{\mathrm{d}\hat{x}}{2\pi^2}\, C_m(\hat{x}\cdot\hat{y})\, C_n(\hat{x}\cdot\hat{z}) \;=\; \delta_{mn}\,\frac{C_m(\hat{y}\cdot\hat{z})}{n+1}, \tag{B.4}$$

where the angular integration measure is defined as $\mathrm{d}\hat{x} \equiv \sin^2\theta\ \sin\phi\ \mathrm{d}\theta\mathrm{d}\xi$.

Having introduced the basic definition of Gegenbauer polynomials, we proceed now to compute the Feynman integral shown in (5.1.80). To simplify notations, let us rename the variables as follows:

$$x_1 \to x \quad , \quad x_2 \to y \quad , \quad x_3 \to z \quad , \quad x_4 \to 0 \tag{B.5}$$

Hence, (5.1.80) is equivalent to

$$I \equiv \int \mathrm{d}^4 y \; \mathrm{d}^4 z \; \frac{1}{(x-z)^2 (y-z)^2 z^2} \; \frac{\partial}{\partial y_\mu} \frac{1}{y^2} \; \frac{\partial}{\partial y_\mu} \frac{1}{(x-y)^2}. \tag{B.6}$$

In order to integrate over the angular variable $\hat{x}$, we redefine $y$ and $z$ as

$$y_\mu \to \sqrt{x^2} y_\mu \quad , \quad z_\mu \to \sqrt{x^2} z_\mu. \tag{B.7}$$

Upon using (B.7) and $\hat{x} = x_\mu / \sqrt{x^2}$, (B.6) becomes

$$\begin{aligned} I &= \frac{1}{x^4} \int \mathrm{d}^4 y \; \mathrm{d}^4 z \; \frac{1}{(\hat{x}-z)^2 (y-z)^2 z^2} \; \frac{\partial}{\partial y_\mu} \frac{1}{y^2} \; \frac{\partial}{\partial y_\mu} \frac{1}{(\hat{x}-y)^2}. \\ &= \frac{1}{x^4} \int \mathrm{d}^4 y \; \mathrm{d}^4 z \; \frac{1}{(y-z)^2 z^2} \; \frac{\partial}{\partial y_\mu} \frac{1}{y^2} \; \frac{\partial}{\partial y_\mu} \left[ \frac{1}{(\hat{x}-z)^2} \frac{1}{(\hat{x}-y)^2} \right]. \end{aligned} \tag{B.8}$$

We observe that this expression is a pure number, so we may choose to average over the angular measure for the $\hat{x}$ variable. Hence, we have

$$\begin{aligned} \int \frac{\mathrm{d}\hat{x}}{2\pi^2} \; \frac{1}{(\hat{x}-z)^2} \frac{1}{(\hat{x}-y)^2} &= \int \frac{\mathrm{d}\hat{x}}{2\pi^2} \sum_{m,n=0}^{\infty} \frac{1}{y_>^2} \left( \frac{y_<}{y_>} \right)^n \frac{1}{z_>^2} \left( \frac{z_<}{z_>} \right)^m C_n(\hat{x} \cdot \hat{y}) C_m(\hat{x} \cdot \hat{z}) \\ &= \sum_{n}^{\infty} \frac{1}{n+1} \; \frac{1}{y_>^2 z_>^2} \left( \frac{y_< z_<}{y_> z_>} \right)^n C_n(\hat{y} \cdot \hat{z}). \end{aligned} \tag{B.9}$$

In the first line, we have used the definition of Gegenbauer polynomials given in (B.1) with $\mathrm{Norm}(\hat{x}) = 1$. In the second line, we have used the orthogonal relation (B.4). The new variables shown in (B.9) are defined as

$$y_> = \begin{cases} y & \text{if } |y| > 1 \\ 1 & \text{if } |y| < 1 \end{cases} \quad , \quad y_< = \begin{cases} 1 & \text{if } |y| > 1 \\ y & \text{if } |y| < 1 \end{cases}$$

$$z_> \;=\; \begin{cases} z & \text{if } |z|>1 \\ 1 & \text{if } |z|<1 \end{cases} \quad , \qquad z_< \;=\; \begin{cases} 1 & \text{if } |z|>1 \\ z & \text{if } |z|<1 \end{cases} \; . \tag{B.10}$$

Next, we insert (B.9) into (B.8) and use (B.1) to expand $1/(y-z)^2$. After integrating over the angular variables, we obtain the expression

$$I \;=\; -8\pi^4 \frac{1}{x^4} \sum_{n=0}^{\infty} \int \mathrm{d}y \; \mathrm{d}z \; z \frac{1}{w_>^2} \left(\frac{w_<}{w_>}\right)^n \; \frac{\partial}{\partial y} \left[ \frac{1}{y_>^2 z_>^2} \left(\frac{y_< z_<}{y_> z_>}\right)^n \right], \tag{B.11}$$

where $y$ and $z$ denote moduli, and $w_>$ and $w_<$ are defined as in (B.2). In order to evaluate this expression, we need to consider six distinct domains. They are

$$\begin{array}{llllll} y > z > 1 & , & y > 1 > z & , & 1 > y > z & , \\ z > y > 1 & , & z > 1 > y & , & 1 > z > y & . \end{array} \tag{B.12}$$

After performing the six simple integrals, we finally obtain

$$\begin{aligned} I \;&=\; -8\pi^4 \frac{1}{x^4} \left[ \sum_{n=1}^{\infty} \frac{1}{2n(n+1)(n+2)} - \frac{5}{8} \right] \\ &=\; 4\pi^4 \frac{1}{x^4}. \end{aligned} \tag{B.13}$$

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