CS 294-6

Lecture 8 Sep. 25.

Review of last time

1. Pin-hole camera model

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} S_x & S_\theta & O_x \\ 0 & S_y & O_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\Pi_0} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

λ: unknown depth

K: intrinsic parameter matrix Π_0: projection matrix

2. Vanishing points and Vanishing lines

① Let X^1 on a line L^1, X^2 on L^2

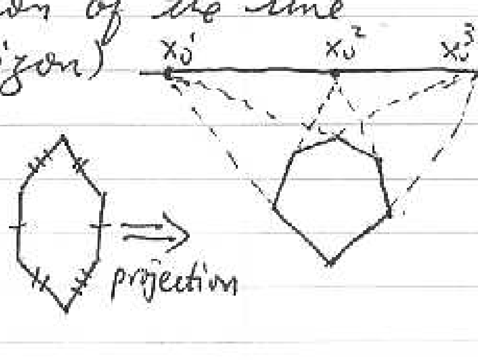
$$\begin{cases} X^1 = X_0^1 + \mu^1 V, & \mu^1 \in \mathbb{R}, \\ X^2 = X_0^2 + \mu^2 V, & \mu^2 \in \mathbb{R} \end{cases}$$ then after projection

$$\begin{cases} \lambda x^1 = K \Pi_0 X^1 \\ \lambda x^2 = K \Pi_0 X^2 \end{cases}, \quad \text{let } \mu^1 \to \infty, \ \mu^2 \to \infty.$$

$\boxed{\text{the intersection } x^1 \sim K \Pi_0 V.}$

② All vanishing points of parallel lines on a plane align on a line in the image plane, called. <u>Vanishing line</u>

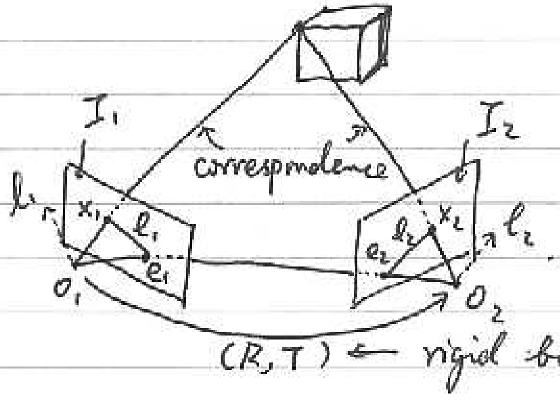
The vanishing Line is the projection of the line at infinity of the plane (horizon)



Suppose x_v^1 and x_v^2 are two vanishing points ~~for~~ on the same plane in space.

$$l_v \doteq x_v^1 \times x_v^2 = \widehat{x_v^1}\, x_v^2$$

3. Epipolar Constraint : $E = \hat{T}R$, $x_2^T E x_1 = 0$



Notice that l_1 l_2 l_2 are abuse of notation
$l_1 = \hat{x}_1 e_1$, $l_2 = \hat{x}_2 e_2$, they are co-images

① what is e_1, e_2 : epipoles.
e_1 is the image of O_2 on I_1
e_2 is the image of O_1 on I_2.
Remember the r.b.m.

$$X_{O_2} = R X_{O_1} + T$$

$$\therefore X_{O_2}(O_1) = R\begin{bmatrix}0\\0\\0\end{bmatrix} + T = T.$$

$$\therefore \lambda e_2 = K \Pi \cdot \begin{bmatrix}T\\1\end{bmatrix}$$, if we assume calibrated.

$K = I$, then $\lambda e_2 = T$.

Similarly,

$$\begin{bmatrix}R & T\\0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}R^T & -R^T T\\0 & 1\end{bmatrix}$$

$$\therefore \lambda e_1 = -R^T T$$

$$\Rightarrow E e_1 = \hat{T} R e_1 = \hat{T} R(-R^T T) = 0 \in \mathbb{R}^3$$

$$e_2^T E = T^T \hat{T} R = 0 \in \mathbb{R}^3$$

② $\begin{cases} l_1^T x_1 = 0 \\ x_2^T E x_1 = 0 \end{cases} \Rightarrow l_1 \sim E^T x_2$

(epipolar lines)

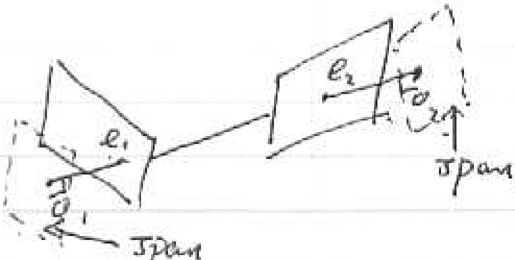
$\begin{cases} l_2^T x_2 = 0 \\ x_2^T E x_1 = 0 \end{cases} \Rightarrow l_2 \sim E x_1$

③ Given multiple points on the rigid body $\{X^1, \dots, X^N\}$
we have corresponding $\{x_1^1, \dots, x_1^N\}$ on I_1
$\{x_2^1, \dots, x_2^N\}$ on I_2
$\{l_1^1, \dots, l_1^N\}$ on I_1
$\{l_2^1, \dots, l_2^N\}$ on I_2

satisfy

$l_i^{j^T} x_i^j = 0$

* However, e_1 and e_2 are uniquely determined by (R, T),
for all l_1^j, $l_1^{j^T} e_1 = 0$
" " l_2^j, $l_2^{j^T} e_2 = 0$



So what is S_1: span $\{l_1^1, \dots, l_1^N\}$
$S_2 = \text{span}\{l_2^1, \dots, l_2^N\}$

$S_1 \perp e_1$, $S_2 \perp e_2$.

④ How to solve e_1, e_2 (ideally)?
given pairs $(x_1^1, x_2^1), \dots, (x_1^N, x_2^N)$.

Step 1. recover E (8-point algorithm)

Step 2. recover $\{l_1^1, \dots, l_1^N\} = S_1$
$\{l_2^1, \dots, l_2^N\} = S_2$.

Step 3: $\text{Null}(S_1) = \text{span}\{e_1\}$
$\text{Null}(S_2) = \text{span}\{e_2\}$ (use SVD!!)

4. SVD of E matrix:

A non-zero matrix E is an essential matrix iff $E = U \Sigma V^T$, such that, $U, V \in SO(3)$, and. $\Sigma = diag([\sigma, \sigma, 0])$

TODAY

1. Eight-point algorithm: solve for E matrix

$$E = \hat{T}R = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

Constraints: ① given a pair (x_1, x_2), $x_2^T E x_1 = 0 \in \mathbb{R}$

Using ① ② $rank(E) = 2$ or $det(E) = 0$

To solve 8 unknowns, we need 8 pairs of correspondence in general position in 3-D.

Degenerate cases?

points all lie on a planar structure. → Homography

① Rewrite $x_2^T E x_1 = 0$ in vector form

Observe: $x_2^T E x_1 = (x_1 \otimes x_2)^T E^s$

where E^s (stacked) $= \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \\ \vdots \\ e_{33} \end{bmatrix} \in \mathbb{R}^9$.

$x_1 \otimes x_2$ (Kronecker product) $= [x_1x_2, x_1y_2, x_1z_2, y_1x_2, \cdots, z_1z_2]^T \in \mathbb{R}^9$

$\therefore \boxed{(x_1 \otimes x_2)^T E^s = 0}$

② Solve for the null space of a data matrix

$D = \begin{bmatrix} (x_1^1 \otimes x_2^1)^T \\ \vdots \\ (x_1^N \otimes x_2^N)^T \end{bmatrix} \in \mathbb{R}^{N\times 9}$, for N pairs of points corresps.

$\Rightarrow DE^s = 0$, and $E^s \in \mathbb{R}^9$ is in the 1-D null space of D.

$[U, S, V] = svd(D)$,

$\Rightarrow E^s \sim V(end)$, denote its "unstacked" as F.

③ Projection onto the essential space.

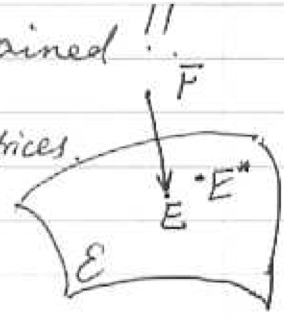
Recall properties of E: ① $E = \hat{T}R$

② $\sigma_1(E) = \sigma_2(E)$, $\sigma_3(E) = 0$.

But F obtained in ② is not constrained !!

That is $F \notin \mathcal{E}$, where

$\mathcal{E}$ is the space of all essential matrices.



let $svd(F) = U \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} V^T$

define its projection

$$E = U \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & 0 & 0 \\ 0 & \frac{\lambda_1+\lambda_2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^T$$

Then $E \in \mathcal{E}$.

Theorem 5.9. $E = \underset{E' \in \mathcal{E}}{\arg\min} \|F - E'\|_f^2$.

④ Recover R and T.

$$R = U R_z^T(\pm\tfrac{\pi}{2}) V^T, \quad \hat{T} = U R_z(\pm\tfrac{\pi}{2}) \Sigma U^T$$

End of the eight-point algorithm.

Discussion:

① Number of points: up to a scale, E has 8 unknowns.
But E only has 5 DOF: 3 rotation and 2 translation.
So called "X-point" algorithms

7-point: consider an extra condition $\det(E)=0$

We may only use 7 pairs of correspondence.

$$D = \begin{bmatrix} (x_1^1 \otimes x_2^1)^T \\ \vdots \\ (x_1^7 \otimes x_2^7) \end{bmatrix} \in \mathbb{R}^{7\times 9}$$

Let $\{E_1^s, E_2^s\}$ be a basis for Null(D)

Then $\exists$ a unique $\alpha \in \mathbb{R}$, such that

$$\det(E_1 + \alpha E_2) = 0 \implies E = E_1 + \alpha E_2.$$

6-point: HW 5-13.

5-point: [Kruppa 1913] But not closed-form.

② Number of solutions:

In total, there are four possible solutions for (R, T), But only one of them guarantees positive depths of all the 3-D points. (HW 5.11)

③ Infinitesimal viewpoint change

✓ if $T=0 \iff E = \hat{T}R = 0$.

Ideally, 8-point algorithm should return 0.

But due to data noise, $E \neq 0$,

$\implies$ T obtained is meaningless $\boxed{\text{Be Careful.}}$

✓ when $\|T\| \to 0$, the two view-points are close

$\implies$ Use <u>continuous epipolar constraints</u>

This is the situation in a video sequence taken by a moving camera.

④ Multiple rigid bodies

We now have K motions: $E_1, \dots, E_K$.

<u>multiple-motion problem</u>

① $\hat{x}_2 T \sim l_2$

② $x_2 \sim R\lambda_1 x_1 + T \quad \therefore\ Rx_1 \sim \lambda_2 x_2 - T$

$\Rightarrow \hat{x}_2 R x_1 \sim \hat{x}_2(\lambda_2 x_2 - T) \sim \hat{x}_2 T$

Hence, if $\hat{x}_2 T \neq 0$, $\text{rank}([\hat{x}_2 R x_1,\ \hat{x}_2 T]) = 1$

2. Solving for unknown depths λ_i.

after (R, T) are recovered.

$$\lambda_2 x_2 = \lambda_1 R x_1 + \gamma T$$

where γ is the scale of the translation (also unknown)

If we assume the scale of 3-D is 1.

$$\lambda_1 \hat{x}_2 R x_1 + \hat{x}_2 T = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} \hat{x}_2 R x_1 & \hat{x}_2 T \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}}_{2\times 1} = 0.$$

Condition on that $[\hat{x}_2 R x_1\ \ \hat{x}_2 T]$ is a rank-1 matrix

3. Bundle Adjustment (nonlinear optimization for E)

(Reading: Bundle Adjustment – A modern Synthesis).

by Bill Triggs, et al.

Goal: Given noisy image pairs $(\tilde{x}_1, \tilde{x}_2)$, solve for the optimal E^* to approximate the epipolar relation. and recover the noiseless positions (x_1, x_2)

① the noise model

$$\tilde{x}_1^j = x_1^j + w_1^j \ ; \quad \tilde{x}_2^j = x_2^j + w_2^j \qquad j = 1, 2, \ldots, N.$$

$w_1^j = \begin{bmatrix} w_{11}^j \\ w_{12}^j \\ 0 \end{bmatrix}$, $w_2^j = \begin{bmatrix} w_{21}^j \\ w_{22}^j \\ 0 \end{bmatrix}$ are localization error.

$$\therefore\ x_2^T \hat{T} R x_1 = 0, \text{ but } \tilde{x}_2^T \hat{T} R \tilde{x}_1 \neq 0.$$

② Objective function. (sum of squares)

$$\phi(x_1, R, T, \lambda_1) = \sum_{j=1}^{N} \left\{ \|\tilde{x}_1^j - x_1^j\|^2 + \|\tilde{x}_2^j - (R\lambda_1^j x_1^j + T)\|^2 \right\}$$

$$(x_1^*, R^*, T^*, \lambda_1^*) = \arg\min \phi(x_1, R, T, \lambda_1)$$

③ optimization techniques: