Students were asked on Fri. 2 Oct. to work out *some* of these problems aided by their own notes and by any texts but by no other person, and to hand in solutions Mon. morning 5 Oct. 1998.

Problem 0: When we see our own images in a mirror, why does it swap Left and Right but not Up and Down?

It doesn't swap Left and Right; it swaps Forward and Backward.

Problem 1: Exhibit two matrices P and Q such that $(PQ)^2 = O \neq (QP)^2$.

Try $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for which $PQ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O$ and $QP = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The necessity of $PQ \neq O$ comes from $O \neq (QP)^2 = Q(PQ)P$; but QP must satisfy $(QP)^3 = Q(PQ)^2P = O$.

Problem 2a: Obtain an explicit formula for $(\mathbf{I} - \mathbf{c}\mathbf{r}^T)^{-1}$ given $\mathbf{c}$ and $\mathbf{r}^T$ and that $\mathbf{r}^T\mathbf{c} \neq 1$.

$(\mathbf{I} - \mathbf{c}\mathbf{r}^T)^{-1} = \mathbf{I} + \mathbf{c}\mathbf{r}^T/(1-\mathbf{r}^T\mathbf{c})$.

Problem 2b: Obtain $(\mathbf{B} - \mathbf{c}\mathbf{r}^T)^{-1}$ explicitly given $\mathbf{B}^{-1}$, $\mathbf{c}$, $\mathbf{r}^T$, and that $\mathbf{r}^T\mathbf{B}^{-1}\mathbf{c} \neq 1$.

$(\mathbf{B} - \mathbf{c}\mathbf{r}^T)^{-1} = \mathbf{B}^{-1} + \mathbf{B}^{-1}\mathbf{c}\mathbf{r}^T\mathbf{B}^{-1}/(1 - \mathbf{r}^T\mathbf{B}^{-1}\mathbf{c})$.

Problem 3: Obtain the n-by-n matrix U from the identity by deleting its first row and appending a row of zeros after its last. Obtain R from $2I - (I-U)^{-1}$ by inserting the scalar μ into its lowest leftmost element. Express the value of μ for which R is not invertible as a function of n , assuming $n > 1$. Hint: Experiment with n = 2, 3, 4, … first.

$\mu = -2^{2-n}$. Here is why: $R = 2I - (I-U)^{-1} + \mu e f^T$ in which e is the column whose last element is 1 and the rest zeros, and f^T is the row whose first element is 1 and the rest zeros. R is not invertible just when $Rx = o$ for some $x \neq o$. Then $x = -\mu\left(2I - (I-U)^{-1}\right)^{-1} e f^T x \neq o$, whence follows $f^T x = -\mu f^T\left(2I - (I-U)^{-1}\right)^{-1} e f^T x \neq 0$, and then $\mu = -1/f^T\left(2I - (I-U)^{-1}\right)^{-1} e$. Now, $\left(2I - (I-U)^{-1}\right)^{-1} = (I-U)(I-2U)^{-1} = I + U + 2U^2 + 4U^3 + 8U^4 + \ldots + 2^{n-2}U^{n-1}$ since $U^n = O$. Consequently $\mu = -1/f^T(2^{n-2}U^{n-1})e = -2^{2-n}$. Determinantal manipulation gives the same result; $0 = \det(R) = \det(R(I-U)) = \det(I-2U+\mu e f^T(I-U)) = \det\left((I-2U+\mu e f^T(I-U))(I+ff^TU)\right) = 1 + 2^{n-2}\mu$.

Problem 4a: Given two different vectors x and y of the same Euclidean length (so $x^Tx = y^Ty \neq 0$), exhibit an elementary orthogonal reflector $W = I - (2/c^Tc)cc^T$ that swaps them.

Choose $c = x-y$; then $Wx = y$ and so $Wy = x$. (Note that this kind of $W = W^T = W^{-1}$.)

Problem 4b: Prove that every n-by-n orthogonal matrix $Q = (Q^T)^{-1}$ can be expressed as a product of at most n elementary orthogonal reflectors like W .

" $Q^TQ = I$ " implies that every column of any orthogonal matrix Q has the same length 1 as every column of the identity I . Choose reflector W_1 to swap the first column of Q with the first column of I . Note that W_1Q is still orthogonal, and its first column (and first row) must be the same as I 's . Choose reflector W_2 to swap the second column of W_1Q with the second column of I . W_2 leaves the first column of W_1Q unchanged because it is orthogonal to the second columns of W_1Q and of I . Therefore the first two columns (and first two rows) of W_2W_1Q , which is still orthogonal, must be the same as I 's . Choose reflector W_3 to swap the third column of W_2W_1Q with the third column of I , and so on. Of course, if a column to be swapped with a column of I already matches it, a reflector can be skipped. So, premultiplying Q by at most n reflectors transforms it into I . Therefore Q equals the inverse of that product, which is the product of the same reflectors in reverse order.

Problem 5: Two proper subspaces of a vector space are *complementary* just when their sum is the whole space and their intersection is { **o** } . Can either determine the other uniquely? Why?

No. Let **E** and **F** be bases for complementary subspaces of a vector space for which [**E**, **F**] must therefore be a basis. Given any nonzero matrix G with as many columns as **E** has, and with as many rows as **F** has columns, we shall show that [**E**+**F**G, **F**] is another basis for the vector space, but Range(**E**+**F**G) ≠ Range(**E**) ; this will confirm that the subspace Range(**F**) cannot determine its complementary subspace uniquely. $[\mathbf{E}+\mathbf{F}G, \mathbf{F}] = [\mathbf{E}, \mathbf{F}]\begin{bmatrix} I & O \\ G & I \end{bmatrix}$ is a basis because the last matrix in the product has an inverse obtained by reversing the sign of G . To see why Range(**E**+**F**G) ≠ Range(**E**) choose any column z for which $Gz \neq o$ and verify that the equation $\mathbf{E}x = (\mathbf{E}+\mathbf{F}G)z$ cannot be solved for x because otherwise $\mathbf{F}Gz = \mathbf{E}(x-z)$ would be a nonzero vector in the intersection of complementary subspaces Range(**F**) and Range(**E**) .

Problem 6: $\boldsymbol{S}$ and $\boldsymbol{T}$ are two subspaces of a vector space $\boldsymbol{V}$, and f is a real scalar-valued function defined for every vector in $\boldsymbol{V}$. Moreover, $f(s) < f(t)$ for every nonzero vector **s** in $\boldsymbol{S}$ and every nonzero vector **t** in $\boldsymbol{T}$. How must Dimension($\boldsymbol{S}$) + Dimension($\boldsymbol{T}$) compare with Dimension($\boldsymbol{V}$) ?

Subspaces $\boldsymbol{S}$ and $\boldsymbol{T}$ can have only the zero vector **o** in their intersection, so Dimension($\boldsymbol{S}$) + Dimension($\boldsymbol{T}$) = Dimension({**o**}) + Dimension($\boldsymbol{S} + \boldsymbol{T}$) ≤ Dimension($\boldsymbol{V}$) .

Problem 7: Let the cubic polynomial whose value at ξ is $p(\xi) = \pi_o + 3\pi_1\xi + 3\pi_2\xi^2 + \pi_3\xi^3$ be represented by a row-vector $p^T := [\pi_o, \pi_1, \pi_2, \pi_3]$ of its coefficients. For constant μ let cubic $b(\xi) := p(\xi + \mu) = ß_o + 3ß_1\xi + 3ß_2\xi^2 + ß_3\xi^3$ be represented by $b^T := [ß_o, ß_1, ß_2, ß_3]$. Exhibit the matrix L that takes p^T to $b^T = p^TL$. This matrix can be factorized; L is the product of three matrices among whose elements only the numbers 0, 1 and μ appear. Find these factors and thus determine how few scalar multiplications suffice to compute p^TL given p^T and μ .

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3\mu & 1 & 0 & 0 \\ 3\mu^2 & 2\mu & 1 & 0 \\ \mu^3 & \mu^2 & \mu & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ 0 & 0 & \mu & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & \mu & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

, so six multiplications and additions suffice to compute p^TL without first computing μ^2 and μ^3 which would cost two more multiplications.

Problem 8: Given matrices E and F with the same number of rows but any numbers of columns (and their columns need not be linearly independent), we seek a matrix S whose range is the intersection of Range(E) and Range(F) . Show how and why S may be constructed if matrices J, L, P, Q and R are found to satisfy EJ + FL = O and E[JP–I, JQ] = R[E, F] .

The range of S := EJ = -FL is contained in both Range(E) and Range(F) , and therefore in their intersection. Any vector w := Eu = –Fv in that intersection can also be found as w = S(Pu + Qv) in Range(S) , implying that Range(S) contains that intersection, because

$$S(Pu + Qv) – w = EJ(Pu + Qv) – Eu = E((JP–I)u + JQv) = R(Eu + Fv) = o .$$

Therefore Range(S) is the intersection of Range(E) and Range(F) , as required. (To find J, L, P, Q and R, which the problem did not request, see the lecture notes titled "Geometry of Elementary Operations and Subspaces" and set $R := EH[I, O]G^{-1}$.)

Problem 9: Given a matrix F whose target-space is Euclidean, and a vector g in that space but not in Range(F) , explain how to find a vector r perpendicular to Range(F) such that g – r lies in Range(F) .

Solve the Least-Squares problem that chooses x to minimize ||Fx - g|| . Then r := g – Fx because $F^Tr = F^Tg – F^TFx = o$. The lecture notes on Least Squares explain why the last equation always has a solution x .