Written solutions for these problems were to be handed in for grading on Tues. 17 Oct. 2000.

1. Suppose an odd number (at least three) of coins have the property that, if any one coin is removed, the rest can be partitioned into two groups each with the same number of coins and also the same total weight. Show that all the coins must have the same weight.

Proof: Let the coins' weights be $x_1, x_2, x_3, \dots, x_{2N+1}$. Then for $i = 1, 2, 3, \dots, 2N+1$ these weights must satisfy $\sum_j h_{ij}x_j = 0$ wherein every $h_{ij} = \pm 1$ except $h_{ii} = 0$, and for each i there are N coefficients $h_{ij} = +1$ and N coefficients $h_{ij} = -1$. Evidently a nontrivial solution for the homogeneous linear equations has $x_1 = x_2 = x_3 = \dots = x_{2N+1} \neq 0$. Can there be any other kind of nontrivial solution? To see why not, we shall show that the deletion of the last row and column from the (2N+1)-by-(2N+1) matrix $\overline{H} := [h_{ij}]$ leaves a 2N-by-2N matrix H with $\det(H) \neq 0$, so the choice of a nonzero x_{2N+1} determines uniquely all the other x's in a nontrivial solution.

What makes the problem interesting is that we do not know the signs of the nonzero h's . Let's perform our computation of $\det(H)$ **mod** 2 . Now every off-diagonal $h_{ij} \equiv 1$ **mod** 2 , and we find that $H \equiv (uu^T - I)$ **mod** 2 where $u^T = [1, 1, 1, \dots, 1]$ and I is the *Identity* matrix, all with 2N columns. Then $H^2 \equiv I$ **mod** 2 , so $\det(H) \equiv 1$ **mod** 2 . Consequently $\det(H) \neq 0$ as claimed.
(Adapted from problem B5 of the 1988 Putnam Exam; *cf.* L-S. Hahn (1992) *Math. Magazine* **65** #2 pp. 111-2.)

2. In an n-dimensional Euclidean space, the vertices of a triangle are at positions joined to the origin by vectors **x**, **y** and **z** . Show that the triangle's unoriented area is $A := \sqrt{(\det(M^TM))}/2$ wherein $M := \begin{bmatrix} 1 & 1 & 1 \\ x\text{-}m & y\text{-}m & z\text{-}m \end{bmatrix}$ and $\mathbf{m} := (\mathbf{x}+\mathbf{y}+\mathbf{z})/3$.

Proof: If $n > 3$ choose a new orthonormal basis whose first three vectors span the subspace containing **x**, **y** and **z** . This reduces n to 1, 2 or 3 ; if $n = 1$ then $A = 0$ trivially. Next, without changing the determinant, subtract the column containing **x** from the others to get $A = \sqrt{(\det(\begin{bmatrix} 1 & 0 & 0 \\ x\text{-}m & u & w \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ x\text{-}m & u & w \end{bmatrix}))}/2$ wherein $\mathbf{u} := \mathbf{y}-\mathbf{x}$ and $w := \mathbf{z}-\mathbf{x}$. Then add a third of each of the second and third columns to the first to get $A = \sqrt{(\det(\begin{bmatrix} 1 & 0 & 0 \\ o & u & w \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ o & u & w \end{bmatrix}))}/2$, which expands to $A = \sqrt{(\mathbf{u}^T\mathbf{u}\cdot\mathbf{w}^T\mathbf{w} - (\mathbf{u}^T\mathbf{w})^2)}/2$ = ||**u**¢**w**||/2 by Lagrange's identity. Since ||**u**¢**w**||/2 is the area of the triangle in question, the proof is complete. An alternative proof moves the origin to **m** , whereupon $\mathbf{x}+\mathbf{y}+\mathbf{z} = \mathbf{o}$; next choose a new orthonormal basis whose first two vectors span the subspace containing **x**, **y** and **z** to reduce n to 2 , and then invoke a well-known formula $A = |\det(M)|/2 = \sqrt{(\det(M^TM))}/2$ now that $\mathbf{m} = \mathbf{o}$.

3. Four ghostly galleons – call them E, F, G and H, – sail on a ghostly sea so foggy that visibility is nearly zero. Each pursues its course steadily, changing neither its speed nor heading. G collides with H amidships; but since they are ghostly galleons they pass through each other with no damage nor change in course. As they part, H's captain hears G's say "Damnation! That's our third collision this night!" A little while later, F runs into H amidships with the same effect (none) and H's captain hears the same outburst from F's. What can H's captain do to avoid a third collision and yet reach his original destination, whatever it may be, and why will doing that succeed?

Answer: H need only change speed while keeping to the same heading. To see why this works, choose a coordinate system centered upon H and moving with it. As seen on a RADAR screen in H , the other galleons trace straight-line courses at constant speed. The courses traced by F and G are straight lines through H , with which they have collided. But F and G have also collided with each other; therefore F and G trace the same straight line through H . Having both suffered three collisions, not two, F and G must have collided with E at different times, so E's course must stay in that same straight line. It cannot be aligned along H's course because H suffered collisions amidships, not by the bow or stern. If H changes speed and E does not, their courses will no longer intersect. E's captain has no reason to change speed since he cannot yet know anything about the course nor speed of H .

4. All the vectors in an abstract n-dimensional *Complex* vector-space can be construed as a *Real* vector-space over the field of real scalars. What is the dimension of this real space, and why? What operator upon this real space is effected by a complex scalar multiplication, say by $\mu + \imath ß$, on the complex space? (Here $\imath^2 = -1$.)

Answer: The dimension is $2n$. To see why, choose any basis $\mathbf{Z} = [\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_n]$ of the complex space. Then $\mathbf{B} := [\mathbf{z}_1, \imath\mathbf{z}_1, \mathbf{z}_2, \imath\mathbf{z}_2, \mathbf{z}_3, \imath\mathbf{z}_3, \dots, \mathbf{z}_n, \imath\mathbf{z}_n]$ constitutes a basis for the real space. The vectors in $\mathbf{B}$ are linearly independent over the field of real scalars because, if $\mathbf{B}x = \mathbf{o}$ for some real column-vector x then $\mathbf{Z}z = \mathbf{o}$ for a complex column-vector z derived in an obvious way from x . Multiplication of the complex space by the scalar $\mu + \imath ß$ corresponds to multiplication of the real space by a linear operator whose matrix, using the basis $\mathbf{B}$, is the diagonal sum of n copies of the 2-by-2 matrix $M := \begin{bmatrix} \mu & -\beta \\ \beta & \mu \end{bmatrix}$ since $(\mu+\imath ß)[\mathbf{z}_k, \imath\mathbf{z}_k] = [\mathbf{z}_k, \imath\mathbf{z}_k]M$.

5. Given a k-by-n matrix R of rank k , describe how to construct a basis for its null-space.

Solution: The null-space of R is a (n–k)-dimensional subspace of the space of n-dimensional columns. Let E be a k-by-k matrix, a product of (invertible) elementary row operations, that exhibits $E\cdot R$ in reduced row-echelon form; this means that some n-by-n permutation matrix P puts $E\cdot R\cdot P = [I, H]$ for some k-by-(n–k) matrix H . Then $B := P\cdot[H^T, -I]^T$ is a n-by-(n–k) matrix of rank n–k , and $R\cdot B = O$, so the columns of B provide the requested basis.