1. Must Triangular Matrices have Triangular Inverses ?

The first problem was to show why every triangular matrix has a triangular inverse whenever the inverse exists. The following demonstration goes by induction. The claim is obviously true for 1-by-1 triangular matrices (scalars). Suppose now for some integer $n \ge 1$ that every n-by-n upper-triangular matrix U has an upper-triangular inverse U^{-1} whenever it exists. Let $\hat{U}$ be any (n+1)-by-(n+1) upper-triangle with an inverse $\hat{A}$. It must satisfy $\hat{U}\cdot\hat{A} = \hat{I}$, the (n+1)-by-(n+1) identity. Partition each matrix in this equation conformally into submatrices thus:

$$\begin{bmatrix} U & u \\ o^T & \mu \end{bmatrix} \cdot \begin{bmatrix} A & c \\ r^T & \beta \end{bmatrix} = \begin{bmatrix} I & o \\ o^T & 1 \end{bmatrix} .$$

Here submatrices U, A and I are n-by-n , columns u, c and o are n-by-1 , rows r^T and o^T are 1-by-n , and scalars μ, ß and 1 are 1-by-1 . These submatrices satisfy

$$U\cdot A + u\cdot r^T = I , \qquad U\cdot c + u\cdot ß = o ,$$
$$o^T\cdot A + \mu\cdot r^T = o^T , \qquad o^T\cdot c + \mu\cdot ß = 1 .$$

Since o^T is a row of zeros, the last two equations simplify to $\mu\cdot r^T = o^T$ and $\mu\cdot ß = 1$. These equations can both be satisfied only if $\mu \ne 0$, which is necessary for $\hat{U}^{-1}$ to exist; and then $ß = 1/\mu$ and $r^T = o^T$. Then $A = U^{-1}$ and $c = -U^{-1}u/\mu$. Consequently

$$\hat{U}^{-1} = \hat{A} = \begin{bmatrix} U^{-1} & c \\ o^T & \beta \end{bmatrix}$$

is upper-triangular because U^{-1} is. This completes the induction for upper-triangles, showing that the ones with nonzero diagonals have upper-triangular inverses. For lower-triangles we transpose the equation $\hat{U}\cdot\hat{A} = \ddot{I}$ to get $\hat{A}^T\cdot\hat{U}^T = \hat{I}$, which shows why every lower-triangle $\hat{U}^T$ with a nonzero diagonal has a lower-triangular inverse $\hat{A}^T$. Note too, from the relation $ß = 1/\mu$, that two triangular matrices inverse to each other have diagonal elements respectively reciprocals of each other.

2. When are Triangular Factorizations Unique ?

The second problem was to show that whenever a square matrix $B = L\cdot U$ has triangular factors, L unit-lower-triangular (with 1's on its diagonal) and U upper-triangular with all its diagonal elements nonzero except perhaps its last diagonal element, then B determines L and U uniquely. For this purpose assume first that the diagonal of U is altogether nonzero, and that another factorization $B = \overline{L}\cdot\overline{U}$ has been found, perhaps by a different method, with unit-lower-triangle $\overline{L}$ and upper-triangle $\overline{U}$. The equation $L\cdot U = \overline{L}\cdot\overline{U}$ implies $\overline{L}^{-1}\cdot L = \overline{U}\cdot U^{-1}$; here $\overline{L}^{-1}\cdot L$ is the lower-triangular product of lower-triangles, and $\overline{U}\cdot U^{-1}$ is an upper-triangle similarly, so the products can be equal only if both are diagonal. This makes $\overline{L}^{-1}\cdot L = I$; it is the product of just the diagonal elements of $\overline{L}^{-1}$ and L , which are 1's. Therefore $\overline{L} = L$; and it soon follows that $\overline{U} = U$ too, which makes the triangular factorization of B unique.

Now consider a triangular factorization when the upper-triangular factor's last diagonal element is zero. For this purpose let us write

$$\begin{bmatrix} B & c \\ b^T & \beta \end{bmatrix} = \begin{bmatrix} L & o \\ r^T & 1 \end{bmatrix} \cdot \begin{bmatrix} U & u \\ o^T & 0 \end{bmatrix} ,$$

in which c , o and u are columns, b^T, r^T and o^T are rows, and the o's contain zeros. This triangular factorization's submatrices satisfy

$$B = L\cdot U , \qquad c = L\cdot u ,$$
$$b^T = r^T\cdot U , \qquad ß = r^T\cdot u + 0 .$$

We have seen that the equation $B = L\cdot U$ determines unit-lower-triangle L and upper-triangle U uniquely when all diagonal elements of U are nonzero, as is assumed here. Then column $u = L^{-1}\cdot c$ and row $r^T = b^T\cdot U^{-1}$ are determined uniquely too by c and b^T , as is the upper-triangle's last diagonal element $ß - r^T\cdot u$. That it happens to vanish does not alter the uniqueness of the triangular factorization up to this point.

= =

The nonuniqueness or nonexistence of LU triangular factorization when any but the last diagonal element of U vanishes is illustrated by examples: First is a nonunique factorization

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 10 \\ 3 & 6 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \beta & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & \mu \end{bmatrix}$$

in which $\mu := 21 - 4ß$. The second example,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

has no triangular factorization.