Assignment Due Fri. 22 March 2002

Suppose the triangular factorization $P \cdot B = L \cdot U$ of a square matrix B has been computed; here P is a *Permutation* matrix, L is *Unit-lower-triangular*, and U is *Upper-triangular*. Desired is a way to compute the *Adjugate* $\mathrm{Adj}(B)$ from only the given factors P, L and U under three different circumstances:

(i) U is invertible.
(ii) Only the last row of U is a row of zeros.
(iii) The last two rows of U are rows of zeros.

Explain how to compute $\mathrm{Adj}(B)$ in each case. (Recall that $\mathrm{Adj}(B)$ is a polynomial function of B satisfying $\mathrm{Adj}(B) = \det(B) \cdot B^{-1}$ when $\det(B) \neq 0$. You may experiment with Matlab.)

Solution:

Case (i): $\det(B) = \det(L) \cdot \det(U)/\det(P) = \pm\det(U) = \pm$(the product of all U's diagonal elements) where $\det(P) = \pm 1$ according as P is an even or odd permutation. And $B^{-1} = U^{-1} \cdot L^{-1} \cdot P$ is computable too. Thus $\mathrm{Adj}(B) = \det(B) \cdot B^{-1} = \pm\mathrm{Adj}(U) \cdot L^{-1} \cdot P$, wherein $\mathrm{Adj}(U) = \det(U) \cdot U^{-1}$ and $\det(P) = \pm 1$ respectively, is computable from just P, L and U.

Case (ii): Conformally partition $U = \begin{bmatrix} \bar{U} & u \\ o^T & 0 \end{bmatrix}$ and $L = \begin{bmatrix} \bar{L} & o \\ r^T & 1 \end{bmatrix}$, noting that $\bar{U}$ is invertible though U is not. For any $\mu \neq 0$ let $U(\mu) := \begin{bmatrix} \bar{U} & u \\ o^T & \mu \end{bmatrix}$ and $B(\mu) := P^T \cdot L \cdot U(\mu)$. Then $B(\mu)$ is a continuous function of μ with $B(0) = B$, so $\mathrm{Adj}(B(\mu))$ must be continuous too with $\mathrm{Adj}(B(0)) = \mathrm{Adj}(B)$; consequently

$$\mathrm{Adj}(B) = \lim_{\mu\to 0} \mathrm{Adj}(B(\mu)) = \pm\left(\lim_{\mu\to 0} \mathrm{Adj}(U(\mu))\right) \cdot L^{-1} \cdot P = \pm\mathrm{Adj}(U) \cdot L^{-1} \cdot P$$

in which $\det(P) = \pm 1$ respectively is independent of μ. To compute $\mathrm{Adj}(U)$ we find first

$$\mathrm{Adj}(U(\mu)) = \det(U(\mu)) \cdot U(\mu)^{-1} = \det(\bar{U}) \cdot \mu \cdot \begin{bmatrix} \bar{U}^{-1} & -\bar{U}^{-1} u/\mu \\ o^T & 1/\mu \end{bmatrix} = \det(\bar{U}) \cdot \begin{bmatrix} \mu\bar{U}^{-1} & -\bar{U}^{-1} u \\ o^T & 1 \end{bmatrix}.$$

Then we let $\mu \to 0$ to get $\mathrm{Adj}(U)$. After finding that the last row of L^{-1} is $[-\bar{L}^{-1} r^T \quad 1]$, we get

$$\mathrm{Adj}(B) = \pm\mathrm{Adj}(U) \cdot L^{-1} \cdot P = \pm\det(\bar{U}) \cdot \begin{bmatrix} -\bar{U}^{-1} u \\ 1 \end{bmatrix} \cdot [-\bar{L}^{-1} r^T \quad 1] \cdot P = \pm \begin{bmatrix} -\mathrm{Adj}(\bar{U}) u \\ 1 \end{bmatrix} \cdot [-\mathrm{Adj}(\bar{L}) r^T \quad 1] \cdot P,$$

which is computable from just P, L and U.

Case (iii): Now $\mathrm{Adj}(B) = O$ because $\mathrm{rank}(B) = \mathrm{rank}(U) \leq$ (dimension of B) $- 2$.