Math. 55 EXAM. Model Solutions

This 75 min. exam can be answered with the aid of any texts, notes and calculating instruments. Feel free to experiment; answers may be worked out as audaciously as you like on scratch paper but must then be reconsidered before being entered on this sheet in the spaces provided. Each correct proof, if required, earns two points; each almost complete proof or other correct answer earns a point; each incorrect answer loses a point; space left blank loses nothing, so **DON'T JUST GUESS.** The last problems are the hardest. Take your time; check your work. Finally hand this sheet and/or more to *your* T. A. to be scored. Your score *will* affect your final grade.

1. A box contains only red and blue socks, and the same number of each. The minimum number of socks that must be picked in the dark in order to be sure of getting at least one pair of matching color is the same as the minimum number that must be picked to ensure getting at least one pair with different colors. How many socks are in the box?

_____ 4 ____ [1 pt.]

2. $\bar{x}$ is the expected value of a random variable x ; prove that the expected value of x^2 cannot be less than $\bar{x}^2$. [2 pts.]

$$0 \le E((x-\bar{x})^2) = E(x^2 - 2x\bar{x} + \bar{x}^2) = E(x^2) - \bar{x}^2 .$$

3. A fair coin marked " 1 " on one side and " 0 " on the other is flipped twice independently and fairly. Random variable X is what shows on the first flip, Y is what shows on the second, and $Z := |X-Y|$. Is Z independent of each of X and Y ?

__ YES __ [1 pt.]

Is Z independent of $X+Y$? __ NO ___ [1 pt.]

X	Y	Z = |X–Y|	X+Y	Probability
0	0	0	0	1/4
	1	1	1	1/4
1	0	1	1	1/4
	1	0	2	1/4

Z is independent of X and of Y , but not of $X+Y$ since $\mathrm{Prob}(Z=1 \ne X+Y) = 0 \ne 1/4$.

4. Violins produced on the island of Grxcd have become collectors' items since it sank into the sea a century ago. All the island's violins were produced by Bropcs or one of his sons, or by Czwyz or one of his sons. Every violin was labelled ostensibly to reveal its maker but, although Bropcs and his sons always labelled their violins truthfully, Czwyz and his sons always labelled their violins with falsehoods. Both families playfully hindered collectors' attempts to establish provenances for their violins. For example, who must have made a violin labelled " This violin was not made by any son of Bropcs." ? ____ Bropcs Sr. _____ [1 pt]

The most desirable violins are so labelled that a connoisseur can tell that it must have been made by one of the fathers, either Bropcs or Czwyz, but cannot tell which. How might these violins be labelled? [1 pt.]

" Made by Bropcs himself, or by a son of Czwyz." (… among other possibilities)

(Over)

5. Given that integers i, I, j and J satisfy

$$I > 1\ ,\ \ J > 1\ ,\ \ GCD(I, J) = 1 = i\cdot J - j\cdot I\ ,\ \ 0 \le i \le I \quad\text{and}\quad 0 \le j \le J\ ,$$

evaluate the constant $(j+i)\cdot(J-I) - (j-i)\cdot(J+I)$ = ____ 2 _____ . [1 pt.]

If $j \ne i$ must $\mathrm{sign}(j-i) = \mathrm{sign}(J-I)$? (YES or NO) __ YES __ [0 pt.]

If you think YES, prove it; if you think NO, give a counterexample. [2 pts]

The constraints can be tightened: $1 \le i < I$ and $1 \le j < J$ because otherwise we would find $GCD(I, J) \ne 1$. For the same reason, $GCD(i, j) = 1$. Therefore $i+j \ge 2$ and $I+J \ge 3$, and $|I-J| \ge 1$. And if $i \ne j$ then $|(j-i)\cdot(J+I)| \ge 3$ and therefore

$$\mathrm{sign}(j-i) = \mathrm{sign}((j-i)\cdot(J+I)) = \mathrm{sign}((j-i)\cdot(J+I) + 2) = \mathrm{sign}((j+i)\cdot(J-I)) = \mathrm{sign}(J-I)\ .$$

6. Frank, who publishes *The Paris County Advertiser*, receives advertisements from veterinary doctors Eloise and Peter, who both treat cattle and swine to the exclusion of all other animals. Eloise claims she has cured a higher percentage of sick animals entrusted to her care than has any other vet. Peter claims he has cured a higher percentage of sick cattle entrusted to his care than has any other vet, and likewise for swine. Frank will not publish claims he thinks contradictory. Must the two doctors' advertisements be contradictory? ___ NO __ [0 pt.]

Justify your answer. [2 pts.]

They need not be contradictory; for instance, suppose …

Eloise has cured 1 of 2 cattle and 76 of 98 swine, totalling 77 out of 100 animals.

Peter has cured 20 of 30 cattle and 9 of 10 swine, totalling 29 out of 40 animals.

$$20/30 > 1/2\ ,\ \text{and}\ \ 9/10 > 76/98\ ,\ \text{but}\ \ 29/40 < 77/100\ .$$

We see here an instance of what is called "Simpson's Paradox", which can arise when Peter and Eloise treat rather different samples of the animal population. No paradox could arise if the ratios of cattle to swine were the same for both vets, but this must not be taken for granted.

7. The following facts about the integrals $J_m := \int_0^{\pi/2} (\sin x)^m dx$ have already been proved:

$$J_0 := \pi/2\ ,\quad J_1 := 1 \quad\text{and}\quad J_m = (m-1)\cdot(J_{m-2} - J_m) \quad\text{for all integers } m \ge 2\ .$$

Using no more information about the integrals than these facts, prove the formulas [2 pts.]

$$J_{2k+1} = (2^k\cdot k!)^2/(2k+1)! \quad\text{and}\quad J_{2k} = (2k)!\cdot(\pi/2)/(2^k\cdot k!)^2 \quad\text{for all integers } k \ge 0\ .$$

The proofs are easy by induction using the deduced fact that $J_m / J_{m-2} = (m-1)/m$ for $m \ge 2$.
The formulas are obviously valid at $k = 0$; and if valid at $k = K-1 \ge 0$ then

$J_{2k+1} = (2K/(2K+1))\cdot J_{2k-1} = (2K/(2K+1))\cdot(2^{K-1}\cdot(K-1)!)^2/(2K-1)! = \dots = (2^K\cdot K!)^2/(2K+1)!$

$J_{2k} = ((2K-1)/(2K))\cdot J_{2k-2} = ((2K-1)/(2K))\cdot(2(K-1))!\cdot(\pi/2)/(2^{k-1}\cdot(K-1)!)^2 = (2K)!\cdot(\pi/2)/(2^K\cdot K!)^2$

proves their validity for $k = K$ too, and therefore for all k .

The maximum score on this exam is 14 points, but 10 points is an excellent score.