How long can one expect to have to wait for a bus at a bus stop? Common experience suggests that transit authorities' reassurances about schedules and frequencies of service are optimistic. Actually, the public shares some of the responsibility for misinterpreting those reassurances in an overly optimistic way whenever traffic congestion and other contingencies introduce unavoidable variations in schedules.

Let w minutes be the intended waiting period between consecutive busses' arrivals at a bus stop. If busses adhered strictly to that intention, a would-be passenger ignorant of their schedule who came to the bus stop at a randomly chosen moment would expect to wait for $w/2$ minutes until the next bus arrived. This figure $w/2$ is the average waiting time, computed by observing that coming at a time $f \cdot w$ after the departure of the previous bus is as likely as coming at a time $f \cdot w$ before the arrival of the next, for $0 < f < 1$, and the average of their two waiting times $w - f \cdot w$ and $f \cdot w$ is $w/2$ for all f. But something else happens if busses arrive at the stop somewhat irregularly. Even if w is the average interval between bus arrivals, the average waiting time for a bus will then exceed $w/2$. This is obvious if busses get bunched up in convoys because then the average waiting time will be at least half the average time between arrivals of convoys instead of busses. What follows explains what happens when arrival times vary less drastically.

Let $w_1, w_2, w_3, \ldots, w_j, \ldots$ be possible intervals between arrivals of consecutive busses, and let p_j be the probability (or relative frequency) of the occurence of w_j. Of course $p_j \geq 0$ and $\sum_j p_j = 1$. The "expected" (average) interval between consecutive busses is $w := \sum_j p_j \cdot w_j$. A would-be passenger who comes to the bus stop at random in an interval of width w_j must expect to wait $w_j/2$ minutes on average. The probability of coming to the bus stop in an interval of width w_j is proportional to w_j and also to the probability p_j with which w_j occurs, so the average waiting time for a random would-be passenger is $W/2 := (\sum_j p_j \cdot w_j \cdot w_j/2)/(\sum_j p_j \cdot w_j)$. What comes next will prove that $W/2 > w/2$ unless $w_j = w_k$ whenever $p_j \cdot p_k > 0$. This proof is traceable to Lagrange:

$$0 \leq \sum_j \sum_k p_j \cdot p_k \cdot (w_j - w_k)^2 = \ldots = 2w \cdot (W - w) \qquad \text{(Fill in the } \ldots \text{ yourself.)}$$
$$= \ldots = 2V^2 ,$$

where $V^2 := \sum_j p_j \cdot (w_j - w)^2$ is the *Variance* in the arrival times w_j; their *Standard Deviation* is V. Therefore $W = w + V^2/w > w$.

Therefore the average waiting time $W/2$ exceeds half the average time w between arrivals by relatively little unless some of the likelier intervals w_j are relatively rather different from the average w of all the w_j's .