Learning in finite games and convex potential games

Walid Krichene

May 12, 2014
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Introduction

- Finite player game, played repeatedly.
- Player $k$ has mixed strategy $p^k(t)$, which obeys an update rule.
Introduction

- Finite player game, played repeatedly.
- Player $k$ has mixed strategy $p^k(t)$, which obeys an update rule.
- Questions:
  - Does $(p(t))_{t \in \mathbb{N}}$ converge to an equilibrium set?
Introduction

- Finite player game, played repeatedly.
- Player $k$ has mixed strategy $p^k(t)$, which obeys an update rule.
- Questions:
  - Does $(p(t))_{t \in \mathbb{N}}$ converge to an equilibrium set?
  - Does $(\bar{p}(t))_{t \in \mathbb{N}}$ converge to an equilibrium set?

$$\bar{p}(t) = \frac{1}{t} \sum_{\tau=1}^{t} p(\tau)$$
Finite player game, played repeatedly.

Player $k$ has mixed strategy $p^k(t)$, which obeys an update rule.

Questions:

- Does $(p(t))_{t \in \mathbb{N}}$ converge to an equilibrium set?
- Does $(\bar{p}(t))_{t \in \mathbb{N}}$ converge to an equilibrium set?

\[
\bar{p}(t) = \frac{1}{t} \sum_{\tau=1}^{t} p(\tau)
\]

Which equilibrium sets? Nash / Correlated (Auman) / Hannan

$\mathcal{N} \subseteq \mathcal{C} \subseteq \mathcal{H}$
## Results

<table>
<thead>
<tr>
<th>Game class</th>
<th>Dynamics</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite</td>
<td>external-regret minimizing</td>
<td>$\bar{p}(t) \to \mathcal{H}$</td>
</tr>
<tr>
<td>finite</td>
<td>internal-regret minimizing</td>
<td>$\bar{p}(t) \to \mathcal{C}$</td>
</tr>
<tr>
<td>2 player zero-sum</td>
<td>external-regret minimizing</td>
<td>$\bar{p}(t) \to \mathcal{N}$</td>
</tr>
<tr>
<td>2 players, 2 actions</td>
<td>fictitious play</td>
<td>$\bar{p}(t) \to \mathcal{N}$</td>
</tr>
</tbody>
</table>
The setting

- $K$ players
- player $k$: finite action set $S_k$
The setting

- $K$ players
- player $k$: finite action set $S_k$
- joint action $i \in S_1 \times \cdots \times S_K$
- loss $\ell_k(i) \in [0, 1]$
The setting

- \( K \) players
- player \( k \): finite action set \( S_k \)
- joint action \( i \in S_1 \times \cdots \times S_K \)
- loss \( \ell_k(i) \in [0, 1] \)
- players randomize: mixed strategy profile \( p \in \Delta^{S_1 \times \cdots \times S_K} \)
- If players randomize independently,
  \[
  p(i) = p_{i_1}^{(1)} \cdots p_{i_K}^{(K)}
  \]
- joint random action \( l \in S_1 \times \cdots \times S_K \), \( l \sim p \)
- expected loss \( \mathbb{E}_p[\ell_k] = \sum_i p(i)\ell_k(i) \)
- write \( \ell_k(p) = \mathbb{E}_p[\ell_k] \)
  (extension of \( \ell \) from \( S_1 \times \cdots \times S_K \) to \( \Delta^{S_1 \times \cdots \times S_K} \))
Outline

1. Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2. Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Nash equilibrium

\[ p \text{ is a Nash eq. if it is a product distribution and for all } k \text{ and all } p' \]

\[ \ell_k(p^{(k)}, p^{(-k)}) \leq \ell_k(p', p^{(-k)}) \]

- No one has an incentive to unilaterally deviate.
Correlated equilibria

Player actions may be correlated.

Correlated equilibria (Aumann)

- Probability distribution $p$ over $S_1 \times \cdots \times S_K$, not necessarily a product distribution
Correlated equilibria

Player actions may be correlated.

Correlated equilibria (Aumann)

- Probability distribution $p$ over $S_1 \times \cdots \times S_K$, not necessarily a product distribution
- Some coordinator gives advice $I \sim P$
- If player unilaterally deviates from $j$, worse-off in expectation.
Correlated equilibria
Player actions may be correlated.

Correlated equilibria (Aumann)

- Probability distribution $p$ over $S_1 \times \cdots \times S_K$, not necessarily a product distribution
- Some coordinator gives advice $I \sim P$
- If player unilaterally deviates from $j$, worse-off in expectation.

“Switching $j$ to $j'$ is a bad idea”. For all $j, j'$

$$\sum_{i: i^k = j} p(i)[\ell_k(j, i^{(-k)}) - \ell(k)(j', i^{(-k)})] \leq 0$$
Correlated equilibria

Player actions may be correlated.

Correlated equilibria (Aumann)

- Probability distribution $p$ over $S_1 \times \cdots \times S_K$, not necessarily a product distribution
- Some coordinator gives advice $I \sim P$
- If player unilaterally deviates from $j$, worse-off in expectation.

“Switching $j$ to $j'$ is a bad idea”. For all $j, j'$

$$\sum_{i : i^k = j} p(i)[\ell_k(j, i^{(-k)}) - \ell^k(j', i^{(-k)})] \leq 0$$

Structure of correlated equilibria

- $p$ vector in $\Delta^{S_1 \times \cdots \times S_K}$
- Linear inequalities in $p \Rightarrow C$ is a closed convex polyhedron
- Convex combination of Nash eq $\Rightarrow$ correlated eq. (but not $\Leftarrow$)
Hannan set

- \( p \in \mathcal{H} \) if for all \( j \)

\[
\ell_k(p) \leq \ell_k(\delta_j \times p^{(-k)})
\]
Hannan set

- $p \in \mathcal{H}$ if for all $j$
  \[ \ell_k(p) \leq \ell_k(\delta^j \times p^{-k}) \]

- Same as the Nash condition but $p$ not necessarily product distribution
- $p \in \mathcal{H}$ and $p$ is a product $\iff p \in \mathcal{N}$
- $\mathcal{H}$ is a convex polyhedron
- $\mathcal{C} \subseteq \mathcal{H}$
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Repeated play

- $k$ maintains $p^{(k)}(t)$, draws action $I^{(k)}(t) \sim p^{(k)}_t$
- observes all players’ actions $I(t) = (I^{(1)}(t), \ldots, I^{(K)}(t))$
Repeated play

- $k$ maintains $p^{(k)}(t)$, draws action $I^{(k)}(t) \sim p^{(k)}(t)$
- observes all players' actions $I(t) = (I^{(1)}(t), \ldots, I^{(K)}(t))$
- **Uncoupled play:** player only knows his loss function.
Regret

Compare expected loss to loss had we played differently.
Let $\psi : S_k \to S_k$.
This defines distribution $\phi(p)$

$$\phi(p)(i) = p(\psi(i^k), i^{-k})$$

Regret:

$$R_{\phi}(t) = \sum_{\tau=1}^{t} \ell_k(p(\tau)) - \sum_{\tau=1}^{t} \ell_k(\phi(p(\tau)))$$
Regret

Compare expected loss to loss had we played differently.
Let $\psi : S_k \rightarrow S_k$.
This defines distribution $\phi(p)$

$$\phi(p)(i) = p(\psi(i^k), i^{-k})$$

Regret:

$$R_\phi(t) = \sum_{\tau=1}^{t} \ell_k(p(\tau)) - \sum_{\tau=1}^{t} \ell_k(\phi(p(\tau)))$$

- Idea: fix a class of functions $\Phi$. This defines a regret vector $R(t) = (R_\phi(t))_{\phi \in \Phi}$.
- We want $\lim_{t \rightarrow \infty} d\left(\frac{R(t)}{t}, \mathbb{R}^{\mid\Phi\mid}\right) = 0$
Regret

By linearity

\[
\frac{1}{t} R_{\phi}(t) = \frac{1}{t} \sum_{\tau=1}^{t} \sum_{i} p(\tau)(i) \left( \ell_k(i) - \ell_k(\psi(i^k), i^{(-k)}) \right) \\
= \sum_{i} \bar{p}(t)(i) \left( \ell_k(i) - \ell_k(\psi(i^k), i^{(-k)}) \right) \\
= \ell_k(\bar{p}(t)) - \ell_k(\phi(\bar{p}(t)))
\]
Regret

External regret

For all $j \in S_k$, let $\psi_j$ map any action to $j$.

$$\phi_j : p \mapsto \delta^j \times p^{(-k)}$$

$$R_{\text{ext}}(t) \in \mathbb{R}^{\left|S_k\right|}$$
Regret

External regret

For all $j \in S_k$, let $\psi_j$ map any action to $j$. 

$$
\phi_j : p \mapsto \delta^j \times p^{(-k)}
$$

$$
R_{\text{ext}}(t) \in \mathbb{R}^{|S_k|}
$$

$$
\frac{1}{t} R_{\phi_j}(t) = \ell_k(\bar{p}(t)) - \ell_k(\delta^j \times \bar{p}^{(-k)}(t))
$$
Regret

External regret

For all $j \in S_k$, let $\psi_j$ map any action to $j$.

$$\phi_j : p \mapsto \delta^j \times p^{(-k)}$$

$$R_{ext}(t) \in \mathbb{R}^{S_k}$$

$$\frac{1}{t} R_{\phi_j}(t) = \ell_k(\bar{p}(t)) - \ell_k(\delta^j \times \bar{p}^{(-k)}(t))$$

Convergence to Hannan set

$$\frac{1}{t} R^{ext}(t) \rightarrow \mathbb{R}^{S_k} \iff \bar{p}(t) \rightarrow \mathcal{H}$$
Regret

**Internal regret**

For all $j, j' \in S_k \times S_k$, let $\psi_{j \rightarrow j'}$ the function play $j'$ instead of $j$.

$$R_{int}(t) \in \mathbb{R}^{\left|S_k\right|(|S_k|-1)}$$
Regret

Internal regret

For all $j, j' \in S_k \times S_k$, let $\psi_{j \rightarrow j'}$ the function play $j'$ instead of $j$.

$$R_{\text{int}}(t) \in \mathbb{R}^{|S_k|(|S_k| - 1)}$$

$$\frac{1}{t} R_{\phi_{j \rightarrow j'}}(t) = \sum_i \bar{p}(t)(i) \left( \ell_k(i) - \ell_k(\psi_{j \rightarrow j'}(i^k), i^{(-k)}) \right)$$

$$= \sum_{i : i^k \neq j} \bar{p}(t)(i) \cdot 0 + \sum_{i : i^k = j} \bar{p}(t)(i) \left( \ell_k(j, i^{(-k)}) - \ell_k(j', i^{(-k)}) \right)$$
Regret

Internal regret

For all $j, j' \in S_k \times S_k$, let $\psi_{j \rightarrow j'}$ the function play $j'$ instead of $j$.

$$R_{int}(t) \in \mathbb{R}^{\mid S_k \mid (\mid S_k \mid - 1)}$$

$$\frac{1}{t} R_{\phi_{j \rightarrow j'}}(t) = \sum_i \bar{p}(t)(i) \left( \ell_k(i) - \ell_k(\psi_{j \rightarrow j'}(i^k), i^{(-k)}) \right)$$

$$= \sum_{i : i^k \neq j} \bar{p}(t)(i).0 + \sum_{i : i^k = j} \bar{p}(t)(i) \left( \ell_k(j, i^{(-k)}) - \ell_k(j', i^{(-k)}) \right)$$

Convergence to Correlated equilibria

$$\frac{1}{t} R_{int}(t) \rightarrow \mathbb{R}^{\mid S_k \mid (\mid S_k \mid - 1)} \Leftrightarrow \bar{p}(t) \rightarrow C$$
### Nash equilibria $\mathcal{N}$

$p$ product distribution. For all $j$

$$\ell_k(p^{(k)}, p^{(-k)}) \leq \ell_k(\delta_j, p^{(-k)})$$

### Correlated equilibria $\mathcal{C}$

$p$ not necessarily a product distribution. For all $j, j'$

$$\sum_{i : i^k = j} p(i) \ell_k(j, i^{(-k)}) \leq \sum_{i : i^k = j} p(i) \ell_k(j', i^{(-k)})$$

### Hannan set $\mathcal{H}$

$p$ not necessarily a product distribution. For all $j$

$$\ell_k(p^{(k)}, p^{(-k)}) \leq \ell_k(\delta_j, p^{(-k)})$$
Fictitious play

On iteration $t$, play best response to cumulative losses

$$L^{(k)}(t)(i^{(k)}) = \sum_{\tau=1}^{t} \ell(\tau)(i^{(k)}, l^{(-k)})$$
Fictitious play

On iteration $t$, play best response to cumulative losses

$$L^{(k)}(t)(i^{(k)}) = \sum_{\tau=1}^{t} \ell(\tau)(i^{(k)}, l^{(-k)})$$

Fictitious play

$$i^{(k)}(t + 1) \in \arg \min_{i \in S^k} L^{(k)}(t)(i)$$

equivalent to

$$\min_{p^{(k)} \in \Delta^k} \langle p^{(k)}, L^{(k)}(t) \rangle$$
Fictitious play

On iteration $t$, play best response to cumulative losses

$$L^{(k)}(t)(i^{(k)}) = \sum_{\tau=1}^{t} \ell(\tau)(i^{(k)}, I^{(-k)})$$

Fictitious play

$$i^{(k)}(t + 1) \in \arg\min_{i \in S^k} L^{(k)}(t)(i)$$

equivalent to

$$\min_{p^{(k)} \in \Delta^{S^k}} \left\langle p^{(k)}, L^{(k)}(t) \right\rangle$$

- Only converges in very special cases (e.g. 2 players, 2 actions each)
- Not Hannan-consistent in general ($\frac{1}{t} R_{ext}(t) \not\to 0$)
Regret-minimization Vs. fictitious play

**Fictitious play**

\[ p^{(k)}(t + 1) \in \arg \min_{p^{(k)} \in \Delta^{s_k}} \langle p^{(k)}, L^{(k)}(t) \rangle \]

**Regret-minimization**

Some algorithms can be written as

\[ p^{(k)}(t + 1) \in \arg \min_{p^{(k)} \in \Delta^{s_k}} \langle p^{(k)}, L^{(k)}(t) \rangle + \frac{1}{\gamma} R(p^{(k)}) \]
Algorithms with sublinear regret

How can we find algorithms such that

\[ \frac{R(t)}{t} \to 0 \]

- Blackwell approachability (reduces the problem)
Outline

1. **Learning in finite games**
   - Equilibria
   - Regret
   - Blackwell approachability

2. **Learning in convex potential games**
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Blackwell approachability

- Vector $r(i, j)$
- Extended bilinearly $r(p, q) = \sum_i \sum_j p(i)q(j)r(i, j)$
- Assume $\|r(i, j)\|_2 \leq 1$
- A closed convex set $S \subseteq B(0, 1)$ is approachable if $\exists$ a randomized strategy such that for all sequences $J(t)$,

$$\lim_{t \to \infty} d \left( \frac{1}{t} \sum_{\tau \leq t} r(I(\tau), J(\tau)), S \right) = 0$$
Blackwell approachability: Half spaces

\[ H_{a,c} = \{ r : \langle a, r \rangle \leq c \} \]
Blackwell approachability: Half spaces

- $H_{a,c} = \{ r : \langle a, r \rangle \leq c \}$
- Define new game: loss of $(i,j)$ is $\langle a, r(i,j) \rangle := (M_a)_{i,j}$. Defines a matrix $M_a$
- This a scalar game
Blackwell approachability: Half spaces

- $H_{a,c} = \{ r : \langle a, r \rangle \leq c \}$
- Define new game: loss of $(i,j)$ is $\langle a, r(i,j) \rangle := (M_a)_{i,j}$. Defines a matrix $M_a$
- This a scalar game

**Half spaces**

$H_{a,c}$ is approachable iff $c \geq V = \min_{p \in \Delta} \max_{q \in \Delta} \langle p, M_a q \rangle$

- in particular, if $H_{a,c}$ is approachable, then $\exists p \in \Delta : \forall q, \langle p, M_a q \rangle \leq c$
- Call it $p^*(H_{a,c})$.
- $\forall j, r(p^*H_{a,c},j) \in H_{a,c}$
Blackwell approachability theorem

Theorem: Blackwell approachability

$S$ is approachable iff every half space containing $S$ is approachable.
Theorem: Blackwell approachability

$S$ is approachable iff every half space containing $S$ is approachable.

$\Rightarrow$ if $S$ is approachable, any super set is approachable.
Theorem: Blackwell approachability

$S$ is approachable iff every half space containing $S$ is approachable.

- $\Rightarrow$ if $S$ is approachable, any super set is approachable.
- $\Leftarrow$ Assume every half space containing $S$ is approachable.
  - average regret $A_t = \frac{1}{t} \sum_{\tau \leq t} r(i_\tau, j_\tau)$
  - at $t$, if $A_{t-1} \notin S$
    - project $\pi_S(A_{t-1})$
    - define $H_{t-1}$: through $\pi_S(A_{t-1})$, $\perp (A_{t-1} - \pi_S(A_{t-1}))$.
  - By assumption $H_{t-1}$ is approachable. Play $p^*(H_{t-1})$
  - Next step: $r(p^*(H_{t-1}), j_t) \in H_{t-1}$
Blackwell approachability theorem

- $A_{t-1}$

Figure: Instantaneous regret $r(p_t, J_t)$ is forced in $H_{t-1}$
Blackwell approachability theorem

Figure: Instantaneous regret $r(p_t, J_t)$ is forced in $H_{t-1}$
Blackwell approachability theorem

\[ r(p^*(H_{t-1}), J_t) \]

\[ \pi_S(A_{t-1}) \]

\[ H_{t-1} \]

\[ A_{t-1} \]

Figure: Instantaneous regret \( r(p_t, J_t) \) is forced in \( H_{t-1} \)
Blackwell approachability theorem

\[ d(A_t, S)^2 \leq \| A_t - \pi_S(A_{t-1}) \|_2^2 \]

\[ = \left\| \frac{t-1}{t} A_{t-1} + \frac{1}{t} r(p_t, j_t) - \left( \frac{t-1}{t} + \frac{1}{t} \right) \pi_S(A_{t-1}) \right\|_2^2 \]

\[ = \left( \frac{t-1}{t} \right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2} \| r(p_t, j_t) - \pi_S(A_{t-1}) \|_2^2 \]

\[ + 2 \frac{t-1}{t^2} \left\langle A_{t-1} - \pi_S(A_{t-1}), r(p_t, j_t) - \pi_S(A_{t-1}) \right\rangle \]

\[ \leq \left( \frac{t-1}{t} \right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2} 2^2 \]
Blackwell approachability theorem

\[ d(A_t, S)^2 \leq \|A_t - \pi_S(A_{t-1})\|_2^2 \]

\[ = \left\| \frac{t-1}{t} A_{t-1} + \frac{1}{t} r(p_t, j_t) - \left( \frac{t-1}{t} \frac{1}{t}\right) \pi_S(A_{t-1}) \right\|_2^2 \]

\[ = \left( \frac{t-1}{t} \right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2} \|r(p_t, j_t) - \pi_S(A_{t-1})\|_2^2 \]

\[ + 2 \frac{t-1}{t^2} \langle A_{t-1} - \pi_S(A_{t-1}), r(p_t, j_t) - \pi_S(A_{t-1}) \rangle \]

\[ \leq \left( \frac{t-1}{t} \right)^2 d(A_{t-1}, S)^2 + \frac{1}{t^2} 2^2 \]

multiply by \( t^2 \), then by induction

\[ t^2 d(A_t, S)^2 \leq 4t \]

thus

\[ d(A_t, S) \leq \frac{2}{\sqrt{t}} \]
Blackwell approachability theorem

- In fact, can show almost sure convergence to $S$. 
Potential based approachability

- $S$ approachable. Want to define a class of algorithms which approach $S$.
- Potential $\phi \geq 0$ convex, $\phi(x) = 0 \iff x \in S$.
- e.g. $\phi(s) = d(x, S)^2$
Potential based approachability

- $S$ approachable. Want to define a class of algorithms which approach $S$.
- Potential $\phi \geq 0$ convex, $\phi(x) = 0 \iff x \in S$.
- e.g. $\phi(s) = d(x, S)^2$

Algorithm

Same idea: if $A_{t-1} \notin S$, use the potential to choose a strategy $p_t$ such that

$$\langle A_{t-1} - \pi_S(A_{t-1}), r(p_t, j) - \pi_S(A_{t-1}) \rangle \leq 0 \; \forall j$$

where $\pi_S$ is the Bregman projection on $S$

$$\pi_S(a) = \arg \min_{r \in S} \phi(r) - \phi(a) - \langle \nabla \phi(a), r - a \rangle$$

$$= \arg \min_{r \in S} - \langle \nabla \phi(a), r \rangle$$
Back to regret

- Choose $S = \mathbb{R}^d$.
- To apply approachability need to guarantee

$$\langle d, r(p_t, j) \rangle \leq 0 \quad \forall j$$

Figure: Instantaneous regret

$r(p^*(H_{t-1}), J_t)$
Back to regret

\[ \langle d, r(p_t, j) \rangle \leq 0 \forall j \]

- Recall

\[ r_i(p, j) = \langle p, \ell(j) \rangle - \ell_i(j) \]

- take \( p = \frac{d}{\|d\|_1} \)

\[ \langle d, r(p, j) \rangle = \sum_i \left( \langle d/\|d\|_1, \ell(j) \rangle d_i - \ell_i(j)d_i \right) = 0 \]
Convergence of $p(t)$?

We have $\bar{p}(t) \to S$. Can we show $p(t) \to S$?
Convergence of $p(t)$?

We have $\bar{p}(t) \rightarrow S$. Can we show $p(t) \rightarrow S$?

Show that $p(t)$ converges.

- If $p(t) \rightarrow p_\infty$, then $\bar{p}(t) \rightarrow p_\infty$, so $\{p_\infty\} \in S$.
- One possible technique:
  - consider a continuous-time version of the dynamics
    $$\dot{p}(t) = F(p(t))$$
  - show that $p$ converges
  - show that discrete trajectories approach continuous trajectories
Extensions

- Continuous time
Extensions

- Continuous time
  Internal regret, external regret: Hart and Mas-Colell (2003)

- Continuum of players
  Potential games in continuous time, dynamics with a positive correlation condition. Sandholm (2001)
  Congestion games with an approximate replicator update
Extensions

- **Continuous time**
  Internal regret, external regret: Hart and Mas-Colell (2003)

- **Continuum of players**
  Potential games in continuous time, dynamics with a positive correlation condition. Sandholm (2001)
  Congestion games with an approximate replicator update

- **Continuous action spaces**
  Compact locally convex, Lipschitz loss functions.
Outline

1. Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2. Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Routing game

Graph \((V, E)\)

source-sink pairs, \((s_k, t_k)\): paths \(P_k\)
Routing game

Graph \((V, E)\)

source-sink pairs, \((s_k, t_k)\): paths \(P_k\)

Players choose a distribution over paths \(\mu^k \in \Delta P_k\),

\(\mu\) determines edge flows \(\phi = M\mu\) (linear function)
Routing game

Graph \((V, E)\)

- source-sink pairs, \((s_k, t_k)\): paths \(P_k\)
- Players choose a distribution over paths \(\mu^k \in \Delta^{P_k}\)
- \(\mu\) determines edge flows \(\phi = M\mu\) (linear function)
- Congestion on edge \(e\): \(c_e : \phi_e \mapsto c_e(\phi_e)\), increasing
- want to minimize personal latency \(\ell^k_p(\mu) = \sum_{e \in P} c_e(\phi_e)\)

Player = infinitesimal amount of flow.

\(\mu = \) combined decision of all players.
More precisely

- Measurable set of players \((\mathcal{X}_k, \mathcal{S}_k, m_k)\), atomless
- \(m_k(\mathcal{X}_k)\) finite
More precisely

- Measurable set of players \((X_k, S_k, m_k)\), atomless
- \(m_k(X_k)\) finite
- Every player \(x \in X_k\) chooses distribution \(\pi^k(x)\)
- This induces population distribution \(\mu^k(x) = \int_{X_k} \pi^k(x) dm(x)\)
Nash equilibria

Nash equilibrium

\( \mu \) is a Nash equilibrium if for all \( k \), for all \( p \in P_k \) with positive mass, \( \ell^k_p(\mu) \) is minimal on \( P_k \)

\[ \ell^k_p(\mu) \leq \ell^k_{p'}(\mu) \quad \forall p' \in P_k \]

- How to compute Nash equilibria?
Nash equilibria

Nash equilibrium

\( \mu \) is a Nash equilibrium if for all \( k \), for all \( p \in P_k \) with positive mass, \( \ell^k_p(\mu) \) is minimal on \( P_k \)

\[ \ell^k_p(\mu) \leq \ell^k_{p'}(\mu) \quad \forall p' \in P_k \]

- How to compute Nash equilibria? Convex formulation

Potential function

\( \mu \) is a Nash equilibrium iff it minimizes a potential function

\[
\text{minimize}_{\phi, \mu \in \Delta P_1 \times \ldots \times \Delta P_K} \quad \sum_{e} \int_{0}^{\phi_e} c_e(u) du \\
\text{subject to} \quad \forall e, \sum_{k} \sum_{p \ni e} \mu^k_p = \phi_e
\]
Nash equilibria

Convex potential function

\[ V(\mu) = \sum_e \int_0^{(M\mu)_e} c_e(u) \, du \]

- \( V \) is convex: composition of \( \phi \mapsto \sum_e \int_0^{\phi_e} c_e(u) \, du \) (strongly convex) with \( \mu \mapsto M\mu \)
- \( \nabla_{\mu_k} V(\mu) = \ell^k(\mu) \)
- \( M \) non injective in general, \( V \) weakly convex, minimizer not unique.
Motivation for a learning model

- How do players find a Nash equilibrium?
Motivation for a learning model

- How do players find a Nash equilibrium?
  Ideally: distributed, and has minimal information requirements.
  ▶ loss on the player’s path
  ▶ loss on population paths
  ▶ all edge losses
  ▶ all congestion functions and edge flows
Motivation for a learning model

- How do players find a Nash equilibrium? Ideally: distributed, and has minimal information requirements.
  - loss on the player’s path
  - loss on population paths
  - all edge losses
  - all congestion functions and edge flows
- Player dynamics: given $\mu^k(t)$, $\ell^k(\mu^{(t)})$, choose $\mu^k(t+1)$
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
The Hedge algorithm

Hedge algorithm

- Update the distribution according to observed loss

\[ \mu_p^{k(t+1)} \propto \mu_p^{k(t)} e^{-\eta \ell_p^{k(t)}} \]
Regret Bound

- Assume losses are in $[0, \rho]$.
- Expected loss is $\langle \mu^k(t), \ell^k(\mu(t)) \rangle$

$$R^k(T) = \sum_{t=1}^{T} \left\langle \mu^k(t), \ell^k(\mu(t)) \right\rangle - \min_p \sum_{t=1}^{T} \ell^k_p(t)$$
Regret Bound

- Assume losses are in $[0, \rho]$.
- Expected loss is $\langle \mu^k(t), \ell^k(\mu(t)) \rangle$

$$R^k(T) = \sum_{t=1}^{T} \langle \mu^k(t), \ell^k(\mu(t)) \rangle - \min_p \sum_{t=1}^{T} \ell^k_p(t)$$

Regret bound

$$\frac{R^k(T)}{T} \leq \frac{\rho \ln \mu^{k(0)}_{\min}}{T \eta} + \rho \eta$$

Motivation for regret: in general finite games, equilibria are characterized in terms of regret (more details later).
Convergence to approximate Nash equilibria

Regret bound

\[ \frac{R^k(T)}{T} \leq \rho \ln \frac{\mu^{k(0)}}{\mu_{\min}} + \rho \eta \]

Convergence to approximate Nash equilibria

If an update satisfies the regret bound, then for all \( \epsilon > 0 \), for \( \eta \) small enough, \( \mu^{(t)} \) converges

\[ \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mu^{(t)} \in N_\epsilon \]

\( N_\epsilon = \{ \mu : V(\mu) < V_N + \epsilon \} \): \( \epsilon \)-approximate Nash equilibrium.
Convergence to approximate Nash equilibria

Regret bound

\[ \frac{R^k(T)}{T} \leq \frac{\rho \ln \mu_{min}^{k(0)}}{T \eta} + \rho \eta \]

Convergence to approximate Nash equilibria

If an update satisfies the regret bound, then for all \( \epsilon > 0 \), for \( \eta \) small enough, \( \mu(t) \) converges

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mu(t) \in \mathcal{N}_\epsilon
\]

\( \mathcal{N}_\epsilon = \{ \mu : V(\mu) < V_N + \epsilon \} \): \( \epsilon \)-approximate Nash equilibrium.

Proof: show

\[
V(\mu(T)) - V(\mu^*) \leq \sum_{k} \frac{R^k(T)}{T}
\]
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Consider the convex problem

$$\text{minimize}_{\mu \in \Delta} V(\mu)$$

**Algorithm 1** Mirror Descent Method

1. **for** $t \in \mathbb{N}$ **do**
2. \[ \mu^{(t+1)} = \arg \min_{\mu \in \Delta} \langle \nabla V(\mu^{(t)}), \mu \rangle + \frac{1}{\eta} D_\psi(\mu, \mu^{(t)}) \]
3. **end for**

where $D_\psi$ is a Bregman divergence

$$D_\psi(\mu, \nu) = \psi(\mu) - \psi(\nu) - \langle \nabla \psi(\nu), \mu - \nu \rangle$$
Mirror descent on potential function

Mirror descent on $V$

- Take $V$ potential function defined earlier. $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$
Mirror descent on potential function

Mirror descent on $V$

- Take $V$ potential function defined earlier. $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$
- Take $\psi(\mu) = \sum_k \sum_p \mu^k_p \ln \mu^k_p$
- Then $D_{\psi}(\mu, \nu) = \sum_k D_{KL}(\mu^k, \nu^k)$
Mirror descent on potential function

**Mirror descent on** $V$

- Take $V$ potential function defined earlier. $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$
- Take $\psi(\mu) = \sum_k \sum_p \mu^k_p \ln \mu^k_p$
- Then $D_\psi(\mu, \nu) = \sum_k D_{KL}(\mu^k, \nu^k)$
- Update:

$$\mu^{(t+1)} = \arg \min_{\mu \in \Delta^1 \times \cdots \times \Delta^K} \sum_k \left( \langle \nabla_{\mu^k} V(\mu), \mu^k \rangle + \frac{1}{\eta} D_{KL}(\mu^k, \mu^k(t)) \right)$$
Mirror descent on potential function

Mirror descent on $V$

- Take $V$ potential function defined earlier. $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$
- Take $\psi(\mu) = \sum_k \sum_p \mu^k_p \ln \mu^k_p$
- Then $D_\psi(\mu, \nu) = \sum_k D_{KL}(\mu^k, \nu^k)$
- Update:

$$\mu^{(t+1)} = \arg \min_{\mu \in \Delta^1 \times \cdots \times \Delta^K} \sum_k \left( \langle \nabla_{\mu^k} V(\mu), \mu^k \rangle + \frac{1}{\eta} D_{KL}(\mu^k, \mu^{k(t)}) \right)$$

- Objective separates

$$\mu^k(t+1) = \arg \min_{\mu \in \Delta^k} \left\langle \nabla_{\mu^k} V(\mu), \mu^k \right\rangle + \frac{1}{\eta} D_\psi(\mu^k, \mu^{k(t)})$$
Mirror descent on potential function

**Mirror descent on** $V$

- Take $V$ potential function defined earlier. $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$
- Take $\psi(\mu) = \sum_k \sum_p \mu_p^k \ln \mu_p^k$
- Then $D_\psi(\mu, \nu) = \sum_k D_{KL}(\mu^k, \nu^k)$
- Update:

$$\mu^{(t+1)} = \arg \min_{\mu \in \Delta_1 \times \cdots \times \Delta_K} \sum_k \left( \langle \nabla_{\mu^k} V(\mu), \mu^k \rangle + \frac{1}{\eta} D_{KL}(\mu^k, \mu^k(t)) \right)$$

- Objective separates

$$\mu^k(t+1) = \arg \min_{\mu \in \Delta_k} \left( \langle \nabla_{\mu^k} V(\mu), \mu^k \rangle + \frac{1}{\eta} D_\psi(\mu^k, \mu^k(t)) \right)$$

- Solution: Hedge algorithm with learning rate $\eta$

$$\mu_p^{k(t+1)} \propto \mu_p^{k(t)} e^{-\eta \ell_p^{k(t)}}$$
Mirror Descent on potential function

Convergence of Mirror Descent method

\[ V \left( \frac{1}{T} \sum_{t \leq T} \mu^{(t)} \right) - V(\mu^{*}) \leq \frac{1}{\eta T} \sum_{k} D_{KL}(\mu^{*k}, \mu^{k(0)}) + \frac{\eta \rho^2}{2} \]

No need to use a regret bound.
Mirror Descent on potential function

Convergence of Mirror Descent method

\[
V \left( \frac{1}{T} \sum_{t \leq T} \mu^{(t)} \right) - V(\mu^*) \leq \frac{1}{\eta T} \sum_k D_{KL}(\mu^* k, \mu^k(0)) + \frac{\eta \rho^2}{2}
\]

No need to use a regret bound.

General result

Convergence of \( \bar{\mu}(T) = \frac{1}{T} \sum_{t \leq T} \mu^{(t)} \) for

- Any potential game (that is, \( \exists V \) convex such that \( \nabla_{\mu k} V(\mu) = \ell^k(\mu) \))
- Any Mirror Descent method (choose your favorite \( \psi \), strongly convex w.r.t. \( \| \cdot \|_1 \))

to \( N_\epsilon \).
Mirror Descent with decreasing rates

**MD with time-varying rates**

\[ \mu^{k(t+1)} = \arg \min_{\mu \in \Delta^k} \left\langle \nabla_{\mu^k} V(\mu), \mu^k \right\rangle + \frac{1}{\eta_t} D_\psi(\mu^k, \mu^{k(t)}) \]
Mirror Descent with decreasing rates

MD with time-varying rates

\[
\mu^{k(\cdot+1)} = \arg \min_{\mu \in \Delta_k} \left\langle \nabla_{\mu^k} V(\mu), \mu^k \right\rangle + \frac{1}{\eta_t} D_{\psi}(\mu^k, \mu^{k(t)})
\]

Then, can show

\[
V\left(\frac{\sum_{t \leq T} \eta_t \mu(t)}{\sum_{t \leq T} \eta_t}\right) - V(\mu^*) \leq \frac{1}{\sum_{t \leq T} \eta_t} D^{(0)} + \frac{\rho^2}{2} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t}
\]
Mirror Descent with decreasing rates

MD with time-varying rates

\[ \mu^{k(t+1)} = \arg \min_{\mu \in \Delta^k} \left\langle \nabla_{\mu^k} V(\mu), \mu^k \right\rangle + \frac{1}{\eta_t} D_\psi(\mu^k, \mu^{k(t)}) \]

Then, can show

\[ V \left( \frac{\sum_{t \leq T} \eta_t \mu(t)}{\sum_{t \leq T} \eta_t} \right) - V(\mu^*) \leq \frac{1}{\sum_{t \leq T} \eta_t} D^{(0)} + \frac{\rho^2}{2} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t} \]

if \( \eta_t = \frac{1}{\sqrt{t}} \), convergence in \( \frac{\ln T}{\sqrt{T}} \)
Discounted hedge algorithm

Have a similar bound for discounted regret

**Discounted regret**

\[
R^k(T) = \sum_{t \leq T} \left< \mu^k(t), \eta_t \ell^k(\mu(t)) \right> - \min_p \sum_{t=1}^T \eta_t \ell^k_p(t)
\]

Interpretation: discount losses over time.

**Discounted regret bound**

\[
\frac{R^k(T)}{\sum_{t \leq T} \eta_t} \leq \rho \log \frac{\mu_{\min}^k}{\mu_{\min}^{k(0)}} + \frac{\rho}{8} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t}
\]
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Convergence of a dense subsequence

\[ V\left(\frac{\sum_{t \leq T} \eta_t \mu^{(t)}}{\sum_{t \leq T} \eta_t}\right) - V(\mu^*) \leq \frac{1}{\sum_{t \leq T} \eta_t} D^{(0)} + \frac{\rho^2}{2} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t} \]

\[ \bar{\mu}^{(t)} \rightarrow \mathcal{N} \]
Convergence of a dense subsequence

\[ V \left( \frac{\sum_{t \leq T} \eta_t \mu(t)}{\sum_{t \leq T} \eta_t} \right) - V(\mu^*) \leq \frac{1}{\sum_{t \leq T} \eta_t} D^{(0)} + \frac{\rho^2}{2} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t} \]

\[ \bar{\mu}(t) \to \mathcal{N} \]

**Theorem**

Under MD with appropriate \( \eta_t \), a dense subsequence of \( (\mu(t))_t \) converges to \( \mathcal{N} \)

- Subsequence \( (\mu(t))_{t \in T} \) converges
- \( \lim_{T \to \infty} \frac{\sum_{t \in T: t \leq T} \eta_t}{\sum_{t \leq T} \eta_t} = 1 \)
Convergence of a dense subsequence

\[
V \left( \frac{\sum_{t \leq T} \eta_t \mu(t)}{\sum_{t \leq T} \eta_t} \right) - V(\mu^*) \leq \frac{1}{\sum_{t \leq T} \eta_t} D^{(0)} + \frac{\rho^2}{2} \frac{\sum_{t \leq T} \eta_t^2}{\sum_{t \leq T} \eta_t}
\]

\[
\bar{\mu}(t) \to \mathcal{N}
\]

**Theorem**

Under MD with appropriate \( \eta_t \), a dense subsequence of \( (\mu(t))_t \) converges to \( \mathcal{N} \)

- Subsequence \( (\mu(t))_{t \in T} \) converges
- \( \lim_{T \to \infty} \frac{\sum_{t \in T: t \leq T} \eta_t}{\sum_{t \leq T} \eta_t} = 1 \)

**Proof.**

- Absolute Cesàro convergence implies convergence of a dense subsequence.
Figure: Example network
Simulations

(a) Trajectories $(\mu^k(t))_t$.

(b) Path flows $\mu^k_p(t)$, $p \in \mathcal{P}_k$

Figure: Constant learning rate $\eta = 0.7$. The trajectories do not converge.
Simulations

(a) Path losses $\ell_p^k(\mu(t))$
(b) Path losses for the means, $\ell_p^k(\frac{1}{t} \sum_{\tau \leq t} \mu(\tau))$

Figure: Path latencies
Simulations

(a) Trajectories $(\mu_k^t)(t)$.  

(b) Path flows $\mu_p^k(t)$, $p \in P_k$

Figure: harmonic sequence of learning rates $\eta_t = \frac{1}{1+t/10}$
Simulations

Figure: Path losses $\ell_p^k(\mu(t))$
Outline

1 Learning in finite games
   - Equilibria
   - Regret
   - Blackwell approachability

2 Learning in convex potential games
   - The routing game and Nash equilibria
   - Regret
   - Mirror descent on potential function
   - Convergence of a dense subsequence
   - Strong convergence
Sufficient conditions for convergence of \((\mu(t))_t\)

- Have \(\bar{\mu}^{(t)} \rightarrow \mathcal{N}\).
Sufficient conditions for convergence of \((\mu(t))_t\)

- Have \(\bar{\mu}(t) \to \mathcal{N}\).

**Sufficient condition 1**

If \((\mu(t))\) converges, then \(\mu(t) \to \mu^* \in \mathcal{N}\).
Sufficient conditions for convergence of \((\mu^{(t)})_t\)

- Have \(\tilde{\mu}^{(t)} \rightarrow \mathcal{N}\).

**Sufficient condition 1**

If \((\mu^{(t)})\) converges, then \(\mu^{(t)} \rightarrow \mu^* \in \mathcal{N}\).

**Sufficient condition 2**

If \(V(\mu^{(t)})\) converges (weaker, \(\mu^{(t)}\) need not converge), then

- \(V(\mu^{(t)}) \rightarrow V^*\)
- \(\mu^{(t)} \rightarrow \mathcal{N}^*\) (\(V\) is continuous, \(\mu \in \Delta\) compact)
Replicator dynamics

Imagine an underlying continuous time. Updates happen at $\eta_1, \eta_1 + \eta_2, \ldots$

\[ 0 \quad \eta_1 \quad \eta_1 + \eta_2 \quad \rightarrow \quad \ldots \]

**Figure**: Underlying continuous time
Replicator dynamics

Imagine an underlying continuous time. Updates happen at $\eta_1, \eta_1 + \eta_2, \ldots$

\[
0 \quad \eta_1 \quad \eta_1 + \eta_2 \quad \cdots
\]

Figure: Underlying continuous time

In the update equation $\mu_p(t + 1) \propto \mu_p(t)e^{-\eta_t \ell_p(t)}$, take $\eta_t \to 0$

We obtain the autonomous ODE:

\[
\forall p \in P_k, \frac{d\mu^k_p}{dt} = \mu^k_p(\langle \ell^k(\mu), \mu^k \rangle - \ell^k_p(\mu))/\rho \tag{1}
\]

Also in evolutionary game theory.
Replicator dynamics

Replicator equation

\[ \forall p \in P_k, \frac{d\mu^k_p}{dt} = \mu^k_p(\langle \ell^k(\mu), \mu^k \rangle - \ell^k_p(\mu))/\rho \]

Theorem

Every solution of the ODE (1) converges to the set of its stationary points.

Proof:

\( V \) is a Lyapunov function.
Replicator dynamics

Replicator equation

\[ \forall p \in \mathcal{P}_k, \quad \frac{d\mu_p^k}{dt} = \mu_p^k(\langle \ell^k(\mu), \mu^k \rangle - \ell_p^k(\mu)) / \rho \]

Theorem

Every solution of the ODE (1) converges to the set of its stationary points.
Replicator dynamics

Replicator equation

\[ \forall p \in \mathcal{P}_k, \quad \frac{d\mu_p^k}{dt} = \mu_p^k(\langle \ell^k(\mu), \mu^k \rangle - \ell_p^k(\mu))/\rho \]

Theorem

Every solution of the ODE (1) converges to the set of its stationary points.

Proof: \( V \) is a Lyapunov function.
REP method

Discretization of the continuous-time replicator dynamics

\[ \mu_p^{k(t+1)} - \mu_p^{k(t)} = \eta_t \mu_p^{k(t)} \left( \frac{\left< \ell^k(\mu(t)), \mu^k(t) \right> - \ell^k_p(\mu(t))}{\rho} \right) + \eta_t U_p^{k(t+1)} \]

\((U^{(t)})_{t \geq 1}\) perturbations that satisfy for all \(T > 0\),

\[ \lim_{\tau_1 \to \infty} \max_{\tau_2 : \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0 \]
Convergence to Nash equilibria

**Theorem**

Consider a potential game with convex potential $V$. Under any MD algorithm which is approximate REP, $\mu(t) \to \mathcal{N}$. 

Proof uses two facts

Affine interpolation of $\mu(t)$ is an asymptotic pseudo trajectory for the ODE.

$V$ is a Lyapunov function for Nash equilibria.
Convergence to Nash equilibria

**Theorem**

Consider a potential game with convex potential $V$. Under any MD algorithm which is approximate REP, $\mu(t) \to N$.

Proof uses two facts

- Affine interpolation of $\mu(t)$ is an asymptotic pseudo trajectory for the ODE.
- $V$ is a Lyapunov function for Nash equilibria.
REP update

In particular for $U = 0$, we obtain

$$
\mu_p^{k(t+1)} - \mu_p^{k(t)} = \eta_t \mu_p^{k(t)} \left( \frac{\langle \ell^k(\mu(t)), \mu^k(t) \rangle - \ell^k_p(\mu(t))}{\rho} \right)
$$
REP update

In particular for \( U = 0 \), we obtain

\[
\mu_p^{k(t+1)} - \mu_p^{k(t)} = \eta_t \mu_p^{k(t)} \left( \frac{\langle \ell^k(\mu(t)), \mu^k(t) \rangle - \ell_p^k(\mu(t))}{\rho} \right)
\]

- Also MD algorithm:

\[
\mu^{(t+1)} \in \arg \min_{\mu \in \Delta} \left\langle \frac{\ell(\mu(t))}{\rho}, \mu \right\rangle + \frac{1}{\eta_t} D(\mu \| \mu(t))
\]

- \( D(\mu \| \nu) = \frac{1}{2} \sum_p \nu_p \left( \frac{\mu_p}{\nu_p} - 1 \right)^2 \) (KKT conditions)
That's all, folks!
A convex formulation of Nash equilibria

\( \mu \) is a Nash equilibrium iff it minimizes a potential function

\[
\text{minimize}_{\mu \geq 0, \phi \geq 0} \sum_e \int_0^{\phi_e} \ell_e(u)du
\]

subject to

\[
\forall e, \sum_{p \ni e} \mu_p = \phi_e \quad \sum_p \mu_p = 1
\]

Walid Krichene  (UC Berkeley)  Learning in finite games and convex potential  May 12, 2014  63 / 64
A convex formulation of Nash equilibria

is a Nash equilibrium iff it minimizes a potential function

\[
\begin{align*}
\text{minimize}_{\mu \geq 0, \phi \geq 0} & \quad \sum_{e} \int_{0}^{\phi_e} \ell_e(u)du \\
\text{subject to} & \quad \forall e, \sum_{p \in e} \mu_p = \phi_e \quad \sum_{p} \mu_p = 1
\end{align*}
\]

partial Lagrangian

\[
\sum_{e} \int_{0}^{\phi_e} \ell_e(u)du + \sum_{e} \nu_e \left( \sum_{p \in e} \mu_p - \phi_e \right) - w \left( \sum_{p} \mu_p - 1 \right) - \sum_{e} \lambda_e \phi_e
\]
A convex formulation of Nash equilibria

$\mu$ is a Nash equilibrium iff it minimizes a potential function

\[
\text{minimize}_{\mu \geq 0, \phi \geq 0} \sum_e \int_0^{\phi_e} \ell_e(u) du
\]

subject to

\[
\forall e, \sum_{p \ni e} \mu_p = \phi_e \quad \sum_p \mu_p = 1
\]

partial Lagrangian

\[
\sum_e \int_0^{\phi_e} \ell_e(u) du + \sum_e \nu_e \left( \sum_{p \ni e} \mu_p - \phi_e \right) - w \left( \sum_p \mu_p - 1 \right) - \sum_e \lambda_e \phi_e
\]
A convex formulation of Nash equilibria

\( \mu \) is a Nash equilibrium iff it minimizes a potential function

\[
\text{minimize}_{\mu \geq 0, \phi \geq 0} \sum_e \int_0^{\phi_e} \ell_e(u) du
\]

subject to

\[
\forall e, \sum_{p \ni e} \mu_p = \phi_e \quad \sum_p \mu_p = 1
\]

partial Lagrangian

\[
\sum_e \int_0^{\phi_e} \ell_e(u) du + \sum_e \nu_e \left( \sum_{p \ni e} \mu_p - \phi_e \right) - w \left( \sum_p \mu_p - 1 \right) - \sum_e \lambda_e \phi_e
\]
A convex formulation of Nash equilibria

$\mu$ is a Nash equilibrium iff it minimizes a potential function

$$\min_{\mu \geq 0, \phi \geq 0} \sum_e \int_0^{\phi_e} \ell_e(u) du$$

subject to

$$\forall e, \sum_{p \ni e} \mu_p = \phi_e, \quad \sum_p \mu_p = 1$$

partial Lagrangian

$$\sum_e \int_0^{\phi_e} \ell_e(u) du + \sum_e v_e \left( \sum_{p \ni e} \mu_p - \phi_e \right) - w \left( \sum_p \mu_p - 1 \right) - \sum_e \lambda_e \phi_e$$
A convex formulation of Nash equilibria

\( \mu \) is a Nash equilibrium iff it minimizes a potential function

\[
\minimize_{\mu \geq 0, \phi \geq 0} \sum_e \int_0^{\phi_e} \ell_e(u) du
\]

subject to

\[
\forall e, \sum_{p \ni e} \mu_p = \phi_e \quad \sum_p \mu_p = 1
\]

partial Lagrangian

\[
\sum_e \int_0^{\phi_e} \ell_e(u) du + \sum_e v_e \left( \sum_{p \ni e} \mu_p - \phi_e \right) - w \left( \sum_p \mu_p - 1 \right) - \sum_e \lambda_e \phi_e
\]

Optimality conditions

- stationarity

\[
\forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0
\]

\[
\forall p, \quad \sum_{e \in p} v_e - w = 0
\]

- complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)
A convex formulation of Nash equilibria

- stationarity

\[ \forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0 \]
\[ \forall p, \quad \sum_{e \in p} v_e - w = 0 \]

- complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)

for all paths \( p \), the cost on the path is

\[ \ell_p(\mu) = \sum_{e \in p} \ell(\phi_e) \]
A convex formulation of Nash equilibria

- stationarity

\[ \forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0 \]

\[ \forall p, \quad \sum_{e \in p} v_e - w = 0 \]

- complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)

for all paths \( p \), the cost on the path is

\[ \ell_p(\mu) = \sum_{e \in p} \ell(\phi_e) = \sum_{e \in p} (v_e + \lambda_e) \]
A convex formulation of Nash equilibria

- stationarity

\[ \forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0 \]
\[ \forall p, \quad \sum_{e \in p} v_e - w = 0 \]

- complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)

for all paths \( p \), the cost on the path is

\[ \ell_p(\mu) = \sum_{e \in p} \ell(\phi_e) = \sum_{e \in p} (v_e + \lambda_e) = w + \sum_{e \in p} \lambda_e \]
A convex formulation of Nash equilibria

- Stationarity

\[ \forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0 \]
\[ \forall p, \quad \sum_{e \in p} v_e - w = 0 \]

- Complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)

For all paths \( p \), the cost on the path is

\[ \ell_p(\mu) = \sum_{e \in p} \ell(\phi_e) = \sum_{e \in p} (v_e + \lambda_e) = w + \sum_{e \in p} \lambda_e \]

- If \( \mu_p > 0 \), \( \ell_p(\mu) = w \)
- If \( \mu_p = 0 \), \( \ell_p(\mu) \geq w \)
A convex formulation of Nash equilibria

• stationarity

\[ \forall e, \quad \ell_e(\phi_e) - v_e - \lambda_e = 0 \]
\[ \forall p, \quad \sum_{e \in p} v_e - w = 0 \]

• complementary slackness: \( \lambda_e \phi_e = 0 \) for all \( e \)

for all paths \( p \), the cost on the path is

\[ \ell_p(\mu) = \sum_{e \in p} \ell(\phi_e) = \sum_{e \in p} (v_e + \lambda_e) = w + \sum_{e \in p} \lambda_e \]

• if \( \mu_p > 0 \), \( \ell_p(\mu) = w \)
• if \( \mu_p = 0 \), \( \ell_p(\mu) \geq w \)

This is the set of Nash equilibria