Robust Convergence of Distributed Routing with Heterogeneous Population Dynamics

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Outline

1. Introduction
   - Routing Game
   - Motivation
   - Existing results

2. Convergence
   - Model
   - Convergence of averages
   - Convergence using Stochastic Approximation
   - Convergence using Stochastic Mirror Descent (SMD)

3. Simulations
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3 Simulations
Routing game

- Directed graph \((V, E)\)
- Population \(k\): paths \(\mathcal{P}_k\)
  - Population distribution over paths \(x_{\mathcal{P}_k} \in \Delta_{\mathcal{P}_k}\)
  - Loss on path \(i\) of population \(k\): \(\ell^k_i(x)\)

Figure: Example network
Routing game

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Figure: Example network
Equilibrium

\(x^* = (x_{P_1}^*, \ldots, x_{P_k}^*)\) is an equilibrium if \(\forall k\),

\[\langle \ell_{P_k}(x^*), x_{P_k}^* \rangle \leq \langle \ell_{P_k}(x^*), x_{P_k} \rangle\]

Losses are minimal on the support of \(x_{P_k}^*\)
The routing game

One-shot routing game

- Well understood
- Useful for characterizing ‘steady-state’ behavior
  - Network performance (price of anarchy)
  - System optimal tolling

Why study dynamics?

- How do players arrive at equilibrium?
- How fast?
- Stability?
- Robustness (noisy measurements)?
## Applications

<table>
<thead>
<tr>
<th>Time scale</th>
<th>Transportation networks</th>
<th>Packet routing</th>
<th>Load balancing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day</td>
<td>Day</td>
<td>minute/second</td>
<td>minute/second</td>
</tr>
<tr>
<td>Measurements</td>
<td>Route delays</td>
<td>Route delays</td>
<td>Job completion</td>
</tr>
<tr>
<td>Distributed</td>
<td>Distributed</td>
<td>Distributed</td>
<td>Can be centralized</td>
</tr>
</tbody>
</table>
Convergence rate

How fast does the system reconverge to equilibrium?

- Catastrophic failure: Mississippi river bridge collapse (2005)
Convergence rate

How fast does the system reconverge to equilibrium?

- Road construction planning.
Convergence rate

How fast does the system reconverge to equilibrium?

- Adding a link to the network: construction of the Millau Viaduct (2004)
Convergence rate

How fast does the system reconverge to equilibrium?

- Tolling: Electronic Road Pricing (ERP) in Singapore.
Existing results

Continuous time:

- General case of potential games, under a positive correlation condition [8]
- Special case of routing games, under replicator dynamics [4]

Discrete time:

- General class of no-regret dynamics, limited convergence result [2]

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Figure: Example network
Online learning model

At iteration $t$
- Players of population $k$ choose routes. Distribution $x^{(t)}$.
- $\ell_{P_k}(x^{(t)})$ is revealed to players of population $k$.
- Players update their distribution.

$$x^{(t+1)}_{P_k} = u_k(x^{(t)}_{P_k}, \text{history})$$

Master problem

Define a class $C$ of algorithms (update rules) such that

$$u_k \in C \quad \forall k \Rightarrow x^{(t)} \rightarrow \mathcal{N}$$

Extension: Losses are noisy $\hat{\ell}_{P_k}(x^{(t)})$ with

$$\mathbb{E}[\hat{\ell}_{P_k}(x^{(t)})|x^{(t)}] = \ell_{P_k}(x^{(t)})$$
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$$\mathbb{E}[\hat{\ell}_{P_k}(x^{(t))}|x^{(t)}] = \ell_{P_k}(x^{(t)})$$
**Convex potential**

**Rosenthal potential**

\[
\begin{align*}
 f(x) & \text{ Convex} \\
 \nabla_{x \mathcal{P}_k} f(x) &= \ell_{\mathcal{P}_k}(x) \\
 \mathcal{N} &= \arg \min_{x \in \Delta \mathcal{P}_1 \times \cdots \times \Delta \mathcal{P}_K} f(x)
\end{align*}
\]

Optimality conditions:

\[
\langle \ell(x^*), x - x^* \rangle \geq 0 \quad \forall x \iff \forall k, \forall x_{\mathcal{P}_k}, \langle \ell_{\mathcal{P}_k}(x_{\mathcal{P}_k}^*), x_{\mathcal{P}_k} - x_{\mathcal{P}_k}^* \rangle \geq 0
\]

- Continuous time: \( f \) used as a Lyapunov function.
- Discrete time: regret.
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3 Simulations
**Regret**

Instantaneous regret:

\[ r^{(t)}(x) = \langle \ell(x^{(t)}), x^{(t)} - x \rangle \]

**Equilibrium**

\[ x^{(t)} \to \mathcal{N} \iff \lim_{t} \sup_{x} \sup_{t} r^{(t)}(x) \leq 0 \]
Regret

Average cumulative regret

\[ R^{(t)}(x) = \frac{1}{t} \sum_{\tau \leq t} r^{(\tau)}(x) \]

Equilibrium

\[ \bar{x}^{(t)} = \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)} \rightarrow \mathcal{N} \iff \limsup_{t} \sup_{x} R^{(t)}(x) \leq 0 \]

By convexity of \( f \),

\[ f \left( \frac{1}{t} \sum_{\tau \leq t} x^{(\tau)} \right) - f(x) \leq \frac{1}{t} \sum_{\tau \leq t} f(x^{(\tau)}) - f(x) \leq \frac{1}{t} \sum_{\tau \leq t} \left\langle \ell(x^{(t)}), x^{(t)} - x \right\rangle = R^{(t)}(x) \]
Regret

- Regret first defined by Hannan (1957) in the context of repeated games [5]
- Large classes of algorithms have “no regret” guarantees, e.g. [3]
- However, only guarantees convergence of $\bar{x}(t)$, not $x(t)$.
- Seek additional conditions to guarantee $x(t) \to N$.

Observation

If $f(x(t))$ is eventually monotone, then $f(x(t)) \to f^*$.


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*Contributions to the Theory of Games, 3:97–139, 1957*

Cambridge University Press, 2006
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Replicator dynamics

Replicator equation

\[ \forall p \in \mathcal{P}_k, \quad \frac{dx^k_p}{dt} = x^k_p \left( \langle \ell_{\mathcal{P}_k}(x), x_{\mathcal{P}_k} \rangle - \ell^k_p(x) \right) \quad (1) \]

Also in evolutionary game theory, Weibull [9].

Theorem: Fischer and Vöcking [4]

Every solution of the ODE (1) converges to the set of its stationary points.

Proof: \( f \) is a Lyapunov function.

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Replicator dynamics

Replicator equation

\[ \forall p \in \mathcal{P}_k, \quad \frac{dx_p^k}{dt} = x_p^k \left( \langle \ell_{\mathcal{P}_k}(x), x_{\mathcal{P}_k} \rangle - \ell_p^k(x) \right) \] (1)

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Proof: \( f \) is a Lyapunov function.


Approximate REPlicator update

Discretization of the continuous-time replicator dynamics

\[ x_p^{(t+1)} - x_p^{(t)} = \eta_t x_p^{(t)} \left( \left\langle \ell^k (x^{(t)}), x_{P_k}^{(t)} \right\rangle - \ell^k (x^{(t)}) \right) + \eta_t U_p^{(t+1)} \]

\((U^{(t)})_{t \geq 1}\) perturbations that satisfy for all \(T > 0\),

\[
\lim_{\tau_1 \to \infty} \max_{\tau_2: \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0
\]

Theorem Krichene et al. [6]

Under AREP updates, if $\eta_t \downarrow 0$ and $\sum \eta_t = \infty$, then

$$x^{(t)} \to \mathcal{N}$$

- Affine interpolation of $x^{(t)}$ is an asymptotic pseudo trajectory.

- $f$ is a Lyapunov function for Nash equilibria in the continuous system.

Convergence to Nash equilibria

Consequence:

- Convergence of any sublinear regret AREP strategy.
- However
  - No convergence rates.
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3 Simulations
minimize \[ f(x) \] convex function
subject to \[ x \in \mathcal{X} \subset \mathbb{R}^d \] convex, compact set

At iteration \( t \)
- have a stochastic subgradient \( \hat{g}^{(t)} \)
- \( \hat{g}^{(t)} \) unbiased: \( \mathbb{E} \left[ \hat{g}^{(t)} | \mathcal{F}_t \right] = g^{(t)} \in \partial f(x^{(t)}) \) (\( \mathcal{F}_t \) natural filtration of \( (x^{(t)}) \))

**Algorithm 1** SMD Method with learning rates \( (\eta_t) \)

1. **for** \( t \in \mathbb{N} \) **do**
2. \( \hat{g}^{(t)} \in \mathcal{F}_{t+1} \)
3. \( x^{(t+1)} = \arg \min_{x \in \mathcal{X}} \langle \hat{g}^{(t)}, x \rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)}) \)
4. **end for**
Stochastic Mirror Descent

minimize \( f(x) \) \hspace{2cm} \text{convex function}

subject to \( x \in \mathcal{X} \subset \mathbb{R}^d \) \hspace{2cm} \text{convex, compact set}

At iteration \( t \)

- have a \textbf{stochastic subgradient} \( \hat{g}(t) \)
- \( \hat{g}(t) \) unbiased: \( \mathbb{E} \left[ \hat{g}(t) | \mathcal{F}_t \right] = g(t) \in \partial f(x(t)) \) (\( \mathcal{F}_t \) natural filtration of \( (x(t)) \))

\textbf{Algorithm 2} SMD Method with learning rates \( (\eta_t) \)

1: \textbf{for} \( t \in \mathbb{N} \) \textbf{do}
2: \( \hat{g}(t) \in \mathcal{F}_{t+1} \)
3: \( x(t+1) = \arg \min_{x \in \mathcal{X}} \left\langle \hat{g}(t), x \right\rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x(t)) \)
4: \textbf{end for}
minimize \( f(x) \) \( \quad \) convex function

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At iteration \( t \)

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Algorithm 3 SMD Method with learning rates \( (\eta_t) \)

1: \textbf{for} \( t \in \mathbb{N} \) \textbf{do}
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3: \( x^{(t+1)} = \arg \min_{x \in \mathcal{X}} \left\langle \hat{g}^{(t)}, x \right\rangle + \frac{1}{\eta_t} D_{\psi_t}(x, x^{(t)}) \)
4: \textbf{end for}
Mirror Descent

\[ f(x(t)) \]

\[ f(x(t + 1)) \]

\[ f(x) \]

\[ f(x(t)) + \langle g(t), x - x(t) \rangle \]

\[ f(x(t)) + \langle g(t), x - x(t) \rangle + \frac{1}{\eta_t} D_\psi(x, x(t)) \]

**Figure:** Mirror Descent iteration
Bregman Divergence

Strongly convex function $\psi$

$$D_\psi(x, y) = \psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle$$

- $\psi(x) = \frac{1}{2} \|x\|_2^2$, $D_\psi(x, y) = \frac{1}{2} \|x - y\|_2^2$ (projected gradient)
- $\psi(x) = -H(x) = \sum_{i=1}^{d} x_i \ln x_i$, $D_\psi(x, y) = D_{KL}(x, y) = \sum_{i=1}^{d} x_i \ln \frac{x_i}{y_i}$. 

Figure: KL divergence
Bregman Divergence

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**Figure:** KL divergence
## Convergence rates

<table>
<thead>
<tr>
<th>$f$</th>
<th>$\eta_t$</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weakly convex</td>
<td>$\frac{c}{\sqrt{t}}$, $\alpha \in (0, 1)$</td>
<td>$\frac{1}{t} \sum_{\tau=1}^{t} \mathbb{E} [f(x^{(\tau)})] - f^* = O\left(\frac{1}{\sqrt{t}}\right)$ [7]</td>
</tr>
<tr>
<td>&amp;</td>
<td>$\mathbb{E} [f(x^{(t)})] - f^* = O\left(\frac{\log t}{t^{\min(\alpha, 1-\alpha)}}\right)$</td>
<td></td>
</tr>
<tr>
<td>Strongly convex</td>
<td>$\eta_t \to 0$, $\sum \eta_t = \infty$</td>
<td>$\mathbb{E} [D_\psi(x^*, x^{(t)})] = O\left(\eta T + e^{-\sum_{\tau=1}^{T} \eta_{\tau}}\right)$</td>
</tr>
<tr>
<td>&amp;</td>
<td>$\mathbb{E} [D_\psi(x^*, x^{(t)})] = O(t^{-\alpha})$</td>
<td></td>
</tr>
</tbody>
</table>

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Regret bound for SMD

Main ingredient:

**Proposition**

Assume $D_\psi$ bounded by $D$ and $\mathbb{E} \|\hat{g}\|^2 \leq G$. SMD method with $(\eta_t)$. $\forall t_2 > t_1 \geq 0$ and $\mathcal{F}_{t_1}$-measurable $x$,

$$\sum_{\tau=t_1}^{t_2} \mathbb{E} \left[ \langle g^{(\tau)}, x^{(\tau)} - x \rangle \right] \leq \frac{\mathbb{E} \left[ D_\psi(x, x^{(t_1)}) \right]}{\eta_{t_1}} + D \left( \frac{1}{\eta_{t_2}} - \frac{1}{\eta_{t_1}} \right) + \frac{G}{2 \ell_\psi} \sum_{\tau=t_1}^{t_2} \eta_\tau \quad (2)$$
Distributed SMD with heterogeneous agents

- $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_K$
- Agent $k$ updates $x^k \in \mathcal{X}_k$
- $D_{\psi_k}$ and $\eta^k_t$ depends on $k$

**Algorithm 4** DSMD Method with learning rates ($\eta^k_t$) and divergences $D_{\psi_k}$

1: for $t \in \mathbb{N}$ do
2: \[ \hat{g}^{(t)} \in \mathcal{F}_{t+1} \]
3: \[ x_{p_k}^{(t+1)} = \arg \min_{x_{p_k} \in \mathcal{X}_k} \left\langle \hat{g}^{(t)}_{p_k}, x_{p_k} - x_{p_k}^{(t)} \right\rangle + \frac{1}{\eta^k_t} D_{\psi_k}(x_{p_k}, x_{p_k}^{(t)}) \]
4: end for
Convergence in DSMD

**Theorem**

Distributed SMD such that $\eta^k_t = \frac{\theta_k}{t^\alpha_k}$ with $\alpha_k \in (0, 1)$. Then

$$\mathbb{E} \left[ f(x^{(t)}) \right] - f(x^*) \leq \left( 1 + \sum_{\tau=1}^{t} \frac{1}{\tau} \right) \sum_{k=1}^{K} \left( \frac{1}{t^{1-\alpha_k}} \frac{D}{\theta_k} + \frac{\theta_k G}{2\ell_\psi (1 - \alpha_k)} \frac{1}{t^{\alpha_k}} \right)$$

This is $O\left( \frac{\log t}{t \min(\min_k \alpha_k, 1 - \max_k \alpha_k)} \right)$. 
Routing game with heterogeneous populations

Under unbiased noisy losses, with heterogeneous update rules with \( \eta_t^k = \theta_k t^{-\alpha_k} \)

\[
\mathbb{E} \left[ f(x^{(t)}) \right] - f^* = O \left( t^{-\min(\min_k \alpha_k, 1-\max_k \alpha_k)} \right)
\]

where \( f \) is the Rosenthal potential function
Simulations

- Centered Gaussian noise on edges.
- Population 1: Hedge with $\eta_t^1 = t^{-0.1}$
- Population 2: Hedge with $\eta_t^2 = \frac{1}{2} t^{-0.5}$

Figure: Example network
One realization

Figure: Population distributions and noisy path losses
One realization

**Figure:** Distance to equilibrium
In Expectation

**Figure**: Population distributions and noisy path losses
One realization

Figure: Expected distance to equilibrium
Summary and Extensions

- Class of algorithms which are guaranteed to converge, convergence rates.
- Robust to unbiased perturbation, e.g. when losses are not known but estimated.
- Provides a model of population dynamics for optimal control problems, e.g. tolling.

Extensions:
- Varying masses.
- Adapt to other problems, such as network consensus.
Thank you.
References I


