Stability of Nash equilibria in Congestion Games under Replicator Dynamics

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Outline

1 Introduction

2 Stability of replicator dynamics

3 Simulations

4 Extensions
Class of Congestion Games, under Replicator Dynamics.

**Congestion games**

- Population of players (non atomic), with action set $\mathcal{A}$
- Mass distribution $x \in \Delta^\mathcal{A}$ determines losses $\ell(x) \in \mathbb{R}_+^\mathcal{A}$
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Replicator dynamics
Players’ mass distribution obeys ODE

$$\dot{x}_a(t) = x_a(t) (\langle \ell(x(t)), x(t) \rangle - \ell_a(x(t)))$$

$x_a(0)$ given

Our goal: study stability of equilibria.
Example: routing game

- Population: packet routers / drivers.
- Action set $A$: paths from 0 to 1
- Mass distribution $x$ determines, edge loads $Mx$, edge costs $c(Mx)$.
- Loss function:
  \[ \ell_a(x) = M_a^T c(Mx) \]

$M \in \mathbb{R}^{E \times A}$: path-edge incidence matrix of the graph.
Nash equilibria

A mass distribution $x^*$ is a Nash equilibrium if $\forall x \in \Delta^A$,

$$\langle \ell(x^*), x^* - x \rangle \leq 0$$
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Replicator dynamics

For $a \in A$

$$\dot{x}_a(t) = x_a(t) \left( \langle \ell(x(t)), x(t) \rangle - \ell_a(x(t)) \right)$$

$x_a(0)$ given

Model:

- Randomly match players
- Compare actions $a$ and $a'$
- Player with higher loss replicates action of other player with probability $\ell_a - \ell_{a'}$. 
Stationary points

Stationary points $\mathcal{R}$

$$\dot{x} = 0 \iff \forall a, \ x_a (\langle \ell(x), x \rangle - \ell_a(x)) = 0$$

$$\iff \ell_a(x) \text{ constant on the support of } x$$

Nash equilibria $\mathcal{N}$

$$x \in \mathcal{N} \iff \langle \ell(x), x \rangle \leq \langle \ell(x), y \rangle \ \forall y$$

$$\iff \ell_a(x) \text{ constant and minimal on the support of } x$$

**Theorem [1]**

Solution trajectories converge to $\mathcal{R}$.

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**Theorem [1]**

Solution trajectories converge to $\mathcal{R}$.

- Stability of equilibria?

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Stability of replicator dynamics

\[ \dot{x}_a(t) = F_a(x(t)) \]
\[ = x_a(t) (\langle \ell(x(t)), x(t) \rangle - \ell_a(x(t))) \]

Instability of non-Nash equilibria

If \( x \in \mathcal{R} \setminus \mathcal{N} \), then \( x \) is unstable.

proof:

- \( \mathcal{H} = \sum_{a \in A} x_a = 0 \), then \( F(\Delta) \subset \mathcal{H} \).
- Derive Jacobian \( \tilde{\nabla}F \) of \( F \) restricted to \( \mathcal{H} \).
- If \( A^* \) is the support of \( x \) and \( A^\circ \) its complement, then
  \[ \text{Sp} (\tilde{\nabla}F(x)) \supset \{ \langle \ell(x), x \rangle - \ell_a(x) \}_{a \in A^\circ} \]
Exponential stability

If $M$ is injective, then

$$x \in \mathcal{N} \iff x \text{ is locally exponentially stable}$$

proof:
If $M$ is injective, can show that $\tilde{\nabla} F(x)$ is negative definite.
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**Figure**: Routing game with $|A| = 4$. 
Simulations

\[ \mathcal{N} = \{x : x_1 = .757, x_2 = 0, x_3 + x_4 = .2426\} \]

**Figure:** Masse trajectories \( x_a(t), a \in \{p_1, p_2, p_3, p_4\} \)
Simulation

Figure: Losses $\ell_a(x(t))$
Simulation

\[ \dot{x}_a(t) = x_a(t) \left( \langle \ell(x(t)), x(t) \rangle - \ell_a(x(t)) \right) \]

\( \mathcal{N} \): Nash equilibria

\( \mathcal{R} \): Stationary points

**Figure**: Mass trajectories in the simplex: convergence to restricted equilibria
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Extension to discrete time

Convergence of discrete time dynamics: approximate replicator dynamics [2]

\[ x_a^{(t+1)} - x_a^{(t)} = \eta_t x_a^{(t)} \left( \langle \ell(x^{(t)}), x^{(t)} \rangle - \ell_a(x^{(t)}) \right) + \eta_t U_a^{(t)} \]

- \( U_a^{(t)} \): stochastic perturbation term.
- \( \eta_t \): discretization time steps, \( \sum_t \eta_t = \infty \).

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Convergence of approximate replicator dynamics

\( x^{(t)} \to N \) almost surely, under mild conditions on \( U^{(t)}, \eta_t \).

E.g. \( \sup_t \mathbb{E} \| U^{(t)} \|^q < \infty \) and \( \sum_t \eta_t^{1+\frac{q}{2}} < \infty \)

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Congestion games, under replicator dynamics

- Trajectories converge to stationary points \( \mathcal{R} \)
- Non-Nash equilibria \( \mathcal{R} \setminus \mathcal{N} \) are unstable
- If injective incidence matrix: \( \mathcal{N} \Leftrightarrow \) loc. exp. stable equilibrium
- Extension to discrete-time dynamics

Thank you!

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