SPECTRAL GRAPH THEORY

Matrix representation of a graph:

- Adjacency matrix of graph $G=(V,E)$

$A \in \{0,1\}^{V \times V}$.

$$\begin{array}{cc} & \quad v \\ v \; u & \begin{pmatrix} 0 & & 1 & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

$A(u,v) = 1$ if $\{u,v\} \in E$

0 otherwise

Can matrix-theoretic notions help shed light on the graph & its properties/structure?

Representations can be a powerful lens/tool on an object, especially from a computational point of view

(Recall FFT algo: coefficient representation $\Longleftrightarrow$ evaluation rep)

Spectral graph theory: Eigenvalues of matrix encode valuable info about the graph.

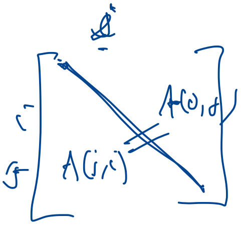
- Useful for structural analysis
- Algorithmically powerful, since spectra of matrices can be computed efficiently

(Full course on Spectral Graph Theory offered regularly by Prof. Gary Miller including this semester)

Interlude on eigenvalues

A = real symmetric $n \times n$ matrix

$$A(i,j) = A(j,i)$$



(Note: Adj matrix of an undirected gh is symmetric)

Defn $\lambda \in \mathbb{R}$ is said to be an eigenvalue of a $n \times n$ matrix M if $\exists \vec{x} \in \mathbb{R}^n$,

s.t. $M\vec{x} = \lambda \vec{x}$

Such an $\vec{x}$ is called an eigenvector

$$\begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix}\begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \lambda\vec{x} \\ \\ \end{bmatrix}$$

Standard Fact: Let A be an $n \times n$ real symmetric matrix

1. Then A has n real eigenvalues (including repetitions). $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$
2. There exist n eigenvectors $v_1, v_2, \dots, v_n$ s.t. $Av_i = \lambda_i v_i$ for $i = 1, 2, \dots n$, and
   - (i) The vectors $\{v_i\}$ span $\mathbb{R}^n$
   - (ii) Eigenvectors corresponding to diff eigenvalues are orthogonal.

$\boxed{\text{Pf of (ii)}}$ $\lambda_i \neq \lambda_j$

$Av_i = \lambda_i v_i$

$Av_j = \lambda_j v_j$

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \quad x, y \in \mathbb{R}^n$$

$$v_j \cdot Av_i = v_j \cdot (\lambda_i v_i) = \lambda_i (v_j \cdot v_i)$$

$$v_j \cdot Av_i = v_j^T A v_i = (Av_j)^T v_i$$

$$= \lambda_j (v_j^T v_i) = \lambda_j (v_j \cdot v_i)$$

$$\Rightarrow \lambda_i (v_j \cdot v_i) = \lambda_j (v_j \cdot v_i)$$

$\rightarrow$ $v_j \cdot v_i = 0$ $\Rightarrow$ v_j & v_i are orthogonal

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Cor: Every vector $v \in \mathbb{R}^n$ can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

(as a linear combination of v_i's).

Lemma: Matrix A has eigenvalues $\lambda_1 \geq \lambda_2 \dots \geq \lambda_n$

Then $\lambda_1 = \max\limits_{\substack{x \neq 0 \\ x \in \mathbb{R}^n}} \dfrac{x^T A x}{x^T x}$ (Rayleigh coefficient)

$x^T A x = \sum A(u,v) x(u) x(v)$

Pf. Take $x = v_1$, $v_1^T A v_1 = \lambda_1 (v_1^T v_1)$

$$v_1^T (\lambda_1 v_1) = \lambda_1 (v_1^T v_1)$$

To prove $\max \leq \lambda_1$, take any x,

$$x = \sum \alpha_i v_i \qquad \frac{x^T A x}{x^T x} = \frac{\sum \alpha_i^2 \lambda_i}{\sum \alpha_i^2} \leq \lambda_1.$$

$\boxed{}$

OK back to graph theory!

Eigenvalues of adj matrix A $\Longrightarrow$ info about graph G properties

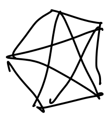
pfs skipped

Thm: G is connected $\Rightarrow$ $\lambda_2 < \lambda_1$

Thm: Let G is connected. Then $\lambda_n = -\lambda_1$ iff G is bipartite.

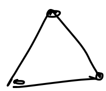
Examples of graph spectra:

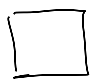
K_n (complete graph on n vertices) : $\lambda_1 = n-1$, $\lambda_2 = \lambda_3 = \cdots = \lambda_n = -1$

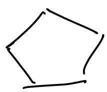


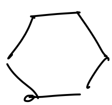
K_5 $(4, -1, -1, -1, -1)$

Cycles:

C_3  $2, -1, -1$ $\left(\begin{array}{l}\lambda_1 = 2\\ \lambda_2 = -1\\ \lambda_3 = -1\end{array}\right)$

C_4  $2, 0, 0, -2$

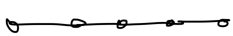
C_5  $2, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}$

C_6  $2, 1, 1, -1, -1, -2$

Paths

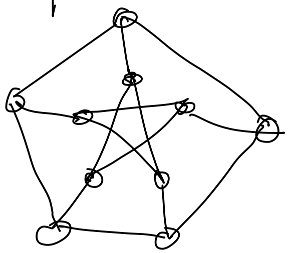
P_2  $1, -1$ $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

P_3  $\sqrt{2}, 0, -\sqrt{2}$

P_5  $\sqrt{3}, 1, 0, -1, -\sqrt{3}$

d-regular graph $:=$ a graph where all vertex degrees $= d$.

All cycles are 2-regular



3-regular gh (Peterson graph)

<u>Fact</u>: For a d-regular Graph, largest eigenvalue of its adj. matrix equals d.

<u>Pf</u>:

$$A = \begin{array}{c} \\ u \\ \\ \end{array}\begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix} \to \text{each row has exactly } d \text{ 1's}$$

$$\begin{pmatrix} \cancel{\;1\;1\;A\;1\;} \\ \\ \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix} = d\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\Rightarrow d$ is an eigenvalue

$(\lambda_1 \geq d)$

Let x be eigenvector with eigenvalue λ_1

Let u be st $x(u)$ max. coordinate of x

$x = (-1\ 2\ 1\ 4)$

$$\begin{array}{c} \\ u \to \\ \\ \end{array}\begin{pmatrix} \\ 1 \quad 1 \quad 1 \\ A \\ \\ \end{pmatrix}\begin{pmatrix} \\ x(u) \\ \\ \\ \end{pmatrix} = \begin{pmatrix} \\ \lambda_1 x(u) \\ \\ \\ \end{pmatrix} \Longleftarrow (Ax)(u)$$

$$= \sum_{(u,v)\in E} x(v)$$

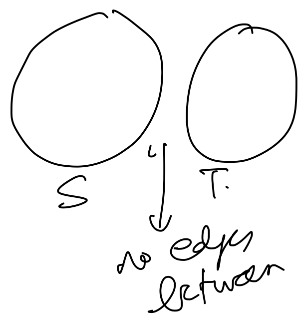
$$\leq d\, x(u).$$

$\therefore \lambda_1 \leq d.$

<u>Theorem</u>: For a d-regular graph G whose adj. matrix has eigenvals $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$,

G is connected if and only if $\lambda_2 < d$.

<u>Proof</u>: ① G is not connected $\Rightarrow$ $\lambda_2 = d$.



$x(u) = 1$ if $u \in S$
$\quad\;\; 0$ otherwise

$y(u) = 1$ if $u \in T$
$\quad\;\; 0$ otherwise.

Both x & y are eigenvectors with eigenvalue d. Plus they are lin. indep (in fact orthogonal)

so there are at least 2 eigenvalues equal to d. $\Rightarrow$ $\lambda_2 \geq d$

$\Rightarrow \lambda_2 = d$

② $\lambda_2 = d \Rightarrow G$ is disconnected

$\vec{1}$ is an eigenvector with eig val $= d$.
(call 1's vector)

$\lambda_2 = d \Rightarrow \exists\, \vec{x} \in \mathbb{R}^n$, $\vec{x} \cdot \vec{1} = 0$
$(\vec{x} \neq \vec{0})$ $\quad (\Rightarrow \sum_{u \in V} x(u) = 0)$

s.t. $A\vec{x} = d\vec{x}$ $\quad \vec{x} = \langle x(u) \rangle_{u \in V}$

Suppose G is connected.

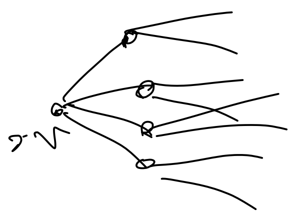
We'll prove all entries of $\vec{x}$ have to be equal, which contradicts $\sum_{u \in V} x(u) = 0$

Let $x(v)$ be max. value in vector $\vec{x}$.

$$\left(x(v) = \max_{u \in V} x(u) \right).$$

$$d \cdot x(v) = (A\vec{x})(v) = \sum_{w \sim v} x(w) \leq d \cdot x(v)$$

Only way equality holds is if $x(w) = x(v)$ for all nbrs w of v



Continuing this argument, because we assumed G to be connected, we eventually reach every vertex $u \in V$, and show $x(u) = x(v)$. $\square$

More generally, There are t connected components
(in a regular graph) $\iff$ t eigenvalues equal to d.

There are easier ways to check connectivity of course, but this spectral perspective allows one to define more quantitative aspects of connectivity.

λ_2 is much smaller than d $\overset{?}{\Longrightarrow}$ G is very well connected?

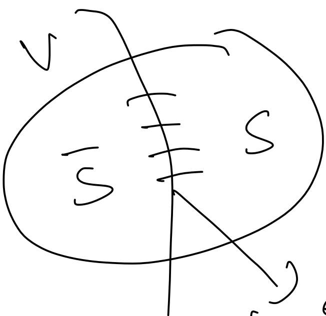
No bottlenecks

is an (expanding graph)

very few edges cross two big halves of graph

"sparse cut"

If $\lambda_2 \approx d$ $\overset{?}{\Longrightarrow}$ Is there a sparse cut?



few edges cross

Yes! Further such a cut can be found using the second eigenvector v_2 corresponding to λ_2.

"Spectral partitioning algorithm"

- very popular heuristic
- very useful in divide & conquer algos.