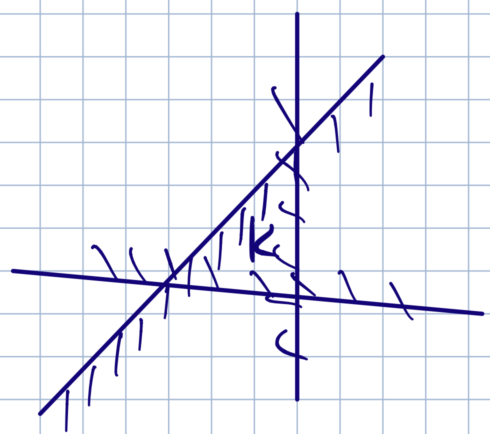
Linear Programming

LP program :

$x = (x_1, \dots x_n) \in \mathbb{R}^n$

$K \subseteq \mathbb{R}^n$ - region of all x satisfying this system

$$K \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m \end{cases}$$



Inequality defines subspace
Equality is a hyperplane

Look at inters of half-spaces

K - polytope

Is K emty?

$a_{11}x_1 + \dots + a_{1n}x_n = a_1^T x = a_1 \cdot x$ $\quad a_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix}$

$$K = \begin{cases} a_1 \cdot x \geq b_1 \\ \vdots \\ a_n \cdot x \geq b_n \end{cases} \qquad A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} \text{ — rows}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$K = \{ Ax \geq b \}$ — LP

Instead of "$\geq$"
Can use "$\leq$" : $a_i \cdot x \leq b_i \iff -a_i \cdot x \geq -b_i$

"$=$" $\quad a_i \cdot x = b_i \iff \begin{cases} a_i \cdot x \geq b_i \\ a_i \cdot x \leq b_i \end{cases}$

Cannot use "$\geq$"

LP questions:

1) Decision problem: is $K \neq \emptyset$?
2) Search problem: find $x \in K$ (if $K \neq \emptyset$)
3) Optimization: $\max c \cdot x$ oder $x \in K$

$\boxed{\text{LP Decision problem}}$

a) if $K \neq \emptyset$: output $x \in K$
b) if $K = \emptyset$: output "proof"

$\boxed{\text{Theorem LP problem can be solved in polynomial time}}$

$\boxed{\text{Khachiyan '79}}$ (Ellipsoid method)

what we consider a proof?

$$\lambda_1 \cdot (a_1 \cdot x \geq b_1)$$
$$+ \lambda_2 \cdot (a_2 \cdot x \geq b_2)$$
$$+ \quad \vdots$$
$$+ \lambda_m (a_m \cdot x \geq b_m)$$

Sum up inequalities scaled by $\lambda_1, \lambda_2 \dots, \lambda_m \geq 0$

$$d \cdot x \geq f$$

Suppose $d = (0, 0, \dots, 0)$ $f = 1$

$$\underline{0 \cdot x_1 + 0 \cdot x_2 \dots + 0 \cdot x_n \geq 1}$$

$\underline{0 \geq 1}$ - contradiction

So if such $\underbrace{\lambda_1, \dots, \lambda_m}_{\text{"proof"}}$ exist $\Rightarrow K = \emptyset$

$\boxed{\text{Theorem}}$ If $K = \emptyset \Rightarrow$ Such $\lambda_1, \dots, \lambda_m \geq 0$ exist.

Farkas Lemma

(LP Duality) $(\Leftarrow)$

So $\lambda_1, \dots \lambda_m$ is a proof of $K = \emptyset$

$$\lambda_1 (a_1 \cdot x) + \lambda_2 (a_2 \cdot x) + \dots + \lambda_m (a_m \cdot x) \geq \lambda_1 b_1 + \lambda_2 b_2 \dots + \lambda_m b_m$$

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

If this inequality is equiv to $0 \geq 1$

$$\boxed{\lambda^T A = 0 \qquad \lambda^T b = 1}$$

Coef before x_1 in sum of ineq.

$$\lambda_1 \cdot a_{11} + \lambda_2 \cdot a_{21} + \lambda_3 \cdot a_{31} \dots + \lambda_m \cdot a_{m1} = 0$$

$$\lambda_2 \cdot a_{12} + \dots + \lambda_m \cdot a_{m2} = 0$$

$$\vdots$$

$$\lambda_m \cdot a_{1m} + \dots + \lambda_m \cdot a_{mn} = 0$$

$$(\lambda_1 \ \lambda_2 \ \dots \ \lambda_m)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & \vdots \\ \vdots & \vdots \\ a_{m1} & \end{pmatrix} = 0$$

Farkas Lemma:

$$K_d = \left\{ \begin{array}{l} \lambda^T A = 0 \\ \lambda^T b = 1 \\ \lambda \geq 0 \end{array} \right\}$$

$$K = \emptyset \iff K_d \neq \emptyset$$

LP algorithm output
a) $x \in \mathbb{Q}^n$
b) $\lambda \in \mathbb{Q}^m$

$\langle X \rangle$ - bit-size of object X

For LP K, has bit-size $\langle K \rangle$

Theorem
a) if $K \neq \emptyset \Rightarrow \exists x \in K,\ x \in \mathbb{Q}^n$,
$\langle x \rangle = \mathrm{poly}(\langle K \rangle)$

b) if $K = \emptyset \Rightarrow \exists \lambda_1 \dots \lambda_m \in \mathbb{Q}$
$\langle \lambda \rangle = \mathrm{poly}(\langle K \rangle)$

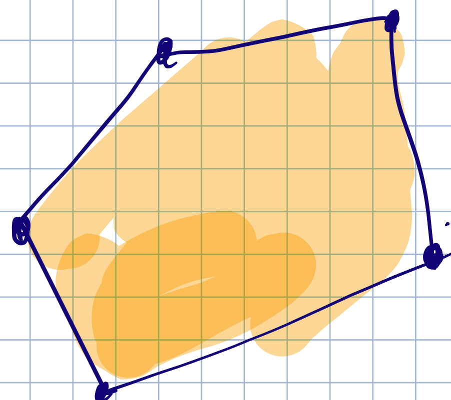
Suppose we proved a)
For b), if $K = \emptyset \Rightarrow K_d \neq \emptyset$
$\langle K_d \rangle \simeq \langle K \rangle$
by part a) $\Rightarrow$

$\exists \lambda \in K_d$ s.t. $\langle \lambda \rangle = \text{poly}(\langle K_d \rangle)$
this proves b)

$\boxed{\text{Pf idea of part a)}}$

Claim $K \neq \emptyset$ "$\Rightarrow$" $\exists$ feasible vertex of K

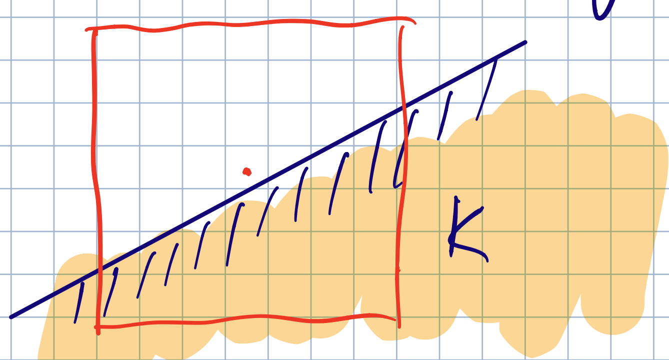
(*)



vertex is unique solution to
n linearly independent equalities from K
(which is in K)

Have Gaussian Elimination to solve
system of n equalities
works in poly-time. (non-trivial)
$\Rightarrow$ Solution is poly-size w.r.t input
$\Rightarrow$ vertex has size poly($\langle K \rangle$)

(*) not always
Work around:
Add $-B \leq x_i \leq B$



If B is large enough then this doesn't change answer to LP (if K is $\emptyset$?)

$K' = K \cup \{-B \le x_i \le B\}$ $\qquad K' = \emptyset \Leftrightarrow K = \emptyset$

$\langle B \rangle = \text{poly}(\langle K \rangle)$

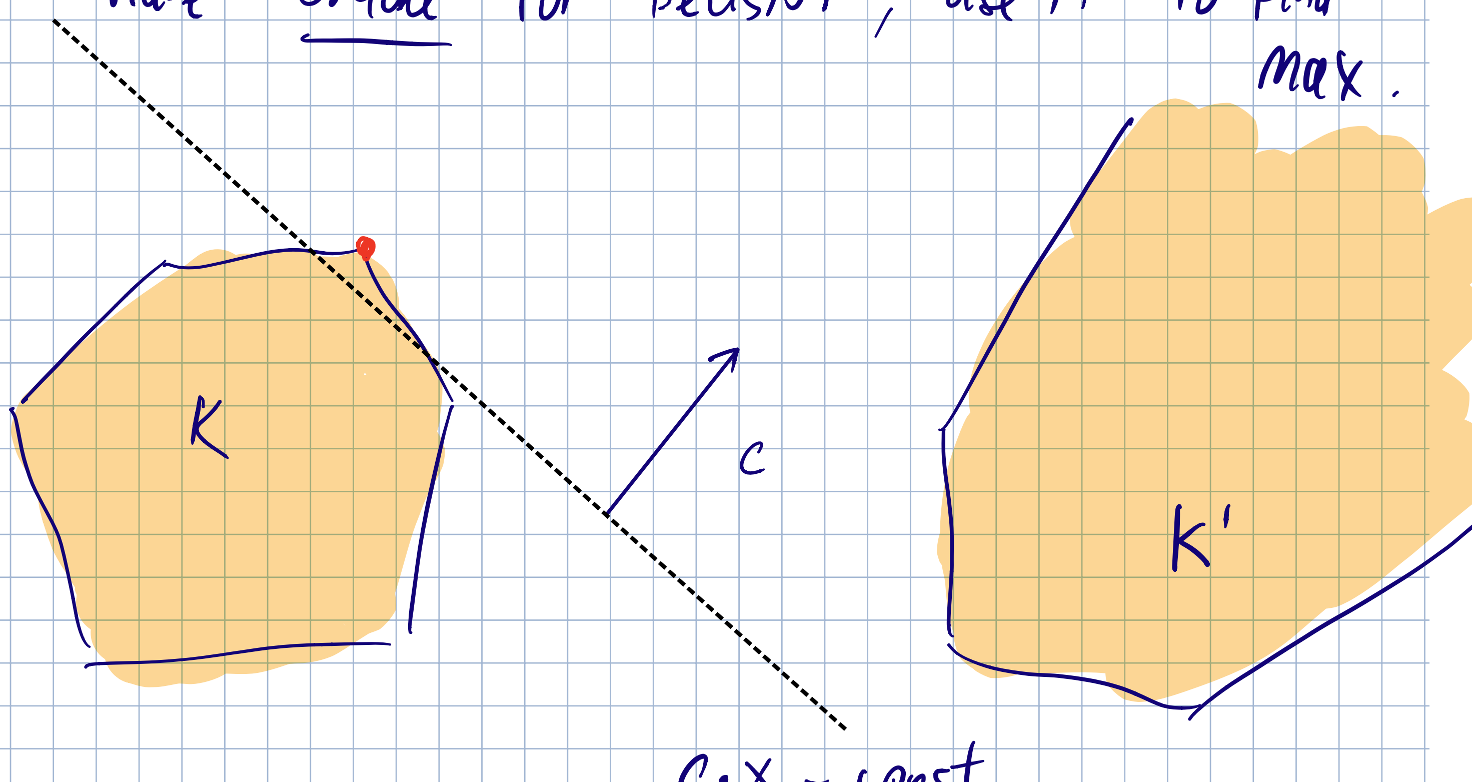
LP Optimization problem

max $c \cdot x$

over $x \in K$

How to reduce Optimization to Decision

Have <u>Oracle</u> for Decision, use it to find max.



<u>Claim</u> If $K \neq \emptyset \Rightarrow c \cdot x$ is maximized at a vertex of K

unless $\max = \infty$

$\max := \max_{x \in K} c \cdot x$

How can detect if max is ∞?

(a) max is ∞

(b) max is not ∞

is achieved at some vertex x^* of K

↳ has size $\mathrm{poly}(\langle K \rangle)$

$\Rightarrow c \cdot x^*$ has size $\mathrm{poly}(\langle K \rangle)$

$\Rightarrow$ Big enough F s.t. $c \cdot x^* \le F$ $\forall x^*$ vertex of K

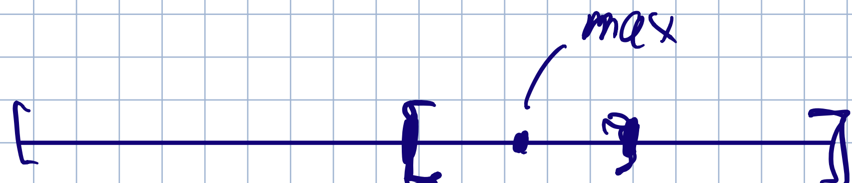
$\Rightarrow \max \le F$

If query the Oracle $K \cup \{c \cdot x \ge F+1\}$ determine which of (a) or (b) holds.

$-F \le \max \le F$

we can ask for any t : is $\max \ge t$?

(by asking Oracle $K \cup \{c \cdot x \ge t\}$)



Binary search: in 1 iteration we halve length of the interval

Start from $2F$

in k iterations: $\underline{L_k \le \max \le U_k}$

$$\varepsilon = (U_k - L_k) = 2F \cdot 2^{-k}$$

take $k = \text{poly}(F, \log \frac{1}{\varepsilon})$

$\text{poly}(\langle k \rangle, \log \frac{1}{\varepsilon})$

brings us ε-close to max.

This is poly-time algorithm for

LP Optimization.

(using Oracle for Decision problem)