Problem 1

1) \[ E = (+1) \Pr(\text{measuring } 10) + (-1) \Pr(\text{measuring } 11) \]
   \[ = |\langle 0 | \psi \rangle|^2 - |\langle 11 | \psi \rangle|^2 \]
   \[ = \langle \psi | 0 \rangle \langle 0 | \psi \rangle - \langle \psi | 1 \times 1 \rangle \langle 1 | \psi \rangle \]
   \[ = \langle \psi | (10 \times 0 - 11 \times 1) | 1 \rangle \]
   \[ = \langle \psi | 0 | \psi \rangle. \]

2) Since \( X = |+\rangle \langle +| - |\rangle \langle -| \), by analogy, measure in \( |+\rangle, |\rangle \) basis and associate \( |+\rangle \) with +1 and \( |\rangle \) with -1.
Problem 2

In the first interpretation of CHSH, on input $x, y \in \{0, 1\}$ our score was $E(u_x \oplus v_y \oplus xy)$. To switch interpretation, we can scale our score to $E(2(u_x \oplus v_y + xy) - 1)$ changing the game. Notice $2(a) - 1 = -(-1)^a$ for $a \in \{0, 1\}$.

So it is equivalent to our score being

$$E(-(-1)^{u_x \oplus v_y \oplus xy}).$$

Let's redefine $u_x, v_y \in \{ \pm 1 \}$ by $u_x = (-1)^{u_x}$, $v_y = (-1)^{v_y}$.

Expanding $E$ over $Pr(x, y) = \frac{1}{4}$ and scaling by 4 our score is equivalent to

$$\sum_{xy} E(-(-1)^{xy} u_x v_y) = E(u_0 v_0) + E(u_1 v_0) + E(u_0 v_1) - E(u_1 v_1).$$
Problem 2 (cont.)

Our transformation of the score was \( 4(2\cdot - 1) = 8\cdot - 4 \).

So the quantum score on the original CHSH is now

\[
8 \cos^2 \frac{\pi}{8} - 4 = 8 \cos^2 \frac{\pi}{8} - 4 \cos^2 \frac{\pi}{8} - 4 \sin^2 \frac{\pi}{8} \\
= 4 \left( \cos^2 \frac{\pi}{8} - \sin^2 \frac{\pi}{8} \right) = 2\sqrt{2}.
\]

In any deterministic strategy, either \( \nu_0 + \nu_1 = 0 \) or \( \nu_0 - \nu_1 = 0 \). Therefore

\[
\nu_0(\nu_0 + \nu_1) + \nu_1(\nu_0 - \nu_1) \leq 2.
\]

A randomized strategy is a linear combination of deterministic strategies.
Problem 3

1. Let $a_x, b_x, c_x, a_y, b_y, c_y$ be the deterministic answers by Alice, Bob and Charlie for inputs $x, y$. Then in order to satisfy w.p. 1,

$$a_x b_x c_x = 1$$
$$a_y b_y c_y = -1$$
$$a_x b_x c_y = -1$$
$$a_x b_y c_y = -1$$

Multiplying all 4 equations together

$$a_x^2 a_y^2 b_x^2 b_y^2 c_x^2 c_y^2 = -1 \quad \leftarrow \text{impossible.}$$

For $\frac{3}{4}$ strategy, everyone always answers $-1$. 
2. wlog assume $a_x, b_x, c_x, a_y, b_y, c_y$ are now $\text{Reals } \mathbb{R} \rightarrow \{\pm 1\}$ for some set $\mathcal{R}$ and assume they show a random sample $\sigma \in \mathcal{R}$. To win with $\Pr(\sigma) = 1$, $\forall \sigma \in \mathcal{R}$ with $\Pr(\sigma) > 0$, $a_x(\sigma), b_x(\sigma), \ldots, c_y(\sigma)$ must satisfy the equations on the previous page which is impossible. Therefore, randomness does not yield $\Pr(\sigma)$ success.

3. Since the eigenvalues of a tensor product are the products of the eigenvalues of the terms, 

$$\langle X \otimes X \otimes X \rangle_\gamma = \Pr(\text{win}) - \Pr(\text{lose}) = 2 \Pr(\text{win}) - 1$$

since we win if the product of the answers is 1. Suffices then to verify $\langle X \otimes X \otimes X \rangle_\gamma = 1$.

4. Analogously, check that $\langle X \otimes Y \otimes Y \rangle_\gamma = -1$.

5. Let $\Pi_\sigma : |a_1, a_2, a_3 \rangle \rightarrow |a_\sigma(1), a_\sigma(2), a_\sigma(3) \rangle$. Easy to check $\Pi_\sigma(\mathbb{R}) = \mathbb{R}$.

Notice that $Y \otimes X \otimes Y = \Pi_{(12)} X \otimes Y \otimes Y \Pi_{(12)}$. Therefore,

$$\langle Y \otimes X \otimes Y \rangle_\gamma = \langle \gamma | \Pi_{(12)} X \otimes Y \otimes Y \Pi_{(12)} | \gamma \rangle = \langle \gamma | X \otimes Y \otimes Y | \gamma \rangle = -1.$$
Problem 3 (cont.)

Therefore, wins with $p_1$. Similar argument holds for $Y \otimes Y \otimes X$. 
Problem 4

(1) This actually holds for general $d$-dim maximally entangled state: $\frac{1}{\sqrt{d}} \sum_i |i\rangle |i\rangle$.

$$U \otimes \mathbb{I} |\Phi\rangle = \frac{1}{\sqrt{d}} \sum_i U(i) \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} |j\rangle \langle j| U(i) \otimes |i\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} |j\rangle \otimes |i\rangle \langle j| U(i) \rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{i,j} |j\rangle \otimes |i\rangle \langle i| U^\dagger |j\rangle$$

b.c. $\langle j| U(i) \rangle = |i\rangle U^\dagger |j\rangle$

$$= \langle i| U^\dagger |j\rangle$$

$$= \mathbb{I} \otimes U^\dagger |\Phi\rangle.$$
Problem 4 (cont.)

Let $\text{Ctrl}_{12} - U = \begin{array}{c}
\text{U}
\end{array}$ and $\text{Ctrl}_{13} - U^\dagger = \begin{array}{c}
\text{U}^\dagger
\end{array}$.  \Rightarrow

\[\text{Ctrl}_{12} - U \ket{0} \ket{\Phi} = \ket{0} \Pi_2 \ket{\Phi}\]
\[= \ket{0} \ket{\Phi}\]
\[= \text{Ctrl}_{13} - U^\dagger \ket{0} \ket{\Phi}\]

\[\text{Ctrl}_{12} - U \ket{1} \ket{\Phi} = \ket{1} U_2 \ket{\Phi}\]
\[= \ket{1} U_2^\dagger \ket{\Phi}\]
\[= \text{Ctrl}_{13} - U^\dagger \ket{1} \ket{\Phi}\]

Holds by linearity for general state on 1st qubit.
Problem 4 (cont.)

(2) We apply (part 1) to the EPR state to move the CNOT to the 3rd qubit.
Now we are measuring one qubit of the EPR state in the computational basis. The
principle of deferred measurement cuts both ways so the circuit is equivalent to with
pr $\frac{1}{2}$ running the circuit in eq. (3) for each $x \in \{0, 1\}$.

(3) $|x\rangle - |0\rangle - \begin{array}{c} X^x \end{array} - |x\rangle - |0\rangle - \begin{array}{c} X^x \end{array} - X^x - |0\rangle - \begin{array}{c} X^x \end{array}$. As CNOT = Ctrl-X, it's easy to see that
commutation gives us

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10 10 10 1 1 1 1
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$10 - \begin{array}{c} X^x \end{array} - \begin{array}{c} X^x \end{array} = 10 - \begin{array}{c} X^x \end{array} - \begin{array}{c} X^x \end{array} - \begin{array}{c} X^x \end{array} = 10 - \begin{array}{c} X^x \end{array}$.
Problem 4 (cont.)

4. \[ |\psi\rangle \rightarrow |H\rangle |X\rangle \cdots |Z\rangle \]

is equivalent to

\[ |\psi\rangle |0\rangle = (\alpha|0\rangle + \beta|1\rangle) |0\rangle \rightarrow \alpha|00\rangle + \beta|11\rangle \]

\[ \rightarrow \frac{1}{\sqrt{2}} (\alpha|00\rangle + \alpha|10\rangle + \beta|01\rangle - \beta|11\rangle) \]

\[ = \frac{1}{\sqrt{2}} \left( |0\rangle |0\rangle \psi\rangle + |1\rangle |Z \psi\rangle \right) \]

\[ = \frac{1}{\sqrt{2}} \sum_{z} |z\rangle |Z^2 \psi\rangle \]

measuring \( Z \) collapses the state to \( |Z^2 \psi\rangle \) so applying \( Z^2 \) produces \( |\psi\rangle \).