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Polytopic Approximations of Reachable Sets applied to Linear Dynamic Games and to a Class of Nonlinear Systems*

Inseok Hwang[1], Dušan M. Stipanović[2], and Claire J. Tomlin[3]

[1] Assistant Professor, School of Aeronautics and Astronautics, Purdue University
ihwang@purdue.edu
[2] Assistant Professor, Department of General Engineering, and Coordinated Science Laboratory, University of Illinois at Urbana-Champaign
dusan@uiuc.edu
[3] Associate Professor, Department of Aeronautics and Astronautics, Stanford University
tomlin@stanford.edu

Summary. This paper presents applications of polytopic approximation methods for reachable set computation using dynamic optimization. The problem of computing exact reachable sets can be formulated in terms of a Hamilton-Jacobi partial differential equation (PDE). Numerical solutions which provide convergent approximations of this PDE have computational complexity which is exponential in the continuous variable dimension. Using dynamic optimization and polytopic approximation, computationally efficient algorithms for overapproximative reachability analysis have been developed for linear dynamical systems [1]. In this paper, we extend these to feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems. Three illustrative examples are presented.

1.1 Introduction

Reachability analysis for continuous and hybrid systems is important for the automatic verification of safety properties and for the synthesis of safe controllers for these systems [2, 3]. Convergent approximations of reachable sets for such systems can be computed by solving a particular Hamilton-Jacobi partial differential equation (PDE) [3, 4]. Numerical methods have been devised to compute these convergent overapproximations [5], which work well in up to four to five continuous variable dimensions, yet these methods are not practical for solving high dimensional problems. Therefore, approximate methods for reachable set computation have been proposed.

Tiwari and Khanna [6] and Alur et al. [7] proposed predicate abstraction for reachable set computation: this method can be used to extract equivalent finite state

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models from complex, infinite state models, which are used to find approximate reachable sets of the original systems. In [8], Hwang et al. have used an augmented form of predicate abstraction to compute reachable sets for a simple biological cell network. However, since the accuracy of reachability analysis using predicate abstraction greatly depends on the choice of polynomials for abstraction, it is important to have information about a given system *a priori* (from analysis and simulations) to get good results in the reachability analysis. Chutinan and Krogh [9, 10] present a method to approximate the flows of autonomous systems with convex polyhedra. An experimental system called **d/dt** [7, 11, 12] has been developed to approximate reachable sets for linear dynamical systems using griddy orthogonal polyhedra. Ideas based on projecting the initial or target set into a lower dimensional subset of the state space, performing the reach set computation in the lower dimensional space, and then back projecting to form an overapproximation of the actual reachable set in the full state space, are presented in [13, 14]. In all of these methods, however, it is difficult to compute the control input which is guaranteed to keep the system on the boundary or inside the set, from the boundary of the overapproximative set.

Varaiya [1] has designed, using techniques from optimal control theory, a polytopic approximation for linear systems. Kostousova [15] has developed two-sided approximations of reachable sets for linear dynamic systems using parallelotopes. Kurzhanski and Varaiya [16, 17] proposed an ellipsoidal approximation for forward and backward reachable sets (a computational tool VeriSHIFT [18] has been developed based on their ideas) and in [19, 20], they define various types of reachable sets for linear time-varying systems with bounded perturbations using both open and closed-loop input laws. In [20], they propose ellipsoidal overapproximations of reachable sets for linear systems under uncertainty via solutions of a particular type of differential equation. In [21, 22], the authors have extended reachable set computations to general nonlinear systems with state constraints and obstacles, using nonstandard Hamilton-Jacobi equations and variational inequalities. Overall, this seminal work in exact and approximate reachable set calculation suggests new research directions in computational methods for such problems. This work was indeed motivation for the current paper.

In this paper, we review the method proposed by Varaiya [1] to compute reachable sets for linear time invariant systems. Inspired by Kurzhanski and Varaiya [16, 17, 19, 20] and by the work of Khrustalev [23], we compute approximate reachable sets for feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems. We present three examples, one of which is a two-aircraft three-dimensional collision avoidance example which we have used in other work [5].

This paper is organized as follows. Motivation for this study is described in Section 1.2. Computations of polytopic reachable sets for linear dynamical systems, feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems are presented in Section 1.3. Examples are presented in Section 1.4. Conclusions are presented in Section 1.5.

1.2 Background and Motivation

Consider a dynamical system,

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), d(t)), \\ x(0) \in X_0 \quad &(\text{or } x(t_f) \in \mathcal{Y}_0), \;\; t \in [0, t_f] \end{aligned} \tag{1.1}$$

where $0 \leq t_f < \infty$, $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ is the control input, $d \in D \subset \mathbb{R}^p$ is the disturbance input, $X_0 = \{x : l(x) \leq 0\}$ is an initial set of states, and $\mathcal{Y}_0 = \{x : y(x) \leq 0\}$ is a target set of states. We assume f to be Lipschitz. The spaces of admissible control input trajectories and disturbance input trajectories are denoted as the spaces of piecewise continuous functions $\mathcal{U} = \{u(\cdot) \in PC^0 | u(t) \in U, 0 \leq t \leq t_f\}$ and $\mathcal{D} = \{d(\cdot) \in PC^0 | d(t) \in D, 0 \leq t \leq t_f\}$ respectively. The forward and the backward reachable sets of the system (1.1) are defined as follows.

Definition 1. *The forward reachable set $\mathcal{X}(\tau)$ at time τ $(0 < \tau \leq t_f)$, of the system (1.1) from the initial set X_0, is the set of all states $x(\tau)$, such that there exists a control input $u(t) \in \mathcal{U}$ $(0 \leq t \leq \tau)$, for all disturbance inputs $d(t) \in \mathcal{D}$ $(0 \leq t \leq \tau)$, for which $x(\tau)$ is reachable from some $x(0) \in X(0)$, along a trajectory satisfying (1.1).*

Definition 2. *The backward reachable set $\mathcal{Y}(\tau)$ at time τ $(0 \leq \tau < t_f)$, of the system (1.1) from the target set $\mathcal{Y}_0$, is the set of all states $x(\tau)$, such that there exists a control input $u(t) \in \mathcal{U}$ $(\tau \leq t \leq t_f)$, for all disturbance inputs $d(t) \in \mathcal{D}$ $(\tau \leq t \leq t_f)$, for which some $x(t_f) \in \mathcal{Y}_0$ are reachable from $x(\tau)$, along a trajectory satisfying (1.1).*

It has been shown that a forward reachable set computation can be formulated as a dynamic optimization problem [17, 23]. The forward reachable set of the dynamical system (1.1) at time τ $(0 < \tau \leq t_f)$ is shown to be [17]:

$$\mathcal{X}(\tau) = \{x : v(x, \tau) \leq 0\} \tag{1.2}$$

where $v(x, \tau)$ is a (viscosity) solution of the Hamilton-Jacobi-Isaacs (HJI) partial differential equation,

$$D_t v(x,t) + \max_{u \in U} \min_{d \in D} \{< D_x v(x,t), f(x,u,d) >\} = 0 \tag{1.3}$$

with $v(x, 0) = l(x)$, $< p, q >= p^T q$ the inner product in $\mathbb{R}^n$, and where $D_{()}$ represents the partial derivative with respect to the subscripted variable. Thus, the forward reachable set of the dynamical system (1.1) is the zero sublevel set of the solution to the HJI equation in (1.3).

Similarly, the backward reachable set of the dynamical system (1.1) at time τ $(0 \leq \tau < t_f)$ is the zero sublevel set of the solution to the HJI equation [17],

$$D_t v(x,t) + \min_{u \in U} \max_{d \in D} \{< D_x v(x,t), f(x,u,d) >\} = 0 \tag{1.4}$$

with $v(x, t_f) = y(x)$.

In [4, 5], a numerical tool for computing convergent approximations for backwards reachable sets is designed and presented. This method is based on the level set method for computing solutions to PDEs [24]. The computational complexity of this tool is exponential in the number of continuous variables dimensions: it has been shown to work well in up to four or five continuous variables dimensions, yet for larger problems computation time is currently prohibitive. Numerical convergence has been demonstrated on several examples; we will use a "benchmark" three-dimensional example from [5] in this paper.

Consider planar kinematic models of two aircraft, labeled 1 and 2. Let the relative position and orientation of aircraft 2 with respect to aircraft 1 be represented by $(x_r, y_r, \psi_r) \in \mathbb{R}^2 \times [-\pi, \pi)$. Given the absolute positions and orientations of the two aircraft, denoted as x_i, y_i, ψ_i for $i = 1, 2$, the relative coordinates are defined as: $x_r = \cos\psi_1(x_2 - x_1) + \sin\psi_1(y_2 - y_1), y_r = -\sin\psi_1(x_2 - x_1) + \cos\psi_1(y_2 - y_1), \psi_r = \psi_2 - \psi_1$. The relative kinematics are thus given by:

$$\begin{aligned} \dot{x}_r &= -\sigma_1 + \sigma_2 \cos\psi_r + \omega_1 y_r \\ \dot{y}_r &= \sigma_2 \sin\psi_r - \omega_1 x_r \\ \dot{\psi}_r &= \omega_2 - \omega_1 \end{aligned} \tag{1.5}$$

where σ_i is the linear velocity of aircraft i and ω_i is its angular velocity. Safety is encoded as a 5 nautical mile radius cylinder "protected zone" centered at the origin of the relative frame. In this paper, following the notation in Definition 2 (which is different from that in [5]), we define the angular velocity of aircraft 2 (ω_2) as the control input that steers the system (1.5) into the target set and the angular velocity of aircraft 1 (ω_1) as the disturbance input that keeps the system (1.5) outside of the target set. Posing this problem as a game, we label aircraft 1 as "evader" and aircraft 2 as "pursuer", and we compute the set of states (x_r, y_r, ψ_r) for which for all possible disturbance inputs, ω_1 action of the evader, there is a control input, ω_2 action of the pursuer, such that the system state enters the protected zone, which we consider the target set of the game. For values $\sigma_1 = \sigma_2 = 5$ and $\omega_i \in [-1, 1]$ ($i \in \{1, 2\}$), the problem has been solved numerically, and the results (solid surface) are shown in Figure 1.4 (Courtesy of I. Mitchell [5]). This computation took approximately 4 minutes to run on a Sun UltraSparc II, in which 50 grid nodes in each dimension were used.

A version of this example may also be solved analytically [25], and it may be verified using this that the average error in computation is less than one tenth of a grid cell, with maximum error always less than one grid cell.

In the following section, we extend Varaiya's method [1] to treat this kind of system and in Section 1.4, we compare the above computation with the resulting approximation.

1.3 Computation of polytopic reachable sets

We first define the overapproximate reachable set [17] (here we specialize to the case of (1.1) in which there are no disturbances). Assume that $x_*(0) \in X_0$ and

$u_*(t) \in \mathcal{U}$ for all $t\varepsilon 0$ such that $x_*(\tau) \in \mathcal{X}(\tau)$ $(0 \leq t \leq \tau)$. Then, an overapproximate solution to the solution of the HJI equation in (1.3) is defined as a function $v^+(x,t)$ satisfying [17, 23]:

$$\begin{array}{l} \frac{dv^+(x,t)}{dt}|_{x=x_*(t),u=u_*(t),\dot{x}=f(x,u)} \\ = D_t v^+(x_*,t) + < D_x v^+(x_*,t), f(x_*,u_*) > \\ \leq D_t v^+(x_*,t) + \max_{u\in U}\{< D_x v^+(x_*,t), f(x_*,u) >\} \\ \leq \mu(t) \end{array} \quad (1.6)$$

where $v^+(x_*,t)$ is a piecewise continuous function, and $\mu(t)$ is a positive-definite, integrable function. By integrating (1.6) from 0 to τ, we obtain an overapproximative reachable set of the dynamical system (1.1) at time τ as:

$$V^+(\tau) = \{x | v^+(x,\tau) \leq \int_0^\tau \mu(t)dt + \max_{x(0)\in X_0} v^+(x(0),0)\} \quad (1.7)$$

Next, we review the polytopic overapproximation of reachable sets for linear dynamical systems and derive computational methods for polytopic overapproximate reachable sets for feedback linearizable nonlinear systems, linear dynamic games, and norm-bounded nonlinear systems.

1.3.1 Linear dynamical systems

In this section, we review the polytopic overapproximation of reachable sets for linear systems from [1]. Consider a time-varying linear dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) \in X_0,\ u(t) \in U \quad (1.8)$$

where the initial set X_0 and the admissible control input set U are assumed to be convex polytopes which have N and N_u faces respectively. In this paper, we assume the initial set X_0 is a polytope, but in general the number of faces of the initial set is a design parameter since X_0 may be a convex compact set and thus the more the number of faces of X_0 the better the overapproximate reachable set.

A convex polytope $\mathcal{P}$ with K faces can be represented in two ways; it can be represented as the bounded intersection of K half spaces,

$$\mathcal{P} = \bigcap_{i=1}^{K} \{x | h_i^T x \leq \gamma_i\} \quad (1.9)$$

where h_i is a normal vector to the i^{th} face of the polytope $\mathcal{P}$. A convex polytope can also be represented as the convex hull of its vertices: if a convex polytope $\mathcal{P}$ has m vertices $\{v^1, \cdots, v^m\}$, then

$$\mathcal{P} = \{x | x = \sum_{i=1}^{m} \alpha_i v^i,\ \alpha_i \varepsilon 0,\ \sum_{i=1}^{m} \alpha_i = 1\} \quad (1.10)$$

Define a set of linear functions as

$$v_i^+(x,t) = h_i^T(t)x, \quad i \in \{1, 2, \cdots, N\} \tag{1.11}$$

These linear functions are used to represent a convex polytope as shown in (1.9). In order to find a polytopic overapproximate reachable set, we solve for $v_i^+(x,t)$ in (1.11) that satisfies (1.6). Then, (1.6) becomes

$$\begin{aligned} &D_t\ v_i^+(x,t) + \max_{u\in U}\{< D_x v_i^+(x,t), f(x,u) >\} \\ &= < \dot{h}_i(t), x(t) > + < A(t)^T h_i(t), x(t) > + \max_{u\in U}\{< h_i(t), B(t)u(t) >\} \\ &\le \mu(t) \end{aligned} \tag{1.12}$$

From optimal control theory [26], the adjoint equation for linear systems when the input set does not depend on x is $\dot{\lambda}(t) = -A(t)^T\lambda(t)$. If we choose $h_i(t) = \lambda(t)$ ($i \in \{1, 2, \cdots, N\}$), then

$$< \dot{h}_i(t), x(t) > + < A(t)^T h_i(t), x(t) >= 0 \tag{1.13}$$

This represents the evolution of the normal vector of the i^{th} face. Let $h_i(0),\ i \in \{1, 2, \cdots, N\}$ be the normal vectors of the faces of the initial set X_0. Then, the solution to (1.13) is

$$h_i(t) = \Phi(t,0)h_i(0), \quad i \in \{1, 2, \cdots, N\} \tag{1.14}$$

where $\Phi(t,0)$ is the state transition matrix satisfying $\dot{\Phi} = -A(t)^T\Phi,\ \Phi(0,0) = I$. If the system dynamics in (1.8) is time invariant, then $\Phi(t,0) = e^{-A^T t}$ and (1.14) becomes

$$h_i(t) = e^{-A^T t}h_i(0), \quad i \in \{1, 2, \cdots, N\} \tag{1.15}$$

Thus, for a linear time invariant system, the evolution of normal vectors can be determined analytically. We denote $\{u^1, \cdots, u^{m_u}\}$ as the vertices of the input set U. Since U is a convex polytope, the following must hold: (for $j \in \{1, \cdots, m_u\}$)

$$\max_{u\in U} < h_i(t), B(t)u(t) >= \max_j < h_i(t), B(t)u^j >\le \mu(t) \tag{1.16}$$

that is, the maximum is achieved at a vertex of U [1]. Furthermore, if the system dynamics in (1.8) is time invariant, (1.16) is simplified to

$$\max_j < h_i(t), Bu^j >= \max_j < e^{-A^T t}h_i(0), Bu^j >\le \mu(t) \tag{1.17}$$

for $j \in \{1, \cdots, m_u\}$. We choose $\mu(t) = \max_j < h_i(t), B(t)u^j >$ and note that $\mu(t)$ is always positive for a properly chosen input set U (e.g., chosen such that $0 \in U$). Then, the linear function $v_i^+(x,t)$ in (1.11) is a supporting hyperplane of the exact reachable set [1]. A polytopic overapproximate forward reachable set $V^+(t)$ for the dynamical system (1.8) is the intersection of half spaces as follows:

$$\begin{aligned} V^+(t) = \bigcap_{i=1}^N\{\ x : v_i^+(x,t) \le \int_0^t \max_j < h_i(s), B(s)u^j > ds \\ + \max_{x(0)\in X_o} v_i^+(x(0), 0)\} \end{aligned} \tag{1.18}$$

$V^+(t)$ is a convex polytope which contains the exact reachable set at time t since each $v_i^+(x,t)$ in (1.18) is a supporting hyperplane of the exact reachable set. If the system dynamics is linear time invariant, $V^+(t)$ becomes

$$V^+(t) = \bigcap_{i=1}^{N} \{ \; x : v_i^+(x,t) \leq \int_0^t \max_j < e^{-A^T s} h_i(0), Bu^j > ds \\ + \max_{x(0) \in X_o} v_i^+(x(0),0)\} \tag{1.19}$$

1.3.2 Feedback linearizable nonlinear systems

In this section, we consider a class of nonlinear systems [27], in which $u(t)$ is a feedback control:

$$\dot{x}(t) = f(x) + g(x)u(t) \tag{1.20}$$

where

$$u(t) = a(x(t)) + b(x(t))v(t) \tag{1.21}$$

We assume that there exists a diffeomorphism T: such that $z = T(x)$, which transforms, with a control input $u(t)$, a nonlinear system (1.20) into an equivalent linear system [27]. Then, we can compute an overapproximate forward reachable set for the nonlinear system (1.20) as follows:

- Step 1: Transform the nonlinear system (1.20) to an equivalent linear system, $\dot{z}(t) = A(t)z(t) + B(t)v(t)$ with appropriate $u(t)$ and T.
- Step 2: Compute a polytopic overapproximate forward reachable set $V^+(t)$ of the linear system following the procedure in Section 1.3.1.
- Step 3: Using the inverse state transformation $x = T^{-1}(z)$, we obtain the overapproximate forward reachable set for the original nonlinear system (1.20) from $V^+(t)$.

Since there is no approximation during the transformation and the transformation is a diffeomorphism on a given domain of interest, the forward reachable set obtained in Step 3 is guaranteed to be an overapproximate forward reachable set of the nonlinear system (1.20).

1.3.3 Linear dynamic games

Now, we consider the linear dynamic game:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t)d(t), \\ x(0) \in X_0, \; u(t) \in U, \; d(t) \in D \tag{1.22}$$

where the initial set X_0, the admissible control input set U, and the disturbance input set D are assumed to be convex polytopes which have N, N_u, and N_d faces respectively. Then, the HJI equation in (1.3) for a forward reachable set computation becomes [19, 20],

$$D_t v(x,t) + \max_{u \in U} \min_{d \in D} \{< D_x v(x,t), \\ A(t)x(t) + B(t)u(t) + C(t)d(t) >\} = 0 \tag{1.23}$$

To find an overapproximate solution to (1.23), we look for a set of linear functions $v_i^+(x,t)$ in (1.11) satisfying (1.13), and compute

$$\begin{aligned} D_t v_i^+(x,t) &+ \max_{u\in U} \min_{d\in D}\{< D_x v_i^+(x,t), \\ &A(t)x(t) + B(t)u(t) + C(t)d(t) >\} \\ &= \max_{u\in U}\{< h_i(t), B(t)u(t) >\} \\ &\quad + \min_{d\in D}\{< h_i(t), C(t)d(t) >\} \\ &\le \mu(t) \end{aligned} \tag{1.24}$$

We denote $\{u^1, \cdots, u^{m_u}\}$ and $\{d^1, \cdots, d^{m_d}\}$ as the vertices of U and D respectively. Since (1.24) is linear with respect to u and d, the maximum and the minimum in (1.24) are achieved at vertices of U and D as follows:

$$\max_j < h_i(t), B(t)u^j > + \min_k < h_i(t), C(t)d^k > \le \mu(t) \tag{1.25}$$

for $j \in \{1, \cdots, m_u\}$, $k \in \{1, \cdots, m_d\}$.

By choice of $\mu(t) = \max_j < h_i(t), B(t)u^j > + \min_k < h_i(t), C(t)d^k >$, the polytopic overapproximate reachable set $V^+(t)$ for the linear dynamic game (1.22) is

$$V^+(t) = \bigcap_{i=1}^{N} \{x : v_i^+(x,t) \le \int_0^t \mu(s)ds + \max_{x(0)\in X_0} v_i^+(x(0),0)\} \tag{1.26}$$

1.3.4 Norm-bounded nonlinear systems

In this section, we consider a norm-bounded nonlinear system,

$$\begin{aligned} &\dot{x}(t) = A(t)x(t) + B(t)u(t) + \phi(x,t), \\ &x(0) \in X_0,\ u(t) \in U,\ \|\phi(x,t)\| \le \beta(t) \end{aligned} \tag{1.27}$$

where the initial set X_0 and the admissible control input set U are assumed to be convex polytopes which have N and N_u faces respectively. $\|\cdot\|$ represents the Euclidean norm; $\beta(\cdot)$ is a positive-definite function. Then, the HJI equation in (1.3) becomes

$$D_t v(x,t) + \max_{u\in U}\{< D_x v(x,t), A(t)x(t) + B(t)u(t) + \phi(x,t) >\} = 0 \tag{1.28}$$

To compute an overapproximate solution to the HJB equation in (1.28), we find the linear functions $v_i^+(x,t)$ in (1.11) satisfying (1.13), and compute

$$\begin{aligned} D_t v_i^+(x,t) &+ \max_{u\in U}\{< D_x v_i^+(x,t), A(t)x(t) + B(t)u(t) + \phi(x,t) >\} \\ &= \max_{u\in U}\{< h_i(t), B(t)u(t) >\} + < h_i(t), \phi(x,t) > \\ &\le \max_{u\in U}\{< h_i(t), B(t)u(t) >\} + \tfrac{1}{2}(\|h_i(t)\|^2 + \|\phi(x,t)\|^2) \\ &\le \max_j\{< h_i(t), B(t)u^j >\} + \tfrac{1}{2}(\|h_i(t)\|^2 + \beta(t)^2) \\ &\le \mu(t) \end{aligned} \tag{1.29}$$

If we choose $\mu(t)$ such that

$$\mu(t) = \max_j < h_i(t), B(t)u^j > + \frac{1}{2}(\|h_i(t)\|^2 + \beta(t)^2) \tag{1.30}$$

then, a polytopic overapproximate reachable set $V^+(t)$ for the norm-bounded dynamical system (1.27) is

$$\begin{aligned} V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x,t) \leq \int_0^t [\max_j < h_i(s), B(s)u^j > \\ + \tfrac{1}{2}(\|h_i(s)\|^2 + \beta(s)^2)]ds + \max_{x(0)\in X_0} v_i^+(x(0),0)\} \end{aligned} \tag{1.31}$$

If $\phi(x,t)$ belongs to a polytope with vertices $\{\phi^1, \cdots, \phi^{m_\phi}\}$, a polytopic overapproximate reachable set $V^+(t)$ becomes

$$\begin{aligned} V^+(t) = \bigcap_{i=1}^N \{x : v_i^+(x,t) \leq \int_0^t [\max_j < h_i(s), B(s)u^j > \\ + \max_k \{< h_i(s), \phi^k >\}]ds + \max_{x(0)\in X_0} v_i^+(x(0),0)\} \end{aligned} \tag{1.32}$$

1.4 Examples

We consider three examples: a linear system, a norm-bounded nonlinear system, and we conclude with the example which motivated this study, a nonlinear, feedback linearizable, dynamic game. Note that equation (1.7) provides overapproximations of the sets of reachable states over a range of times (the flow). In the implementation, we compute overapproximations of the reachable sets at specific instants of time without interpolation between the sets.

1.4.1 Linear dynamical systems

In this section, we consider a linear dynamical system $\dot{x} = Ax + Bu$, $x(0) \in X_0$ where the control input $u(t)$ can vary inside a convex polytope U and the initial set X_0 is also a convex polytope. The system parameters (A, B, X_0, and U) given in [11] are used. Figure 1.1 shows the evolution of the projection on x_3 and x_4 over time. This result is similar to that in [11], yet computation time with the method shown in Section 1.3.1 is 1.17 seconds (which includes plotting the result shown in Figure 1.1) using MATLAB on a 700MHz Pentium III PC. For comparison, the algorithm proposed in [11] takes 18 seconds using the same parameters.

1.4.2 Norm-bounded nonlinear systems

We consider a norm-bounded nonlinear system

$$\dot{x} = A(t)x + B(t)u(t) + \phi(x,t), \ x(0) \in X_0, \ u(t) \in U \tag{1.33}$$

where the initial set X_0 and the control input set U are convex polytopes. The nonlinear function $\phi(x,t)$ is assumed to be norm-bounded i.e., $\|\phi(x,t)\| \leq \frac{1}{3}t$ where $t > 0$. The system parameters are defined as follows:

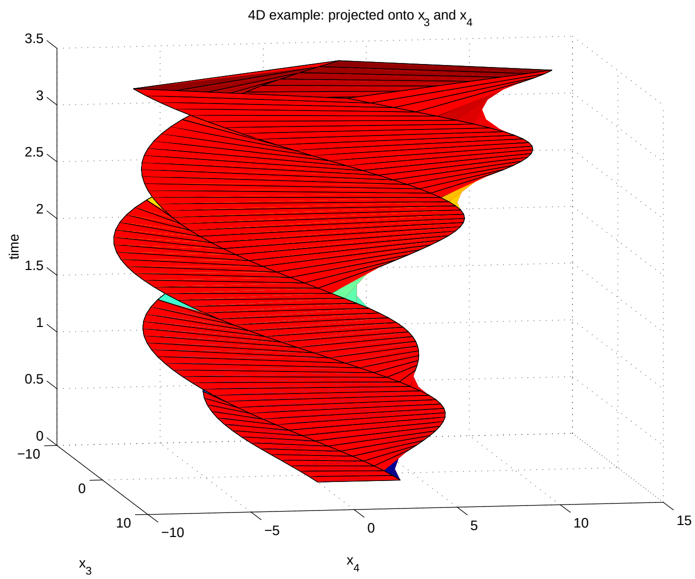

Fig. 1.1. The forward reachable set of a four dimensional linear dynamical system (projection onto x_3 and x_4).

$$A = \begin{bmatrix} -0.5 & 4.0 \\ -3.0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$X_0 = [4, 5] \times [4, 5], \; U = [-0.1, 0.1]$$

The evolution of the forward reachable set over time is shown in Figure 1.2 and its computation time is 0.87 seconds (including plotting the result) using MATLAB on the same PC.

1.4.3 Conflict resolution between two aircraft

Last, we consider the two aircraft collision avoidance problem, as an example of feedback linearizable nonlinear systems and linear dynamic games. This is the same problem (the motivation for this research) described in Section 1.2. Figure 1.3 shows the relative configuration between two aircraft showing the protected zone.

Aircraft 1 tries to avoid a conflict with aircraft 2 within the limits of its capability. Thus, we want to compute a backward reachable set (unsafe set) from the target set (protected zone). The target set represents the states from which the two aircraft would eventually have a conflict no matter how aircraft 1 tries to avoid it [5].

Using dynamic extension [27] with σ_i as a new state variable (compared to (1.5)), we obtain a new nonlinear model which is feedback linearizable [28],

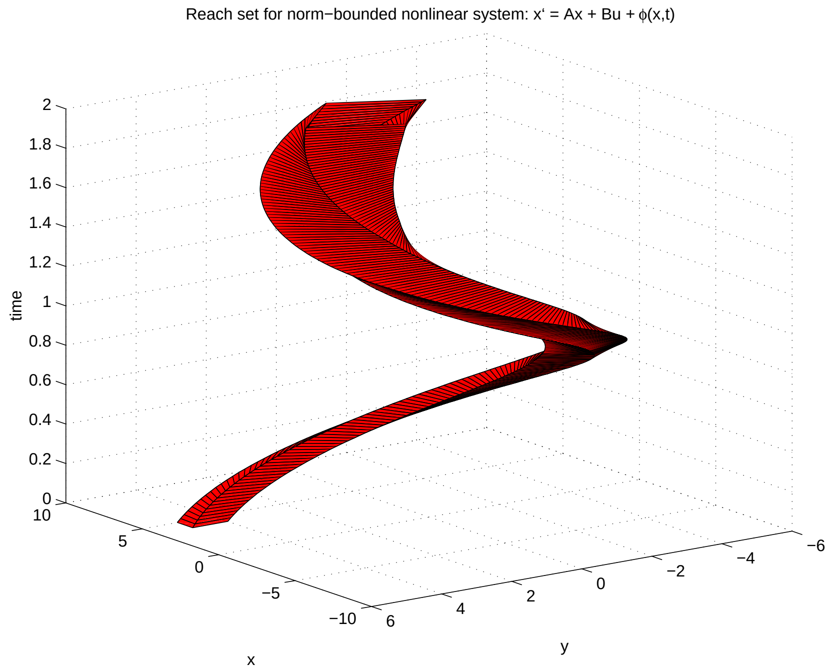

Fig. 1.2. The forward reachable set of a norm-bounded nonlinear system.

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\psi}_i \\ \dot{\sigma}_i \end{bmatrix} = \begin{bmatrix} \sigma_i \cos \psi_i \\ \sigma_i \sin \psi_i \\ \omega_i \\ a_i \end{bmatrix}, \ (i \in \{1, 2\}) \tag{1.34}$$

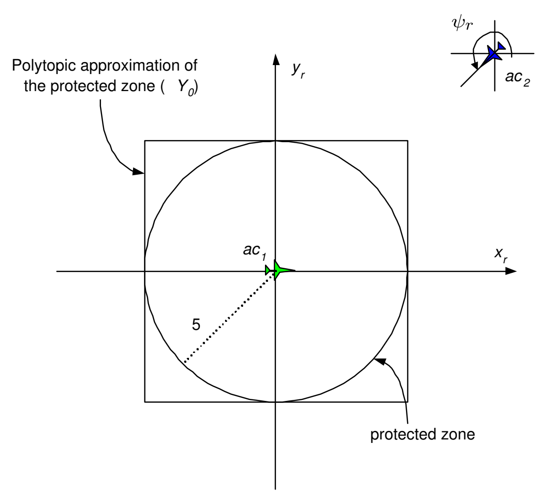

Fig. 1.3. Relative configuration of two aircraft showing the protected zone.

where a_i is the acceleration of aircraft i and is a new control input. Thus, the new state and input variables are $\xi_i := [x_i \, y_i \, \psi_i \, \sigma_i]^T$ and $\eta_i := [a_i \, \omega_i]^T$ respectively. We introduce a change in state variables, $z_i = T(\xi_i)$, and a change of the input variables, $\eta_i = M(\xi_i)u_i$, as in [28]. We denote that T and M are diffeomorphisms everywhere except at $\sigma_i = 0$. Then, the feedback linearized model of the nonlinear kinematic aircraft model in (1.34) obtained through the transformations T and M is [28]:

$$\dot{z}_i = \frac{\partial T}{\partial \xi_i} \dot{\xi}_i \Rightarrow \dot{z}_i = Az_i + Bu_i \tag{1.35}$$

with A and B defined in [28].

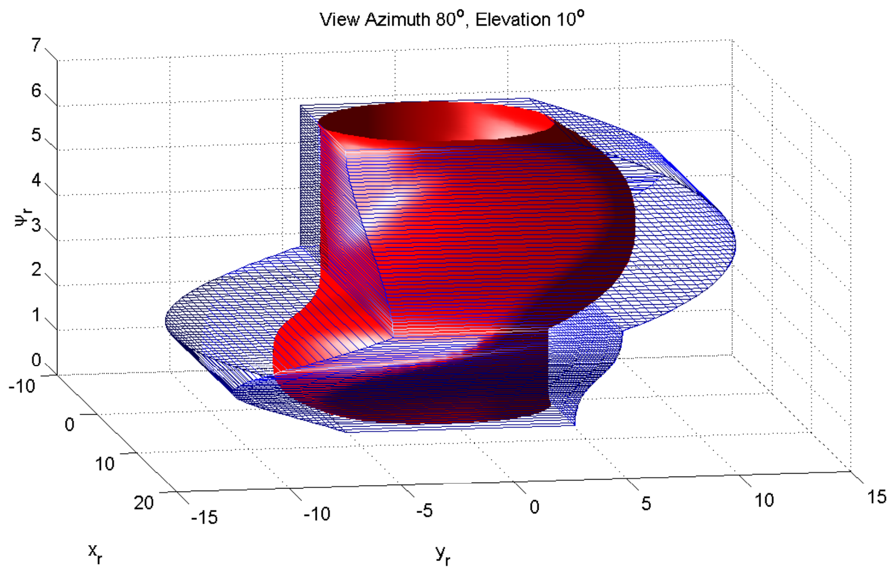

Fig. 1.4. Comparison between overapproximate (grid) and exact (solid) backward reachable sets (unsafe sets) of conflict resolution between two aircraft.

The relative kinematic aircraft model between two aircraft can be obtained by introducing new states $\xi_r := \xi_2 - \xi_1$ in the original nonlinear state space and $z_r := z_2 - z_1$ in the linearized state space. Thus, a linearized relative kinematic aircraft model is

$$\dot{z}_r = Az_r + Bu_2 - Bu_1, \quad u_2 \in U, \, u_1 \in D, \tag{1.36}$$

where the admissible control input set U and the disturbance input set D are polytopes. This is a linear dynamic game since aircraft 1 (u_1) tries to keep aircraft 2 from entering into its protected zone (target set) to prevent a conflict, but aircraft 2 (u_2) tries to enter the protected zone of aircraft 1. A target set (protected zone) is assumed to be $\mathcal{Y}_0 = [-5, 5] \times [-5, 5] \times [-\pi, \pi]$. Using dynamic extension, we have performed the computation in four dimensions (1.36) and projected the result onto the relative coordinate in three-dimensional space. A polytopic overapproximate backward reachable set is first computed in the linearized space, and then the overapproximate backward reachable set in the original state space is obtained through the transformations T and M. The overapproximate backward reachable set for conflict resolution with heading changes only, using the target set $\mathcal{Y}_0$, normalized aircraft speeds $\sigma_1 = \sigma_2 = 5$, angular velocities $|\omega_1| \leq 1$ and $|\omega_2| \leq 1$ is compared with the exact solution in [4] in Figure 1.4.

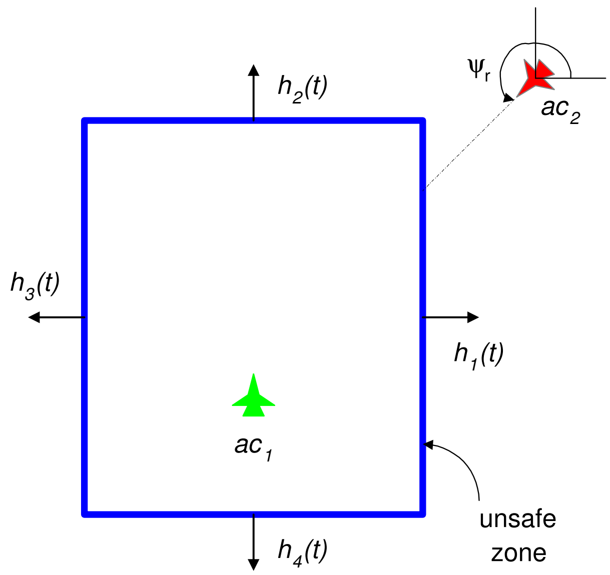

Fig. 1.5. Conflict scenario: Aircraft 2 reaches the boundary of the unsafe zone of aircraft 1 with a given initial relative angle ψ_r.

The backward reachable set obtained by using the polytopic approximation is overapproximate of the exact reachable set and its computation time is about 1.0 seconds (including plotting the result as shown in Figure 1.4) using MATLAB on the same PC, where the numerical solution to the exact PDE [5] takes approximately 4 minutes on a Sun UltraSparc II with 50 grid nodes in each dimension. Figure 1.5 shows a conflict scenario in which aircraft 2 tries to enter the unsafe zone. When aircraft 2 reaches the boundary of the unsafe zone, the optimal control input for aircraft 1 can be easily obtained as follows:

$$\begin{aligned} u_1^*(t) &= \arg\max\nolimits_{u_1 \in D} \{ < D_x v(x,t), -B(t) u_1(t)) > \} \\ &= \arg\max\nolimits_j < e^{-A^T t} h_1(0), -B u_1^j > \end{aligned} \tag{1.37}$$

Figure 1.6 shows a simulation for conflict resolution between the two aircraft with the initial condition $(x_r = 10,\ y_r = -20, \psi_r = 115^o)$. Since both aircraft behave optimally, the relative position of aircraft 2 moves along the boundary of

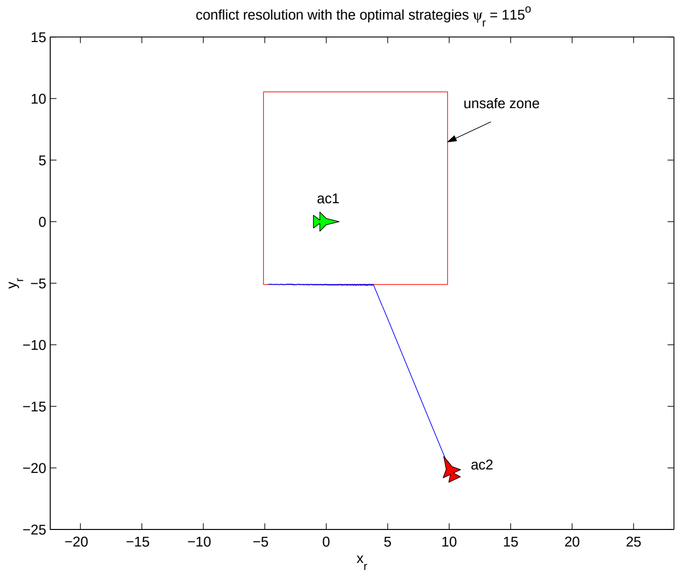

Fig. 1.6. Conflict resolution simulation with relative initial states $(x_r = 10,\ y_r = -20, \psi_r = 115^o)$. Aircraft 1 tries to avoid a conflict with aircraft 2 with the optimal strategy.

the unsafe set. As expected, chattering occurs along the boundary. To avoid such a phenomenon, one would introduce a buffer zone around the boundary so that the control inputs change smoothly as aircraft 2 approaches the boundary.

Using similar analysis to the above, we many obtain the underapproximate backward reachable set. This is obtained for the collision avoidance example, using the same parameters, and compared in Figure 1.7 with the overapproximate set.

1.5 Conclusions

The polytopic approximation gives an overapproximation of the exact reachable set and is computationally efficient: it requires solving matrix exponentials instead of a Hamilton-Jacobi partial differential equation. The data structure of the polytopic approximation method becomes more complicated than that of the ellipsoidal approximation method [17] as the number of faces of the polytope increases, yet the computation of the matrix exponential is easier than solving the (usually Riccati type) differential equation required for the ellipsoidal methods. The optimal control input can be easily computed from the Hamiltonian since the Hamiltonian is linear with respect to the control, and the control input set is a convex polytope. The polytopic approximation method can be applied to high dimensional systems which may not be solved exactly without substantially increasing the computational time. This may be done by decomposing the computation of an approximation (over or under)

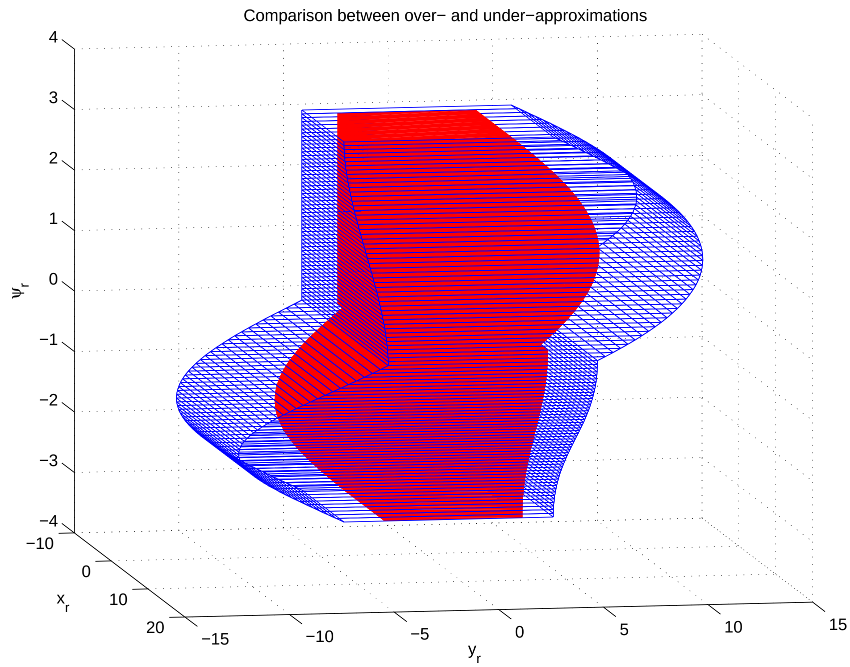

Fig. 1.7. Comparison between the under and overapproximate backward reachable sets for conflict resolution between two aircraft.

of the reachable set into a number of computations of approximations of subsystem reachable sets [29].

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