CS 172: Computability and Complexity

Equivalence of CFGs and PDAs & CFL Pumping Lemma

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Formal Definition of **Acceptance**

PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts a word $w \in \Sigma^*$

where $w = w_1 w_2 w_3 \ldots w_m$ with $w_i \in \Sigma_\varepsilon$

if *there exists* a sequence

$(q_0, s_0) \rightarrow (q_1, s_1) \rightarrow (q_2, s_2) \rightarrow \ldots \rightarrow (q_m, s_m)$

where

$s_i \in \Gamma^*$ (represent the stack), with $s_0 = \varepsilon$,

$q_m \in F$,

$(q_{i+1}, b) \in \delta(q_i, w_{i+1}, a)$

where $s_i = at$ and $s_{i+1} = bt$, $a,b \in \Gamma_\varepsilon$, $t \in \Gamma^*$

Theorem

Suppose L is generated by a CFG G = (V, Σ, R, S)

Construct P = (Q, Σ, Γ, δ, q, F) that recognizes L

A Language is generated by a CFG

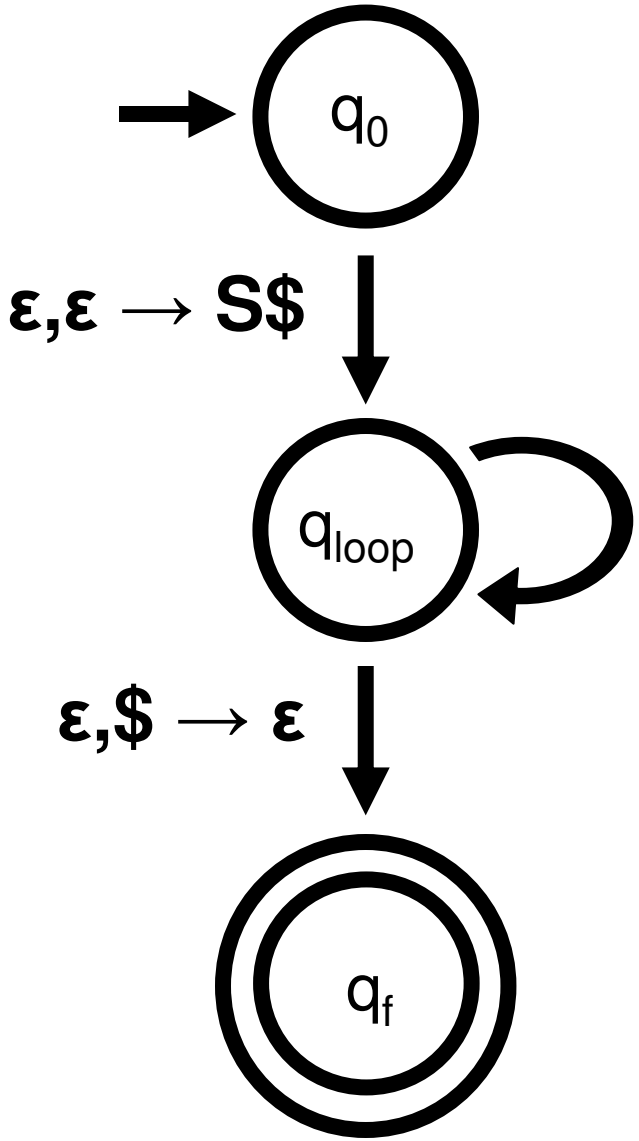
$\Leftrightarrow$

It is recognized by a PDA

A Language is generated by a CFG

$\Rightarrow$

It is recognized by a PDA

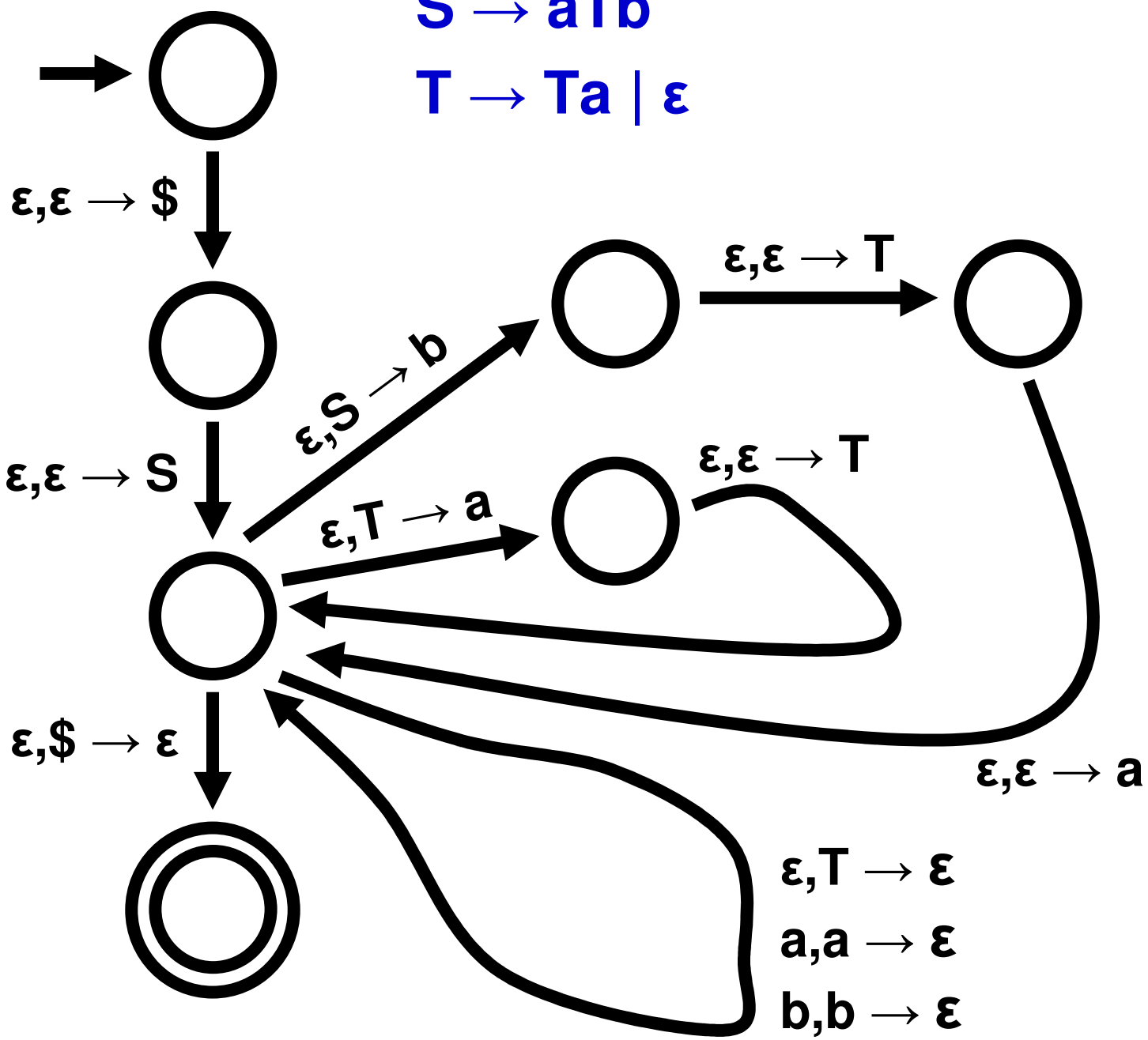


ε,A → w for rule A → w

a,a → ε for terminal a

$S \rightarrow aTb$

$T \rightarrow Ta \mid \varepsilon$



Theorem

Suppose PDA $P = (Q, \Sigma, \Gamma, \delta, q, F)$ recognizes L

Construct CFG $G = (V, \Sigma, R, S)$ that generates L

A Language is generated by a CFG

$\Leftarrow$

It is recognized by a PDA

Proof Ideas

- A_{pq} = variable generating all x that takes P from (p, ε) to (q, ε)
- Formal construction:
  - $V = \{ A_{pq} \mid p, q \in Q \}$
  - $S = A_{q0qf}$
  - Defining R:

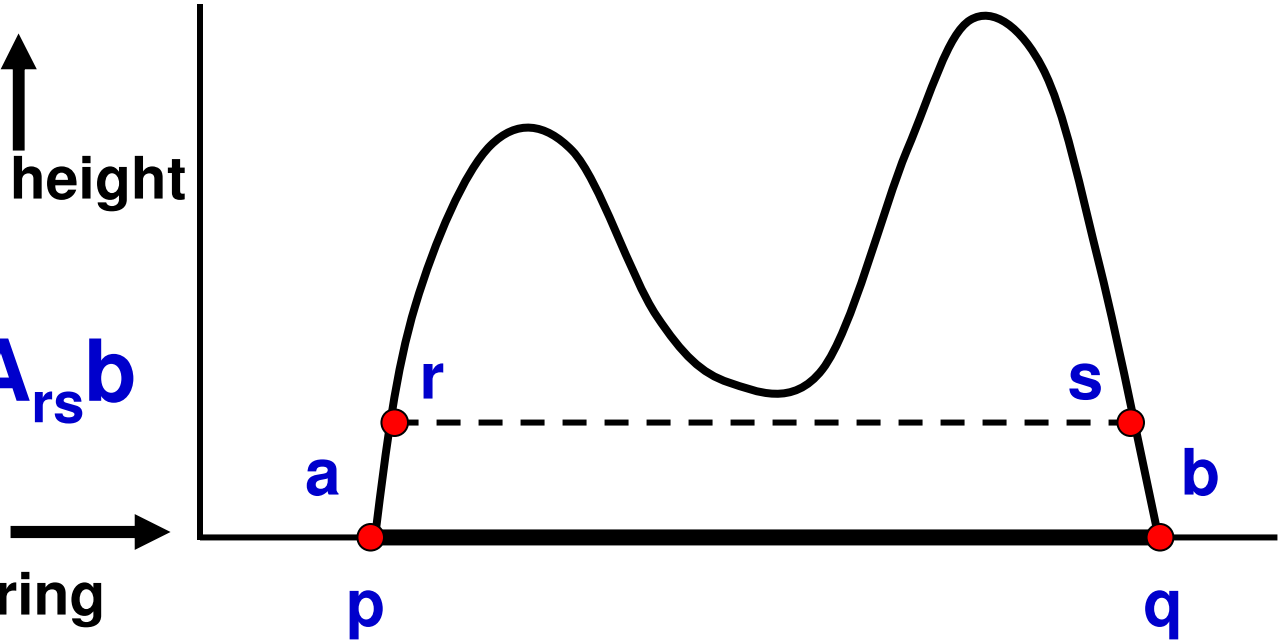
    Intuition: Derivations correspond to computations of P;

    There are two cases for a derivation from A_{pq}

    1. Stack is empty only at the beginning and end of the derivation from A_{pq}
    2. Stack becomes empty somewhere in between

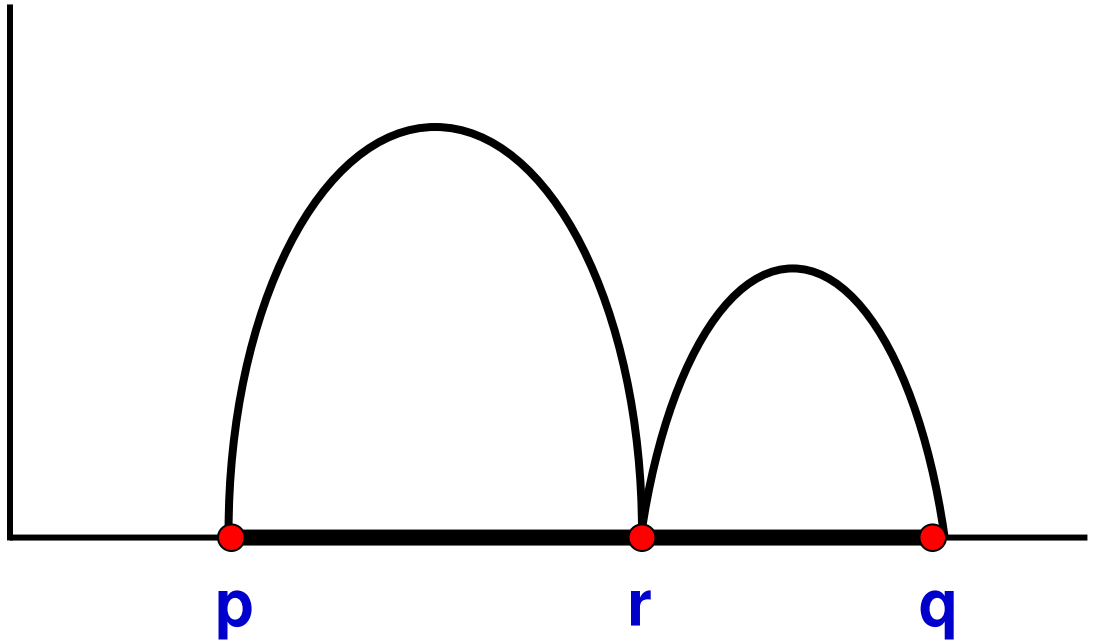
Case 1

$A_{pq} \rightarrow aA_{rs}b$



Case 2

$A_{pq} \rightarrow A_{pr}A_{rq}$



Recap of Proof Ideas

- A_{pq} = variable generating all x that takes P from (p, ε) to (q, ε)
- Formal construction:
  - $V = \{ A_{pq} \mid p, q \in Q \}$
  - $S = A_{q0qf}$
  - R defined as follows:
    - $A_{pp} \rightarrow \varepsilon \qquad \forall\, p \in Q$
    - $A_{pq} \rightarrow a\, A_{rs}\, b \qquad \forall\, p, q, r, s \in Q$
      s.t. $(r, t) \in \delta(p, a, \varepsilon)$
      $(q, \varepsilon) \in \delta(s, b, t)$
    - $A_{pq} \rightarrow A_{pr}\, A_{rq} \quad \forall\, p, q, r \in Q$

[Proof sketched on whiteboard, see textbook for details]

CFL Pumping Lemma

Let L be a context-free language

Then there exists p such that
for all $w \in L$ and $|w| \geq p$

we can write $w = uvxyz$, where:

1. $uv^ixy^iz \in L$ for any $i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

“Surgery” on Parse Trees

Idea: If w is long enough, then any parse tree for w must have a path that contains a variable more than once