CS 172: Computability and Complexity

SAT: Typical-Case Complexity

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k-SAT

- A SAT problem with input in CNF with at most k literals in each clause
- Complexity for non-trivial values of k:
  - 2-SAT: in P
  - 3-SAT: NP-complete
  - > 3-SAT: ?

3-SAT: Complexity Bounds
(circa 2006/07)

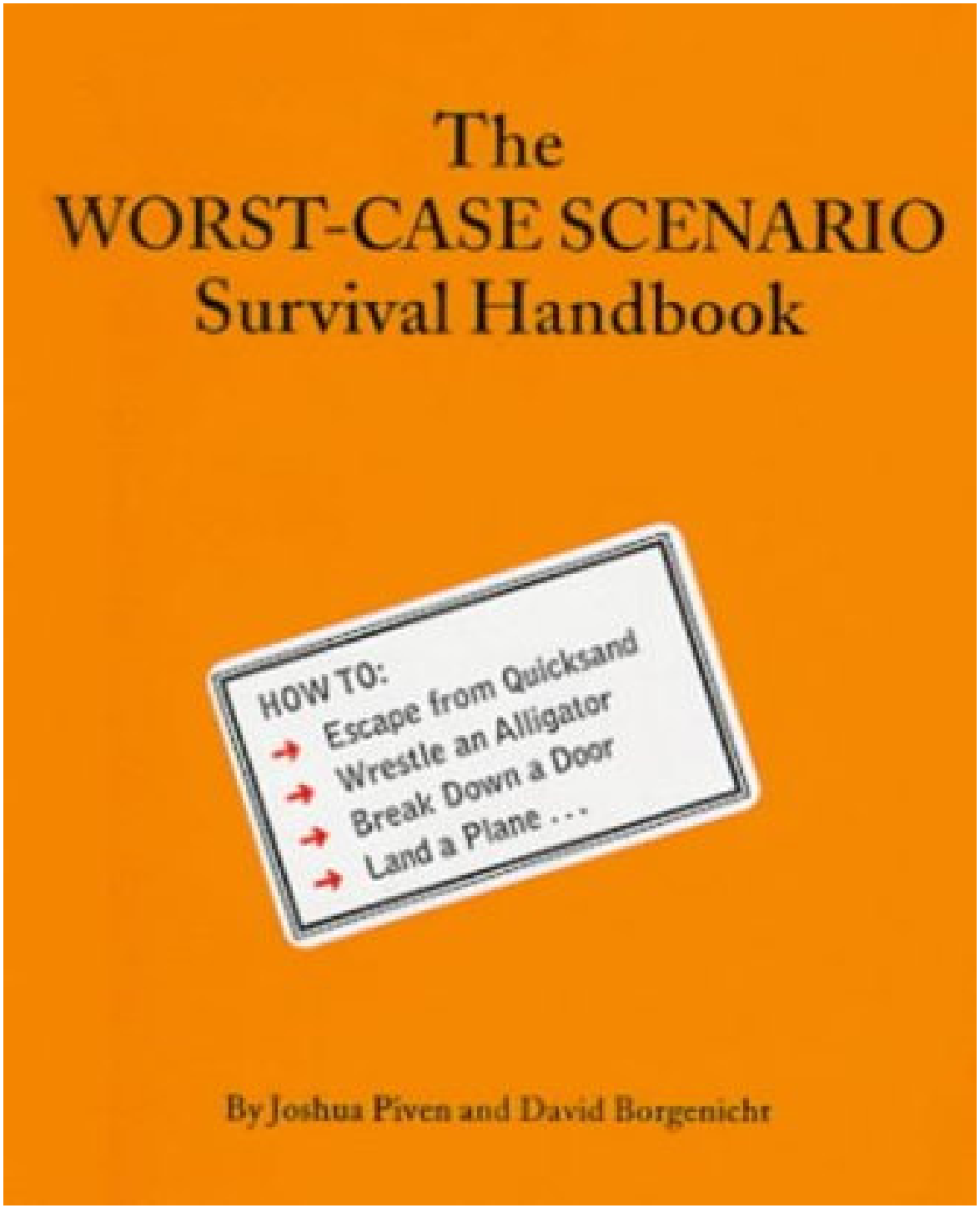
- Obvious upper bound on run-time?
  - In terms of n, the number of Boolean variables
- Best known deterministic upper bound

  1.473^n
- Best known randomized upper bound

  1.324^n
- Best known lower bound

  $n^{1.801}$
  - If limited to sub poly space

Worst-Case Complexity



Beyond Worst-Case Complexity

- What we really care about is “typical-case” complexity
- But how can one measure “typical-case”?
- Two approaches:
  - Is your problem a restricted form of 3-SAT? That might be polynomial-time solvable
  - Experiment with (random) SAT instances and see how the solver run-time varies with formula parameters (#vars, #clauses, …)

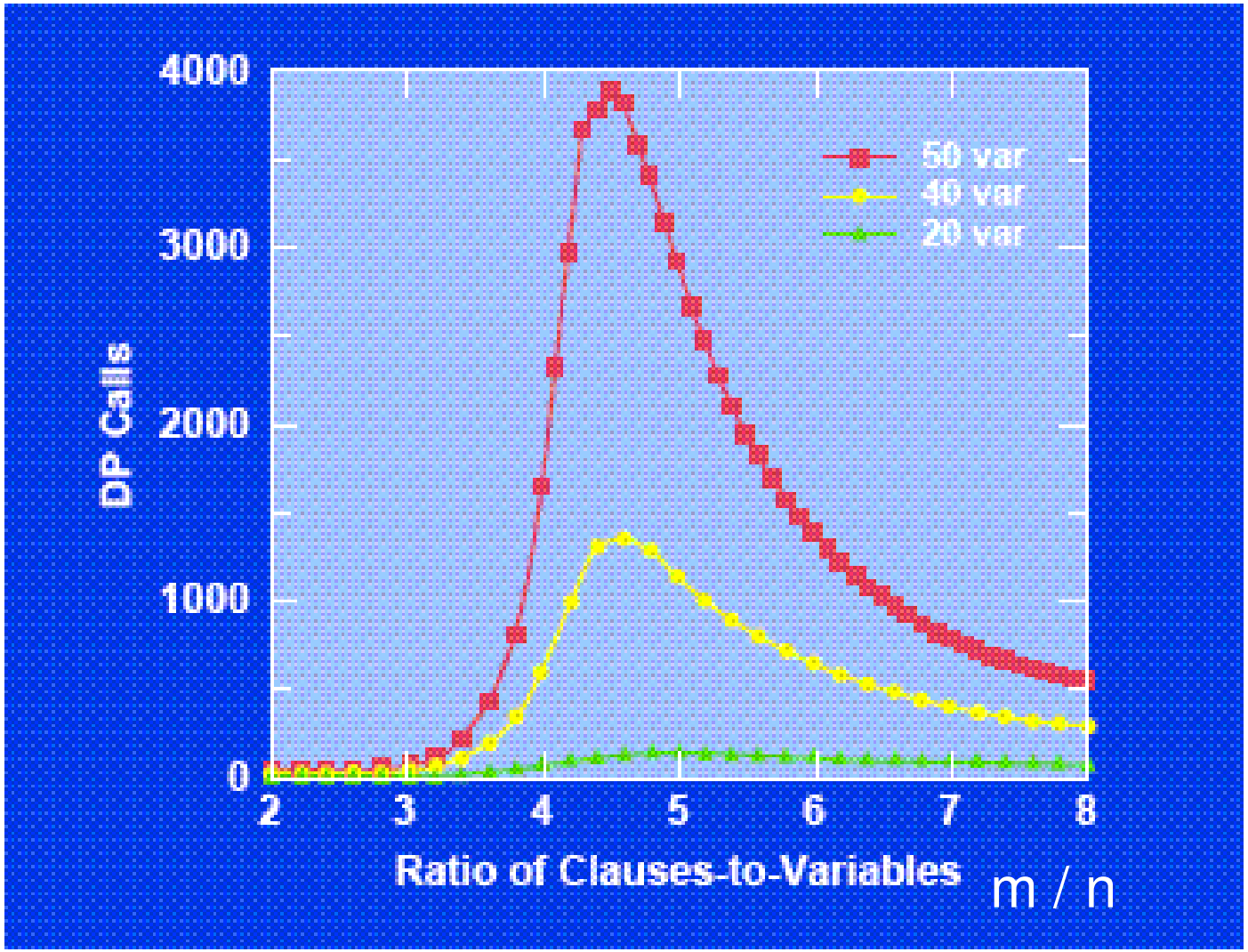
Special Cases of 3-SAT

- You already know two poly-time solvable cases:
  - 2-SAT
  - Horn-SAT

Phase Transitions in k-SAT

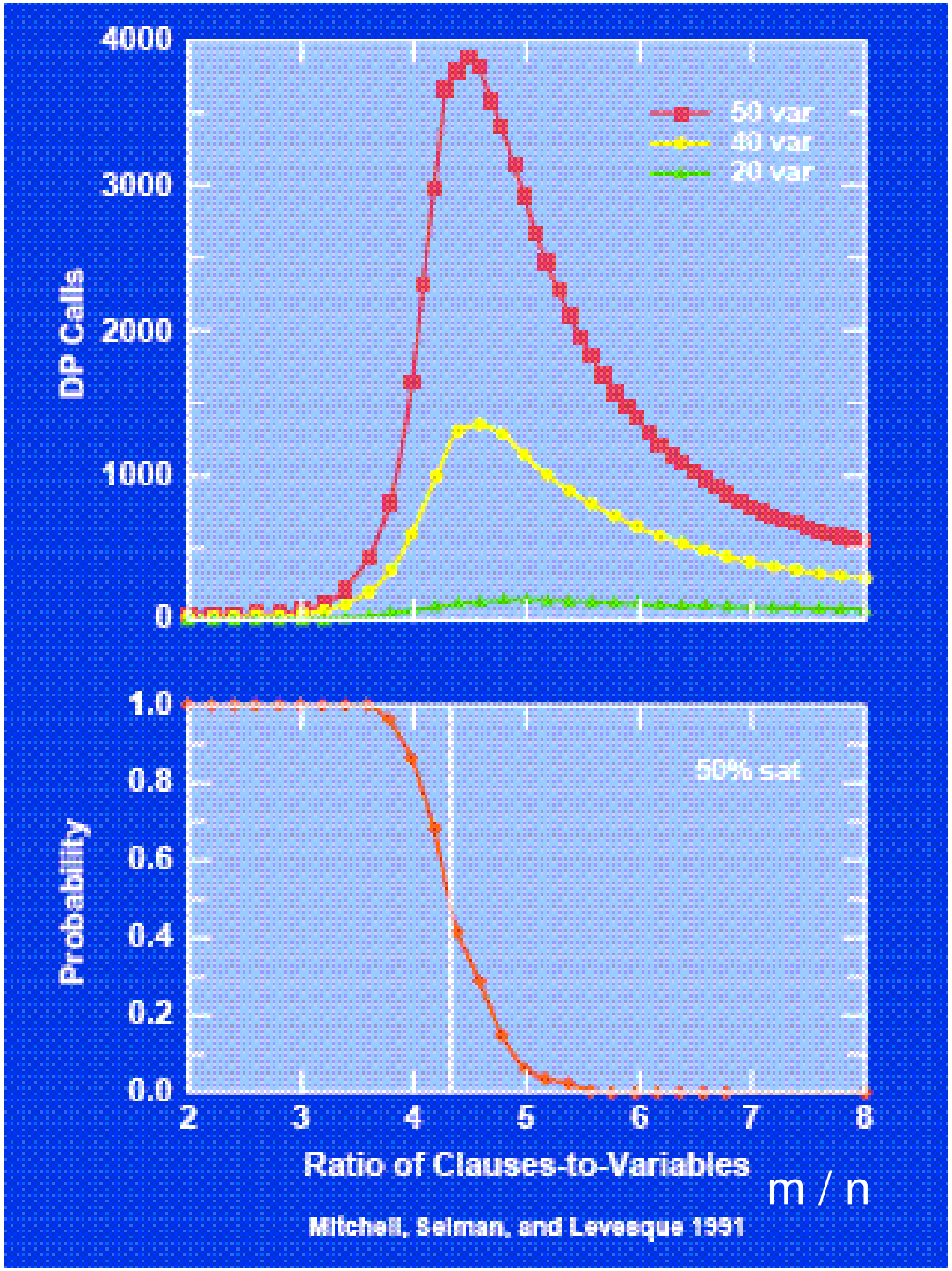
- Consider a fixed-length clause model
  - k-SAT means that each clause contains *exactly* k literals
- Let SAT problem comprise **m** clauses and **n** variables
  - Randomly generate the problem for fixed k and varying m and n
- Question: How does the problem hardness vary with m/n ?
  - Why is m/n interesting?

3-SAT Hardness



As n increases hardness transition grows sharper

Transition from SAT to UNSAT at m/n $\simeq$ 4.3



Threshold Conjecture

- For every k, there exists a c^* such that
  - For $m/n < c^*$, as $n \rightarrow \infty$, problem is satisfiable with probability 1
  - For $m/n > c^*$, as $n \rightarrow \infty$, problem is unsatisfiable with probability 1
- Conjecture proved true for $k=2$ and $c^*=1$
- For $k=3$, current status is that c^* is in the range 3.42 – 4.51

The (2+p)-SAT Model

- We know:
  - 2-SAT is in P
  - 3-SAT is NP-complete
- Problems are (typically) a mix of binary and ternary clauses
  - Let $p \in \{0,1\}$
  - Let problem comprise (1-p) fraction of binary clauses and p of ternary
  - So-called (2+p)-SAT problem

Experimentation with random (2+p)-SAT

- When $p < \sim 0.41$
  - Problem behaves like 2-SAT
  - Linear scaling
- When $p > \sim 0.42$
  - Problem behaves like 3-SAT
  - Exponential scaling

Backbones and Backdoors

- Backbone [Parkes; Monasson et al.]
  - Subset of literals that must be true in every satisfying assignment (if one exists)
  - Empirically related to hardness of problems
- Backdoor [Williams, Gomes, Selman]
  - Subset of variables such that once you've given those a suitable assignment (if one exists), the rest of the problem is poly-time solvable
  - Also empirically related to hardness
- But no easy way to find such backbones / backdoors! ☹

A Classification of SAT Algorithms

- Davis-Putnam (DP)
  - Based on **resolution**
- Davis-Logemann-Loveland (DLL/DPLL)
  - Search-based
  - Basis for current most successful solvers
- Stalmarck's algorithm
  - "Different" kind of search, proprietary algorithm
- Stochastic search
  - Local search, hill climbing, etc.
  - Unable to prove unsatisfiability (incomplete)

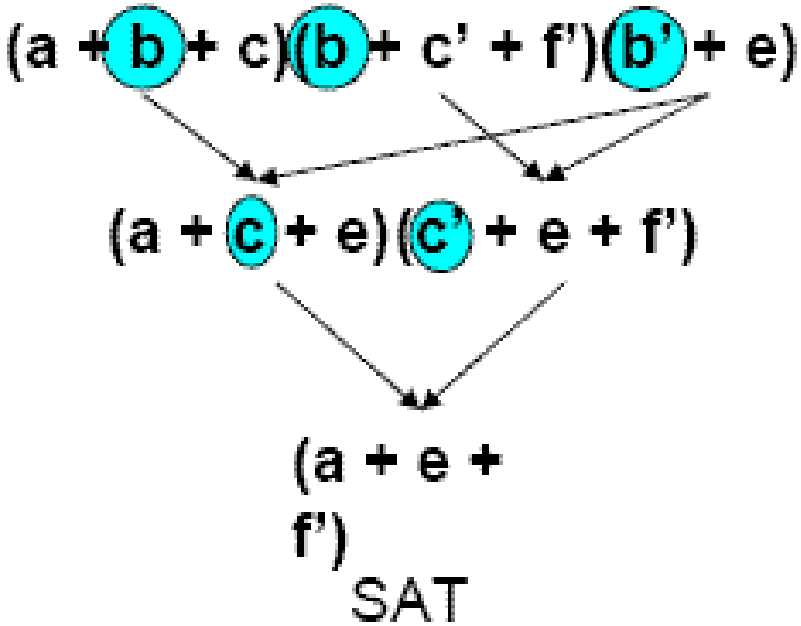
Resolution

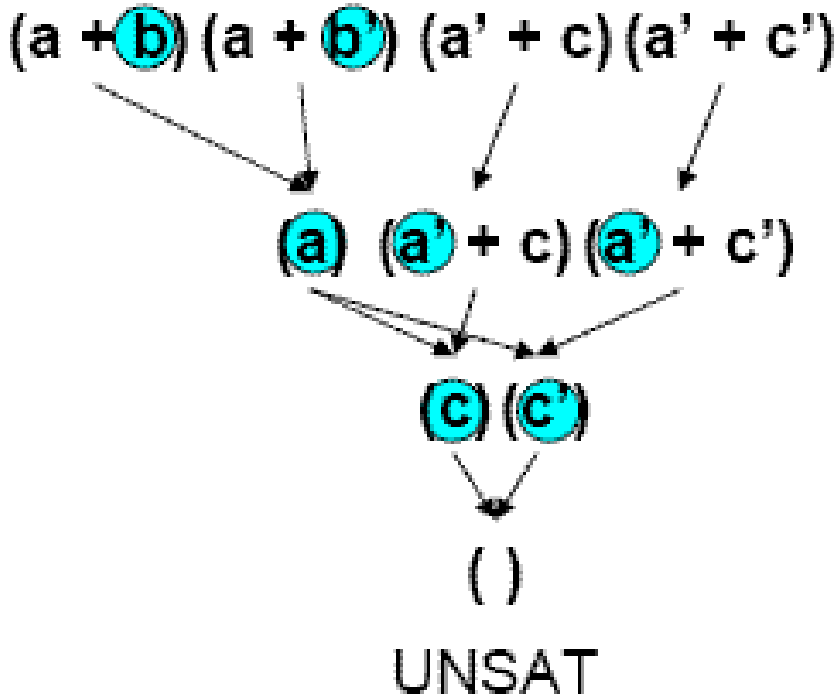
- Two CNF clauses that contain a variable x in opposite phases (polarities) imply a new CNF clause that contains all literals except x and x'
- $(a + b)\ (a' + c) = (a + b)(a' + c)(b + c)$
- Why is this true?

The Davis-Putnam Algorithm

- Iteratively select a variable x to perform resolution on
- Retain only the newly added clauses and the ones not containing x
- Termination: You either
  - Derive the empty clause (conclude UNSAT)
  - Or all variables have been selected

Resolution Example





How many clauses can you end up with?
(at any iteration)

Further Reading

- Handbook of Satisfiability, IOS Press, published 2009.
- Article on Phase Transitions in SAT (linked on website)