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Proper Rendering of the Pacioli Rhombicuboctahedron

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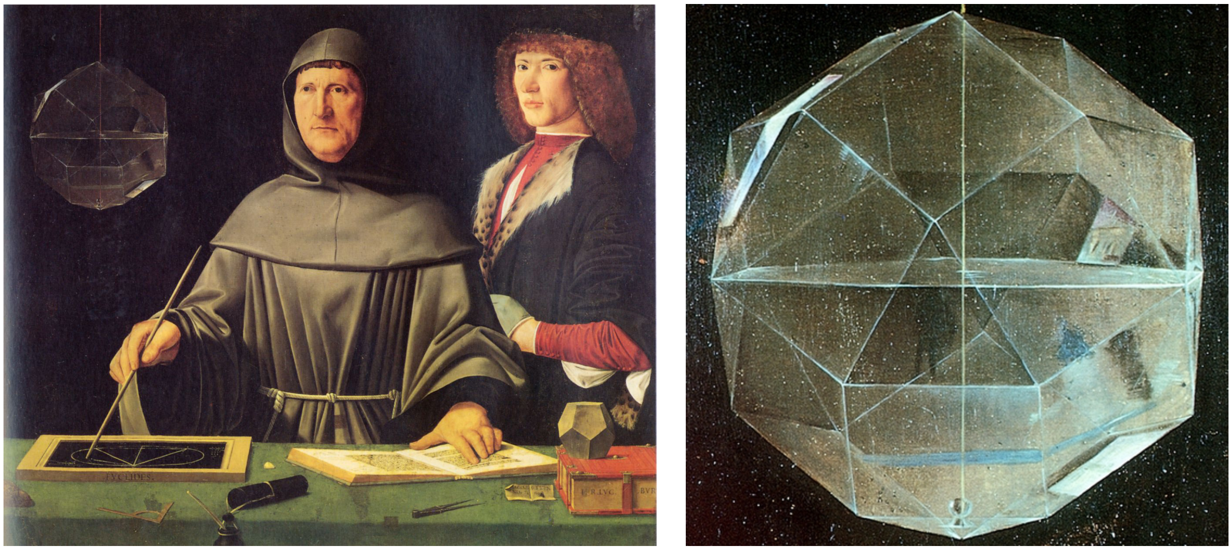
Abstract

The depiction of a glass rhombicuboctahedron (RCO), half-filled with water, appearing in a famous painting of Pacioli (1495) is analyzed, and deviations from a physically correct rendering are studied. A detailed computer rendering is presented that takes into account multiple internal and external reflections and refractions. Detailed rendering studies on simpler glass and water geometries were carried out to assure that the resulting rendering does indeed depict physical reality.

1. Introduction

The "rhombicuboctahedron" (RCO), which is the focus of our attention in this article, appears in the painting "Ritratto di Frà Luca Pacioli" exhibited in the Museo e Gallerie di Capodimonte in Naples, Italy. The painter is unknown, and there is some speculation that the painting is the work of more than one artist. The central person in Figure 1(a) is Frà Luca Pacioli, a famous mathematician of the Renaissance period. The RCO suspended at the top left is one of the thirteen semi-regular Archimedean solids. Its surface is composed of 18 squares and 8 equilateral triangles, and each vertex is shared by three squares and one triangle. The painting implies that this object has been realized with 26 glass plates fused together well enough, so that this shell can be filled with water up to its centroid.



(a) (b)

Figure 1: *(a) Pacioli painting; (b) Enlarged and enhanced view of the RCO.*

The painter seems to have captured the reflections and refractions in an exquisite manner (Figure 1b). This has prompted glowing comments from many admirers and art historians. Mackinnon [3] attributes this part of the painting to Leonardo da Vinci himself. George Hart on his page devoted to Luca Pacioli's Polyhedra states [2]:

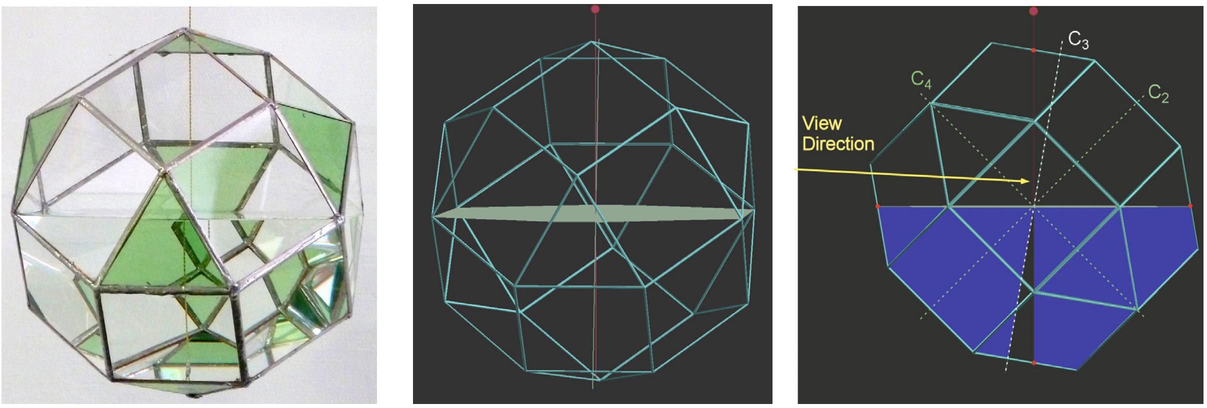
> *The polyhedron in the painting is a masterpiece of reflection, refraction, and perspective. (Davis states that the bright region on its surface reflects a view out an open window, showing the Palazzo Ducale in Urbino.) Certainly an actual glass polyhedron was used as a model. (Pacioli states in his books that he constructed several sets of glass polyhedra, but I know of no other information about them.) The polyhedron in the painting is beautifully positioned, suspended with a 3-fold axis vertical, out of physical contact with the other objects in the scene. I suspect that Pacioli chose it for the portrait because he discovered this form and was quite proud of it. (Presumably Archimedes first discovered it, but that wasn't known in Pacioli's time.) The painting is the earliest known image of the rhombicuboctahedron.*

2. Critical Observations

But on a closer, more critical inspection some things seem not quite right: The reflection of the (invisible) window in the upper left seems more like a pasted-on sticker image that bends around one of the edges of the RCO, rather than like two separate reflections in the two differently angled RCO facets [7]. We later found that already in 2007, Joost Rekveld raised some doubts in a publication [4] and on his website [5]:

> *Last week in Napoli I revisited the Capodimonte museum and its amazing collection of paintings. At some point I found myself face to face with this canvas, attributed to Jacobo de Barbari, a portrait of the mathematician Luca Pacioli painted in 1495.*
>
> *... In said painting I suddenly noticed the mysterious reflections in the gorgeous mathematical shape at the top left, a rhombicuboctahedron, made of glass and half filled with water. I took a picture, and when I zoom in on these painted reflections we see buildings and sky, as if to suggest that an open window in the room is being reflected in the glass facets. The direction doesn't seem right though; am I wrong or do the reflections show that the window is high up towards the left? Or rather towards the bottom right? Both do not really make sense. Also in either case the light in the painting does not seem to be much affected by the source of these reflections.*
>
> *... To me the way the same window seems to be reflected three times within this shape doesn't seem terribly true to the laws of optics. Or more precisely: they seem true to a textbook notion of reflection and refraction of light in water, but the way the images are formed doesn't seem very realistic at all.*
>
> *... Its depiction doesn't seem based too much on observation or the tracing of reality through a camera obscura, but then again, I've never actually seen a glass rhombicuboctahedron half filled with water, so who am I to judge?*

Thus it seems worthwhile to investigate these issues. In May 2011, Claude Boehringer [1], an artist working with various materials, produced a first physical glass model of an RCO shell held together with lead, which was strong enough to be half filled with water. Under the guidance of Herman Serras [6] the model was suspended in the proper place to obtain the same orientation as the polyhedron in the painting. Serras then took the photograph shown in Figure 2a. This image looks quite different from the one in the painting, and so it is difficult to draw conclusions about the realism of the depiction of 1495. The surroundings where this model was photographed are entirely different, and there are no dominant reflections of a single, brightly lit window.



(a) (b) (c)

Figure 2: *(a) Model by Claude Boehringer photographed by Herman Serras; (b,c) viewing geometry.*

Just as disturbing as the “pasted-on” reflection of the window discussed above is the fact that there are no visible effects of refraction in the water body! The RCO edges on the back surface appear in the painting in exactly the places where the computer rendering of a thin wire-frame object shows them. The Boehringer model (Figure 2a) confirms our intuition that those edges seen through the water body would be seriously altered in their rendered positions.

3. Viewing Geometry

There are other issues. If the RCO is indeed suspended along one of its 3-fold symmetry axes piercing the centers of the top and bottom triangles, then these two faces would be truly horizontal. In the painting we see both of these triangles from below, which would imply that the eye of the observer lies below the bottom of the RCO. On the other hand, the water surface clearly is seen from above, which requires an eye point above the center of the RCO. So, how is this RCO suspended, and where is the observer's eye?

Careful inspection of the enlarged view of the painted RCO (Figure 1b) shows that the water surface passes through 4 of the 24 vertices of the RCO and cuts the vertical, square facets on the left and right sides along their horizontal diagonals (Figures 2b, 2c). This implies that the suspension line must form angles of 45° with the planes of the two squares immediately in front and in the back of the top triangle. The C_3 symmetry axis deviates by 54.74°–45° = 9.74° from this suspension line. A few trigonometric calculations then reveal that this suspension line cuts the height line of the top triangle in the ratio $1 : \sqrt{2}$, i.e., a fraction of 0.4142 away from the top vertex in the painting. Herman Serras [6] also had calculated the distance of the suspension line penetration from the closest triangle vertex as $\sqrt{3} *(\sqrt{2}-1)/2 = 0.3587$ times the RCO edge length. Fortunately these two calculations agree. In addition, the water boundary intersects two more pairs of triangle edges at a fraction of $\sqrt{2}-1 = 0.4142$ away from the shared vertex (red dots in Figure 2c).

With the angle of suspension unambiguously resolved based on the boundary of the water level, we now can try to locate the eye point of the observer. The viewer must be looking towards the center of the RCO with a slight downward angle that must lie between 0° and 9.74°. The best match between an appropriately tilted computer model and the rendering in the painting indicates a downward angle of about 3° (Figure 2c).

Now let’s review this in the context of the complete painting. First we try to find the viewer's eye level with respect to the painting. Our intuition tells us that the observer’s eyes are about at the height of Pacioli's nose or eyes, and that the person on the right is looking slightly down on the viewer. This would place the view center properly above the water level. We can also try to find a horizon compatible with the objects on the table, and this is what we found:

1.) a vanishing point on top of Pacioli's head for the slate tablet;

2.) a vanishing point somewhat above the middle of the RCO (but way out to the left) for the open book (with substantial error margins);

3.) a vanishing point about half a head-height above the top of Pacioli's head for the red box (also with substantial error margins).

All of these vanishing points are higher than the center of the RCO, and this is compatible with our view of the water surface from above. But then, the fact that we can see the lower side of the bottom triangle implies that it must be slanted upwards towards the viewer. This confirms the tilted suspension of the RCO established above.

Now let's take a closer look at the projection used in the painting: Clearly there is some perspective involved; the back-face triangle (point down) is a few percent smaller than the size of the front triangle (point up). The same interactive computer rendering program that lets us find the best match for the view angles also lets us find the parameters for the perspective projection in which all the rendered edge crossings of the wire frame model best match the locations in the painting; this occurs when the eye is about 14 RCO diameters away from its center. Now, based on its position and comparing it to the hands and head of Pacioli, we may estimate that it is about 8 to 10 inches in diameter. Thus, based on its own perspective, it must have been drawn as it would look from about 10-11 feet away. But Pacioli in the painting seems only about 5 to 6 feet away from the viewer, and the object, if it is indeed above the table, would then be only 4 to 5 feet away. Thus the perspective of the rhombicuboctahedron is not compatible with the rest of the scene. The RCO should exhibit a much stronger perspective! This raises a strong suspicion that this RCO was drawn quite carefully, but separately, – perhaps from a model or from a geometrical construction – and subsequently copied into the Pacioli painting.

4. Detailed Computer Rendering of the RCO

So let's assume that the RCO was painted in a separate sitting, possibly in a room with an open window in the left wall, offering a view of a palazzo and some bright sky, and perhaps with some more local illumination coming from the lower right. Can we find an environment and some suitable material constants and illumination levels that will produce an image closely resembling the depiction in this famous painting?

Producing physically and optically correct renderings is far from trivial. Several attempts by Berkeley students as well as remote collaborators have produced only rather disappointing results, which did not closely resemble the painted RCO. It was not even clear that these rendering attempts were truly modeling a water-filled glass container in the shape of an RCO. Some efforts lead to fairly convincing results concerning the reflections, but left the refraction calculations incomplete. Others showed reflections as well as refractions, but it was not clear that they all appeared in the right place and with the right intensity. Therefore in this latest attempt, we took a very careful incremental approach to model each physical effect first with extremely simple geometries and in an environment that made it possible to trace every reflected or refracted feature through the RCO back to a location in the surroundings.

We have used Autodesk Maya [9] as our modeling tool and Mental Ray [10] as our rendering engine. A great deal of effort was devoted to understanding and verifying the correct behavior of these tools to the extent we needed them for our studies. The photo-realistic renderings were produced in a ray-tracing mode, where we could control the number of consecutive photon/matter interactions for all the rays cast. Since the water-filled glass RCO is a complex object in which multiple reflection/refraction events occur between different interfaces, we started out with simple prisms and thick glass plates immersed in a "Cornell Box" [8] with distinct, high-contrast textures on its inside walls. We started out with simple ray-casting, having disabled all secondary ray segments. Then we gradually increased the number of allowable reflections and/or refractions and watched extra features appear in the rendered image. This incremental approach allowed us to identify each newly appearing feature by its sequence of refractions and internal or external reflections. We also built a 2-dimensional simulator program that allowed us to see how different light rays travel through our test-setup and split into different reflected and refracted components. In Section 5 we describe some of these preliminary tests; and in Section 6 we then show this incremental approach applied to the complete RCO geometry.

5. Preliminary Rendering Tests

We started by studying the reflections and refractions occurring in simple rectangular prisms (thick glass plates). Our techniques employed to understand the rendered images included: Limiting the depth of the tree of reflection/refraction events, comparing side-by-side two images that differ in only one parameter; gradually and interactively varying the angle of orientation of the prism or of one of its surfaces; and creating meaningful backdrops (environments) to determine the final targets of the rays traced from the eye point. In this context, the Mental Ray documentation [10] proved very useful for understanding how rays move through different interfaces and for controlling the behavior of refracted/reflected rays at the depth limit.

5.1 Surface Reflections

Pure reflections on mirror surfaces are easy to understand and do not require intensive examination. But it was still useful to set up a demonstration that models the reflection of a single window across a convex polyhedral edge, to show how unrealistic the reflections are that were painted onto the top left of the RCO. For this demonstration we just use two mirrors sharing a common edge (Figure 3) and orient them so that the blue window in the left wall of the Cornell Box gets reflected across that edge. In Figure 3(a) the two facets are coplanar and the window is reflected as a single rectangle across the common mirror edge. In Figure 3(b) the dihedral angle between the two facets has been reduced to 170°. The reflection is now broken into two disjoint trapezoidal elements, which get sheared farther apart as the dihedral angle gets reduced further. In Figure 3(c) the dihedral angle comprises only 146°, and it becomes rather difficult to make one and the same window edge appear as a reflection in both mirror facets. Under the viewing conditions given for the RCO in the Pacioli painting, it is impossible to see a vertical window edge reflected in the two facets selected by the painter, which incur a dihedral angle of 144.74° between them.

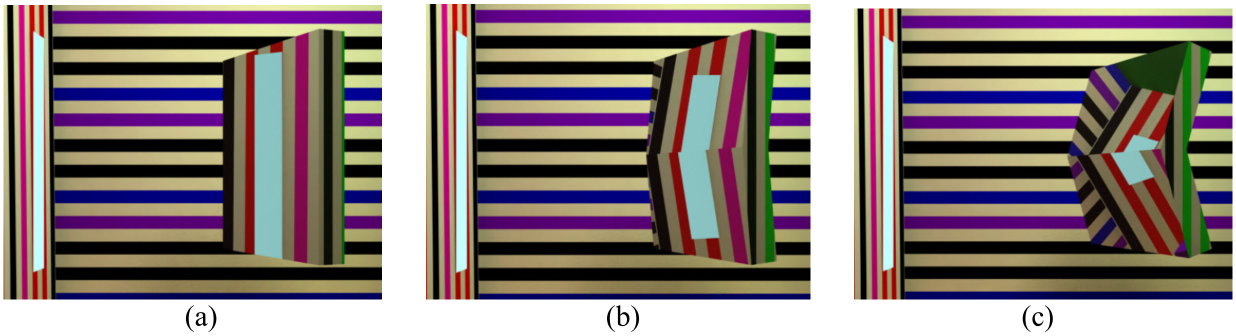

(a) (b) (c)

Figure 3: *Reflection studies of straight vertical contours across a common horizontal edge with varying dihedral angles: (a) 180°; (b) 170°; (c) 146°.*

5.2 Refraction in the Water

Simple views through parallel-plate glass blocks (Figure 6) or through a triangular prism, involving just two refractive events (one on entry and one on exit of the glass body), are also simple to model and to understand. Thus we could progress quickly to studying the primary effect of refraction in the water: i.e., the displacement of the locations of the edges in the back wall of the RCO shell. For this purpose we just used the geometry of the water body alone, properly surrounded by the filler framework. Figure 4(a) shows a non-physical depiction in which the water body has been given a bluish transparency, but has the same refractive index as air. The struts of the back surface of the RCO thus appear tinted, but in the same locations as they are seen in the empty filler framework (Figure 2b).

In Figure 4(b) water is assigned the known index (n=1.333), and the refractive processes are modeled properly; two levels of refraction are needed to see the opaque back struts; two consecutive refractive incidents are also sufficient to see the displaced stripes on the back wall of the Cornell Box. If we also include the glass plates (Figure 4c), then four consecutive refractive incidents are needed to see the back wall through the water-filled half of the RCO. Note that the back struts appear in quite different locations from Figure 2(b) and from what is seen in the painting (Figure 1b). This raises serious doubts that the painter actually had a chance to observe a glass polyhedron partly filled with water. It should also be noted how closely the geometry of the rendered back edges of the RCO in Figure 4(b) resemble the geometry in Figure 2(a), which gives us further confidence that our approach to modeling simple refractions is under control.

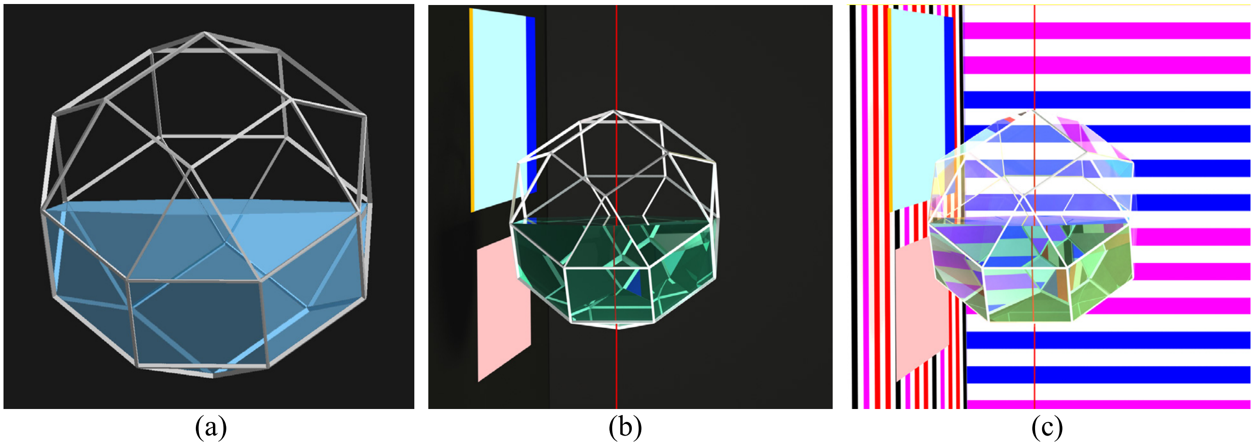
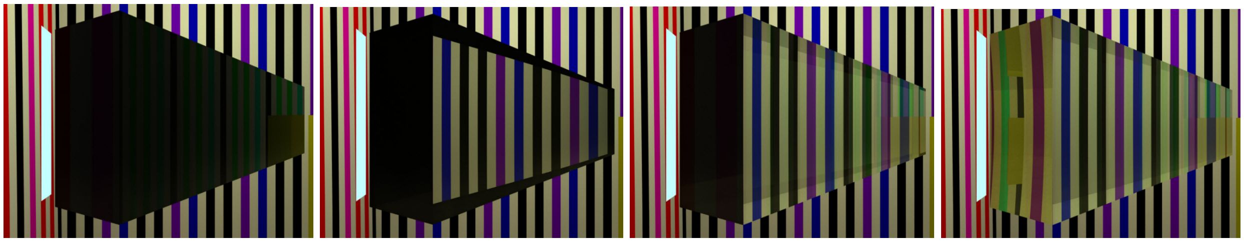

(a) (b) (c)

Figure 4: *Refractions in the water-body: (a) transparency with no refraction; (b) proper refractions in the water body showing displacement of the backside filler struts; (c) full set of refractions and reflections in the RCO inside the Cornell Box.*

5.3 Multiple Refractions and Reflections

Even in scenes of modest complexity, it is difficult to anticipate where certain features in the rendering will appear when there are multiple refractive incidents and possible total internal reflections (depending on the exact angle under which some interface is hit by the ray). Selectively restricting the number of reflection and refraction events admitted and processed is a crucial tool to understand what is going on. Figure 5 comprises a simple rectangular glass prism through which we can see the striped walls of the Cornell Box. In Figure 5(a) we have limited the ray tracing tree to just the first mirror event (M), and a faint reflection of the right wall can be seen in the large, frontal prism face. There are even fainter reflections (invisible in print) in the front left wall.

In Figure 5(b) we have limited ray-tracing to only the first two refraction events (R); this allows us to see through the prism and to observe the stripes on the back wall in horizontally shifted positions. In Figure 5(c) we allow one mirror event and two refraction events to take place; the most pronounced effect is total reflection on the top and bottom surfaces of the prism, after which those rays exit through the back face. There are also total reflections occurring in the right/back face of the prism, which then shows a thin red sliver (not visible in printed rendering) of the left room wall. Finally, in Figure 5(d) we allow as many as 8 mirror and 8 refractive events; this now allows rays entering the front/left face of the prism to exit again after multiple total internal reflections and to display stripes from the right-side room wall. The displayed bars that result from rays that undergo true prismatic refraction, i.e., exit from the glass in a direction different from the one that they entered, will show up slightly curved. The yellow patches seen in the left front face and faintly in the right front face result from a short yellow pillar we placed in the right corner of the room.



(a) (b) (c) (d)

Figure 5: *Refractions and reflections in a brick of glass with a varying depth of the ray-tree: (a) M=1; (b) R=2; (c) M=1, R=2; (d) M=8, R=8.*

5.4 Internal Reflections at the Glass/Water Interface

The real difficulties appear when we start to combine refractions and reflections. Main difficulties encountered in earlier modeling and rendering attempts were caused by the glass/water interfaces in the lower half of the RCO shell and by the handling of the possible total internal reflections at these interfaces. Often proper results cannot be obtained, if one just models a body of glass with a body of water next to it. It all depends on whether the ray-tracer will properly combine the exit event from the glass with the entry event into the water. If these two surfaces are not properly fused into a single interface, then under certain angles of incidence (more than 41.8° away from the normal) a total internal reflection in the glass will be introduced, based on the ratio of the refractive indices of glass (n=1.5) and vacuum (n=1.0). Such rays will then not be pursued into the water body, even though a proper calculation based on the ratio of the refractive indices of glass (n=1.5) and water (n=1.33) would allow rays to propagate, as long as they deviate from the interface normal by less than 62.5°. The situation is even worse when exiting from the water-body: At angles greater than 48.7° a ray cannot exit into air; but it can always exit into glass, since glass has a higher index of refraction.

To make sure that our rendering engine does the right thing, we stacked two thick transparent plates (half-cubes) of different materials behind each other and traced rays through this "sandwich" at various angles of incidence. We found that the most reliable way to obtain the proper results expected from physics is to model this composite of two plates with a single shared interface between the two half-bodies – rather than as two separate, self-contained bodies. This interface must then be properly tagged with the relative index of refraction between the two materials, i.e., the ratio of the individual indices on either side of the interface. This is demonstrated in Figure 6; all shadows have been suppressed for this investigation to minimize unnecessary clutter in the images.

In Figure 6(a), a simple, solid glass cube has been suspended in a Cornell Box with striped walls, with one edge of the cube pointing towards the camera. In both front faces, near the front edge we can see the horizontal stripes of the back wall; they result from rays that pass through the cube as if it were a thick glass plate, with the exiting ray parallel to the incident ray. In the top and bottom portions of these areas on the front faces we can also see bands that result from total internal reflections on the top and bottom faces of the cube. Roughly half-way towards the left and right outer edges of the two front faces, the images change abruptly into sets of vertical stripes; they result from rays that experience one internal total reflection on one of the back faces and then exit towards the opposite side wall through one of the cube's back faces.

In Figure 6(b), a water-body half-cube has been placed behind a glass half-cube. The display in the right front plane has not changed much, except that it has taken on a greenish tint because all the rays, whether internally reflected or not, pass also through the water half-cube. At the left end of the left front panel we can now see some of the stripes on the back wall. This could be because the lower refractive index of water creates a small zone where that edge of the water cube acts as a 90º prism. But it might also be that rays entering in this zone get reflected on the back wall, but then experience a second reflection at the water/glass interface and still exit towards the back wall from the water body. By judiciously controlling the number of allowed reflections (as in Section 5.3), and by verifying the ray path with our 2D ray tracker (Section 5.5), we could determine that the latter explanation is indeed correct. A little further inwards, the rays experience a total internal reflection on the back wall of the cube and then exit through the right back face of the cube towards the right wall.

In the front half of this left face, rays enter the glass half cube. All of them experience total reflection at the back glass wall – which is not acting as a proper interface to the water body. Again we see two subdomains: In the left domain rays experience a second internal reflection at the right front face and exit through the right back face towards the back wall. In the front-most domain the glass body acts as a prism with one internal reflection: Rays experience one total reflection at the back of the glass body and then exit through the right glass back face towards the right wall.

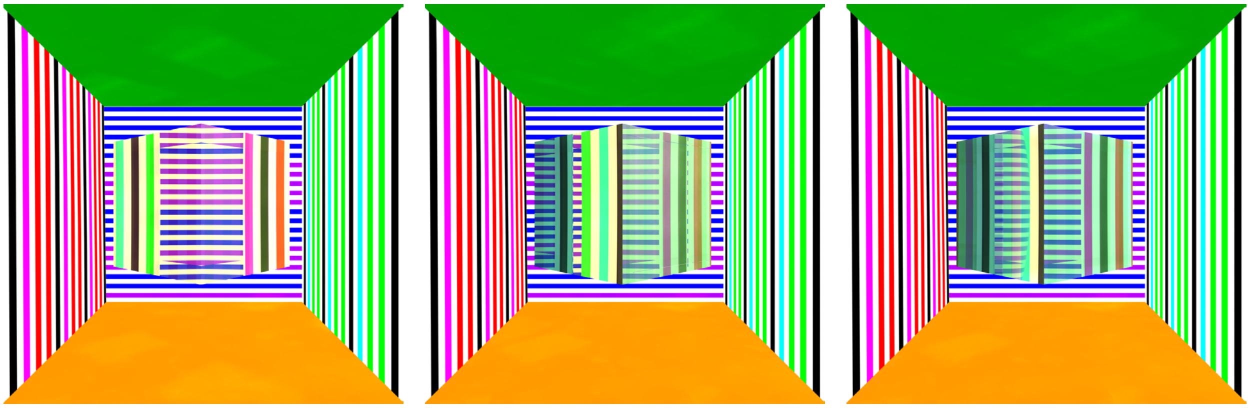

(a) (b) (c)

Figure 6: *Refraction through glass/water cube: (a) one solid glass cube; (b) a water-body half-cube behind a glass half-cube; (c) the two half-cubes properly merged with a single shared interface.*

Figure 6(c) shows the result when the glass/water interface has been properly merged. Nothing significant changes in the right front face, because none of the rays there have experienced an inappropriate internal reflection at the water interface. But in the left face we see some significant changes: In the front half, a wide swath of rays hit the glass/water interface at a steep enough angle, so that they can now exit through the water body and reach the back wall. In this domain, where we now see rays passing properly through the glass/water interface, just short of avoiding total internal reflection, we see distinct signs of the result being a superposition of two sub-trees of rays pursued by the rendering system: We are still seeing some weakened internal reflection showing the green and yellowish stripes of the right wall. Note that the boundary showing the onset of total internal reflection is curved, since the critical angle where this happens also involves the elevation angle of the ray. Meanwhile, in the back half of the left face, we see two sub-regions: The right sub-region demonstrates the same effect as seen in the corresponding region in Figure 6(b), whereas the left sub-region shows an entirely different effect. This is due to rays that experience one internal reflection at the back water face, transition through the interface, experience two more internal reflections at the back right and front right glass faces, transition again through the interface, and exit through the front left water face towards the left wall – in total, a series of 4 refractions and 3 reflections (Figure 7a).

Since it is important to obtain the proper physical behavior at these glass/water interfaces, let's briefly review one example of how a proper model can be generated in Maya [9]. A suitable process is called *Merge Vertices*. It joins non-manifold edges by matching coinciding vertices and then merging these vertices. For instance, our test "sandwich" with one shared vertical interface in the center can be modeled as follows:

1. Create two half-cubes of equal size and stack them behind each other.
2. Delete the back face of the front half-cube.

3. Combine the two half-cubes into a single mesh, by merging four coinciding vertex pairs.
4. Double check that all face normal vectors of the resulting mesh are oriented properly.

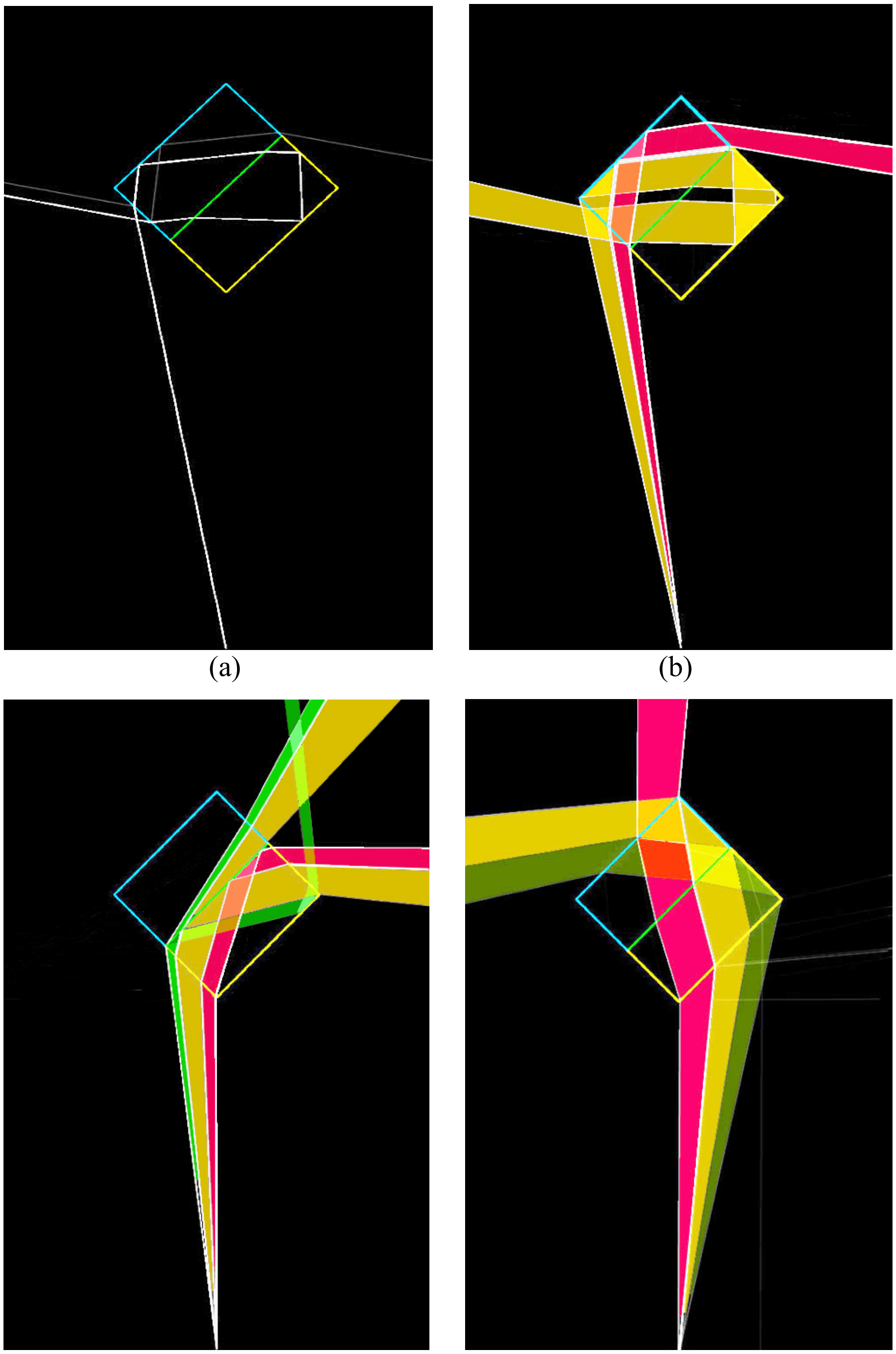
Maya then allows us to set the reflective, refractive, and diffusive coefficients separately for each material. It also provides a utility to use the Fresnel equations to determine the relative strengths of reflection vs. refraction depending on the incident angle of a ray, and it automatically ensures that light energy is conserved. In order to preserve proper physical behavior as expected from glass and water, we set the reflective and refractive coefficients equal for all materials used: glass-air interface; glass-water interface; and water-air interface. This is to prevent the water or glass surfaces from looking either super-reflective or not reflective enough, since the appearances of both, in actuality, are governed by the Fresnel equations. The Maya documentation shows how to create a glass material, suggesting precisely this relationship between reflective and refractive coefficients. Additionally, we tune the diffuse coefficient, which controls the fraction of light energy not devoted to reflection and refraction effects, so as to best match the slightly "dusty" look of the painted RCO. For a more detailed explanation, please refer to the Mental Ray [10] documentation, section 2.6.2.6.

5.5 Two-Dimensional Ray-Path Visualizer

To further verify the correctness of our environment and material settings in Maya, we independently built a 2-D visualizer application to show the path of reflected and refracted rays, as they interact with different transparent bodies. The application launches rays from a single location corresponding to the lens of the camera and sweeps them through an angular range that allows us to sample the whole width of the target. For the sample traces shown in Figure 7 we chose the camera and body locations, as well as the body's size, to be proportional to the environment seen in Figures 6(a-c), so that we can make meaningful comparisons. Blue lines denote a water/air boundary; green lines, a water/glass boundary; and yellow lines, a glass/air boundary. White/grey lines correspond to traced rays, and their brightness indicates the amount of light energy carried by that particular ray. We assume no light energy is lost by diffuse scattering, so that the rays' light energies are split between reflection and refraction purely based on the Fresnel equations. To reduce uninformative clutter, we set a minimum light energy threshold (1%) as a stopping condition for any refraction or reflection branch, after which we no longer trace or display that particular ray or any of its children rays. The main algorithm of the program is similar to that of most ray-tracers:

1.) If the recursion depth has been exceeded or the ray's light energy is below the minimum threshold, return.
2.) Recursively cast a ray and find the next collision point with some interface.
3.) With brightness proportional to the amount of light energy carried by the ray, draw a line segment starting at the ray's origin and ending at the collision point. If there is no collisions, choose the end point of the line segment to be the boundary of the simulator window and return.
4.) Use the Fresnel equations to calculate the percentage of light energy split between reflection and refraction.
5.) Initialize two new rays with origin at the collision point, representing the reflected and refracted components of the original ray after interacting with the interface. For each of these rays, store the amount of light energy carried and go back to step 1.

Figure 7 shows some consolidated outputs from our visualizer for the correctly merged glass/water cube. They show a horizontal plane through our scene, seen from above, with the camera at the bottom of the frames and the cube near the top. In Figure 7(a), we show the normal operation of the application tracing a single ray and its refracted and reflected sub-trees. In each of Figures 7(b) through 7(d), we show the result of merging multiple screenshots into a single composite between boundary angles where the rays undergo a serious change in the sequence of refractions and reflections; ray bundles experiencing the same series of reflection and refraction events share the same color. Using these composites, we are able to identify the wall that each bundle of rays ultimately ends up striking, and thus we can explain the appearances of each region seen in Figure 6(c). Our analysis confirms the explanations given in Section 5.4 for the behavior of individual rays within each distinct region.



(a) (b)

(c) (d)

Figure 7: *Tracing light rays through the correctly merged glass/water cube: (a) a single ray experiencing 4 refractions and 3 reflections; (b) ray bundles entering through the left water face; (c) ray bundles entering through the left glass face; (d) ray bundles entering through the right front glass face.*

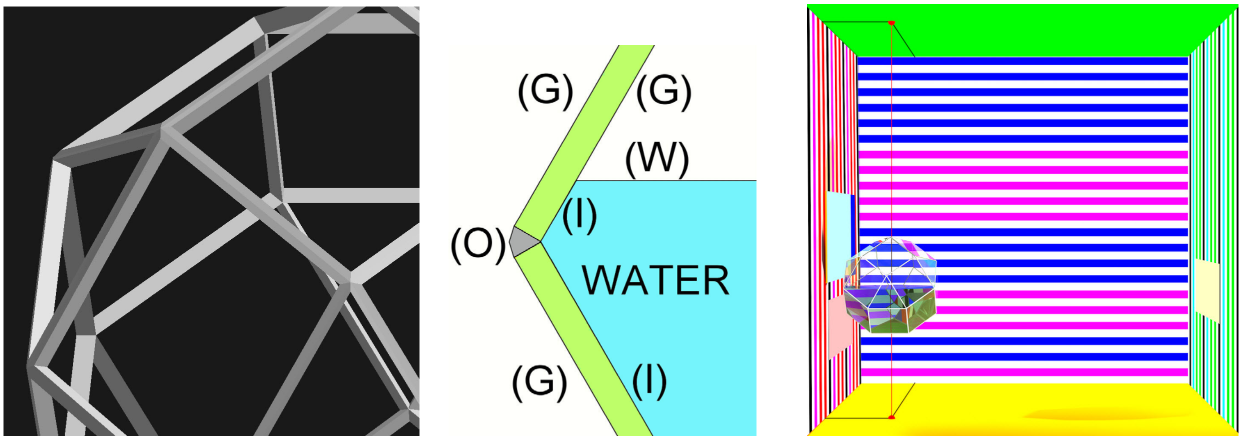
5.6 Tinting Due to the Passage Through Water

Maya also allows rays that travel through some medium to take on the color of that medium by specifying "Max Distance" and "Color at Max Distance" attributes for that material. For the sake of brevity, let us abbreviate "Max Distance" as *md* and "Color at Max Distance" as *color_md*. Thus, a ray, after entering a homogeneous body of material, will be attenuated continually by subtracting small fractions of the complement of *color_md*. At *md*, the ray's color will be the color the ray had before entering the medium minus the complement of *color_md*. This attenuation is equal to an exponential decay curve that starts at the ray's beginning color and passes through the point (*md*, *color_md*), and then continues past the specified *md,* until the ray exits the body. For more details, refer to the Mental Ray [10] documentation, section 2.6.2.5. While this is a gross simplification of the actual physical interactions between light rays and the medium that they are passing through, it was useful for producing convincing tinting within water (blue-green tint) and within glass (yellow tint).

6. Rendering the Complete RCO in the Cornell Box

Now we put it all together. The opaque filler material between the edges of adjacent glass plates is modeled as a separate "fattened wire-frame" object of genus 25 (i.e., 26 facet openings). Figure 8(a) shows an enlarged partial view of this framework; to make its geometry more visible, all the cross sections have been enlarged by a factor of 2. All these surfaces are opaque (O), and any ray that intersects this frame geometry is terminated at the intersection point.

All other surface and interface elements in the RCO are transparent and are rendered explicitly as elements belonging to one of 3 classes: (W): water/air (1 element); (I): glass/water (17 elements); and (G): glass/air (17 + 26 elements). These surfaces permit refraction as well as reflection processes to occur and, in general, may split an incoming ray of order *n* into two sub-rays of order *n*+1.



(a) (b) (c)

Figure 8: *Modeling the complete RCO: (a) Filler framework; (b) cross section showing the types of interfaces; (c) RCO inside a virtual Cornell Box.*

This RCO model is now placed inside a special virtual Cornell Box with distinctly striped walls (Figure 8c), so that the final stopping points of any rays being reflected on or refracted through the RCO can be identified more easily. The RCO is placed in the shown off-center position, so that a camera looking down the central axis of the Cornell Box will see the RCO under the angles indicated in Figure 2(b) while the suspension line shown in Figure 2 remains vertical and the front-most horizontal edge remains parallel to the back wall. The "windows" in the left wall have been positioned so that partial reflections of the light blue window appear in the two top/left panels of the RCO. In subsequent figures we will typically zoom-in onto the RCO alone to show reflections and refractions in greater detail, but the surroundings for these experiments are the ones shown in Figure 8(c). In this setting we now use ray-traced renderings with a controlled depth of the refraction/reflection event tree.

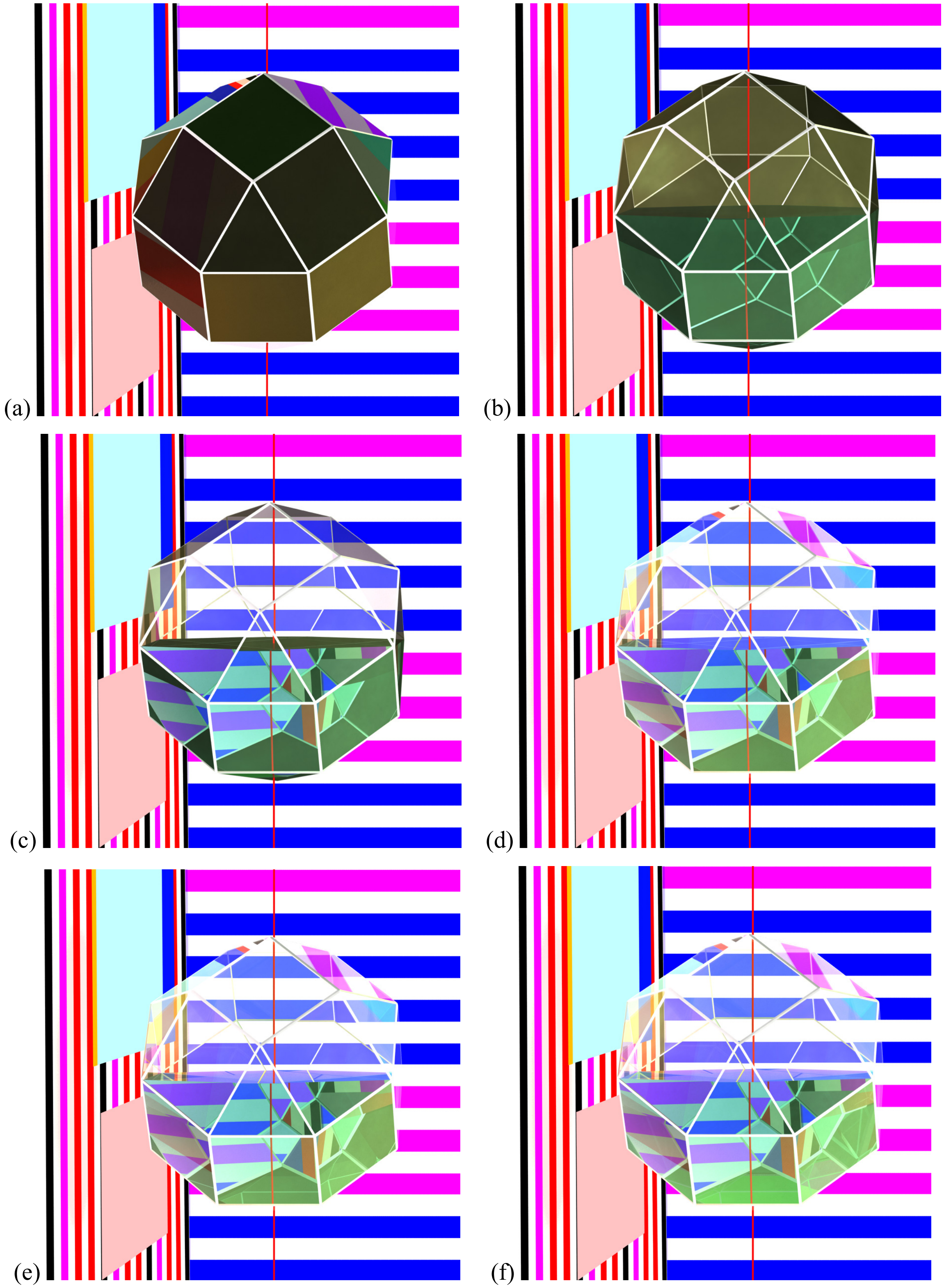


Figure 9: *Ray-tree limits: (a) $M=1$; (b) $R=3$; (c) $R=4$; (d) $M=1$, $R=4$; (e) $M=1$, $R=6$; (f) $M=R=10$.*

Figure 9(a) shows the result when we allow only a single reflection (M=1). Reflections of the walls can be seen most clearly in those glass panels that have a shallow grazing angle of incidence, e.g., the top two panels.

Figure 9(b) allows up to three refractions. Three refractions are necessary to allow the filler framework at the back to be seen through the water (Air / Glass / Water / Glass # Filler). No internal or external reflections are allowed. Note the displacement of the red string inside the water, compared with the location above water – a similar effect is observed with the filler framework.

With four refractions (Figure 9c) things get interesting, since rays can also exit from the back glass plate and thus produce images of the walls of the Cornell Box. Most prominently we now see the horizontal stripes on the back wall. We also can see a portion of the dark blue window back edge, and the adjoining red stripe, in the bottom center quad facet, to the right of the center of the facet. Some of the filler framework from the bottom of the RCO begins to appear on the water surface (not visible in the printed version, but in the online color display at 300% magnification), resulting from rays that were refracted through this top surface and then reached filler material in the bottom half of the RCO. Additionally, in the bottom right quad facet, we can now see a portion of the light blue window.

With four refractions and one reflection allowed (Figure 9d), we see again the surface reflections that were already visible in Figure 9(a). We also can identify an instance of total internal reflection: A light orange patch appearing in the upper right corner of the lower right face, resulting from one of the red stripes on the left wall. Moreover, the water surface that has remained mostly dark so far now reflects part of a blue horizontal stripe from the back wall as well as portions of the filler frame on the back of the RCO. We can also see the same faint additional lines that were visible in Figure 9c. Interestingly, the refracted images that previously appeared in the bottom-most triangular facet and right-most quad facet are now almost completely washed out by the strength of reflection at such a glancing angle.

In Figure 9(e) we add two levels of refraction and see some additional complex effects in the bottom facets of the RCO, which all contribute to the final image in Figure 9(f). The water surface also changes color due to the refracted component of rays hitting this top surface, which can now travel all the way through the bottom glass plates of the RCO to the orange floor (Figure 8c).

With Figure 9f we are trying to establish ground truth, using up to 10 levels of refraction and 10 levels of refraction. However, by fixing levels of refraction at 10 and varying the permitted levels of reflection, we saw a continuous increase in the number of features – in particular, some of the white filler struts continued “marching” forward, with new sections appearing, as the number of permitted reflections increased. Unfortunately, Maya’s ray-tracer limits us to a maximum of 10 levels of refraction and reflection, so this image probably does not show ground truth, but rather our best approximation. The primary differences between Figures 9(e) and 9(f) are seen in the bottom facets: Even more of the filler framework can now be seen. These features result from some very complex interactions, which are impossible to track by a human mind without computer assistance.

7. Rendering the RCO in an Environment Similar to the Pacioli Painting

Having thoroughly tested our rendering system, we can now rely on it to give us a depiction that is close to physical truth. Of course, there are still many parameters that need adjustment: the refractive index of glass; the amount of tinting imparted by the glass plates and the water body; and the amount of diffuse scattering on the outer surfaces – perhaps due to the presence of dust. We chose some of these parameters to make our rendering look somewhat similar to the depiction in the Pacioli painting, but we exaggerated some of the tinting to give a better understanding of the various visible effects. The main focus was on showing the geometrical issues related to the placement of the reflected and refracted geometrical features. In Figure 10 we have suspended the RCO in a darkened room with three bright, colored windows. The blue one was carefully placed so that portions of it appear in both the upper left panels of the RCO, where the original painting shows the primary window reflections. In the upper panel, we can see the dark blue back-edge of that window; in the lower facet, the yellow front-edge appears. This disparity in reflected window positions illustrates the physical impossibility of a single continuous image appearing across both faces, as depicted in the Pacioli painting.

A human artist standing at the location of the camera would appear in a mirror placed at the RCO as if the artist were standing an equal distance behind the RCO. Thus to a first approximation, the reflection of the artist should be about the same size as Pacioli or his companion. The small black reflection shown in the front triangular face was produced by a small puppet approximately 6 inches tall located in the plane of the camera, roughly 5 feet below and to the left of it. Thus, if the humanoid shape visible in the central triangle of the RCO (Figure 1b) is supposed to be a reflection of the painter, then this artist must have been very small indeed!

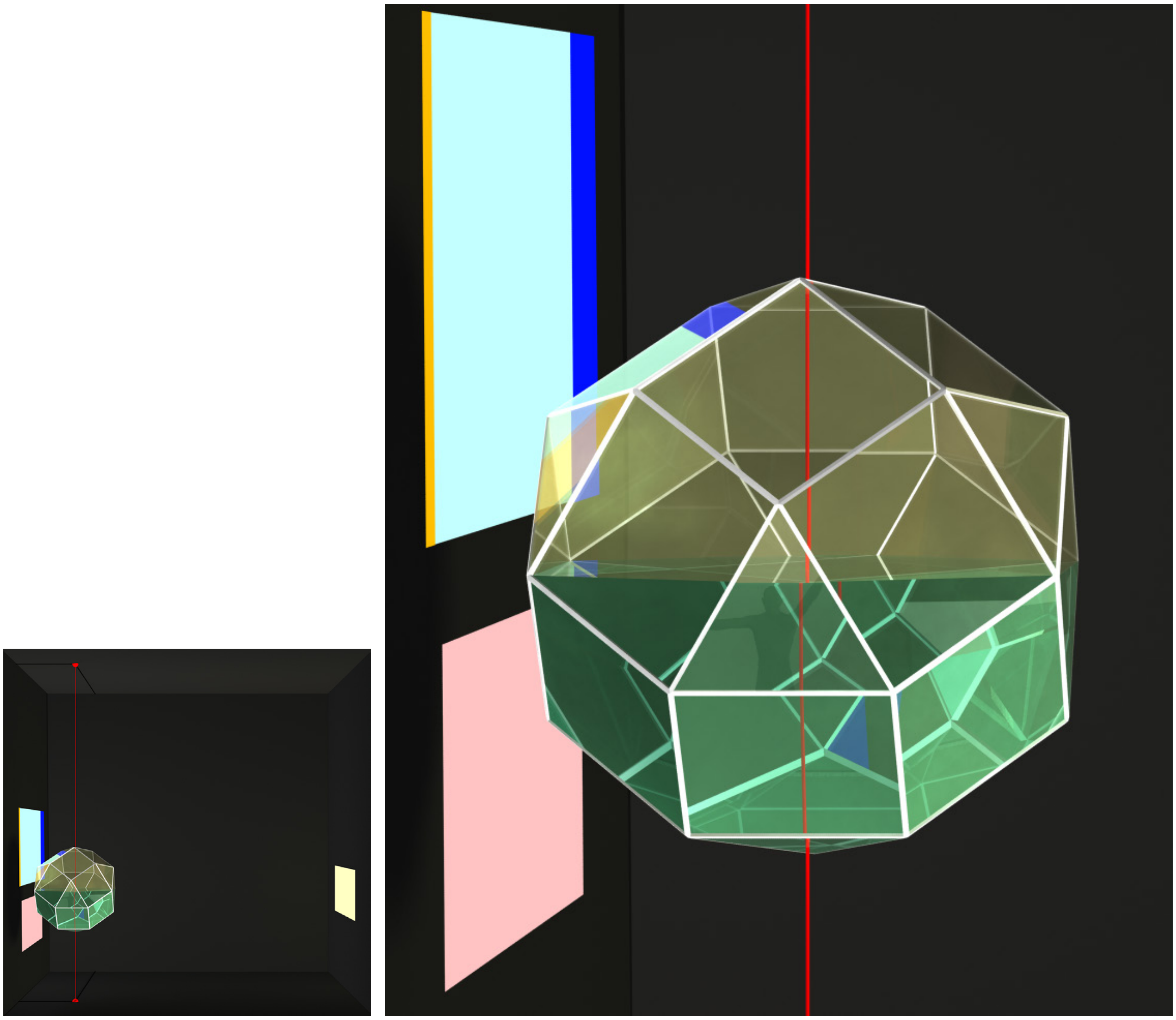

(a) (b)

Figure 10: *The RCO in a dark room with windows: (a) the basic set-up; (b) all-inclusive rendering.*

We also see that internal reflections appear in vastly different locations in Figure 10(b) compared with the painted RCO. For instance, we see portions of the pink window, light blue window, and dark blue window edge in the bottom right quadrant of the ray-traced RCO, but none of these have a similar appearance in the painted RCO, even accounting for the murkiness of the painted water. Conversely, the painted RCO shows some internal reflections that are not seen anywhere in the ray-traced RCO: above the water surface, in the back right center quad facet, and along the bottom right of the RCO. In attempting to replicate these extra internal reflections seen in the painted RCO, we expanded the windows to fill the entire left wall, but still we were not successful.

As mentioned before, the refracted locations of the string and the filler framework also appear in very different locations. In the painting, the string appears as a continuous line above and below the water surface, whereas in the computer rendering the string is discontinuous at the water surface. The same applies to the appearance of the filler framework; our ray-traced RCO exhibits framework locations strikingly similar to the physically realized model shown in Figure 2(a) – in contrast to the painted RCO. These ray-traced effects are caused by the different indices of refraction of air and water, and are what we would expect to see in a real physical model.

8. Discussion and Conclusions

Our analysis indicates that it is highly unlikely that at the time that the Pacioli painting was created any artist had the opportunity to observe a physical glass RCO partly filled with water. We are not aware of any evidence that indicates that in 1495 a technology existed that could have made a polyhedral glass container in the shape of an RCO strong enough to hold about three to four kilograms of water. On the other hand, it is quite likely that artists had access to an empty model made of triangular and square glass plates. It also seems quite plausible that the depiction of the RCO was first sketched or painted separately, and then later copied into the Pacioli painting; this would explain some of the discrepancies in the perspective projections of the RCO and its surroundings discussed at the beginning of this article.

The primary window reflection across the top left facets is so far off from any possible reality that it cannot have been drawn based on an actual observation. It may, however be possible, that the painter observed the general nature of some window reflections on various polyhedral glass models and decided that reality was much too confusing for most observers to yield a pleasing painting. The artist may then have made a conscious decision to render a window reflection that would be quite plausible to most observers, and which would be much more “understandable” than the actual broken-up mosaic of reflections. Moreover, to emphasize the transparency and 3-dimensionality of the RCO, two other internal reflections of the same window were added in two facets on the right hand side, one just above the water surface, and one on the bottom right in the water. The first one occurs where one would assume to see a secondary image, if this were the result of an intersection of a parallel beam of light coming through the assumed window. Based on the enthusiastic and complimentary comments about these reflections found on the web, the desired effect seems to have been accomplished spectacularly well.

Though we have focused in this article on the defects with respect to a perfectly realistic rendering of the RCO, we should consider that this was probably not the primary objective of the painter. The central star of this painting is clearly the figure of the famous mathematician Frà Luca Pacioli, not the RCO. The RCO is simply there to indicate the nature of some of the achievements of Pacioli, and to yield a nice and balanced overall composition. We would like to agree with many art historians that this goal has been achieved besides the discussed rendering flaws.

Acknowledgements

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