From Seifert Surfaces to Star Cinders

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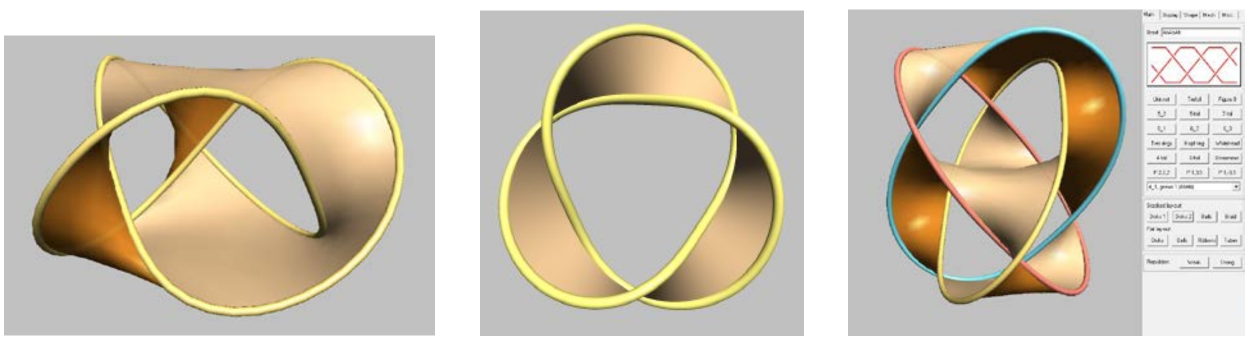
Abstract

Highly symmetrical "Orderly Tangles", composed of multiple liked loops or mathematical knots, are used as the supporting boundaries for single-sided or double-sided soap-film-like surfaces. The resulting shapes are called "Star Cinders" following an inspiring model by Charles O. Perry.

Introduction: Seifert Surfaces

Seifert surfaces [13] are intersection-free, orientable, minimal surfaces with mathematical knots or links as their supporting border curves. A program by J.J. van Wijk [14] makes it easy to explore such surfaces for simple knots and links and, with the right degree of smoothing, can result in attractive 2-manifolds. Figure 1a shows the example of a simple trefoil knot and its Seifert surface. Seifert surfaces, by definition, are required to be orientable. But, on most knots and links, it is actually easier to form non-orientable spanning surfaces of zero mean curvature, and they may be even more aesthetically pleasing. For instance, the same trefoil border curve can also support a triply twisted Möbius band (Fig.1b).



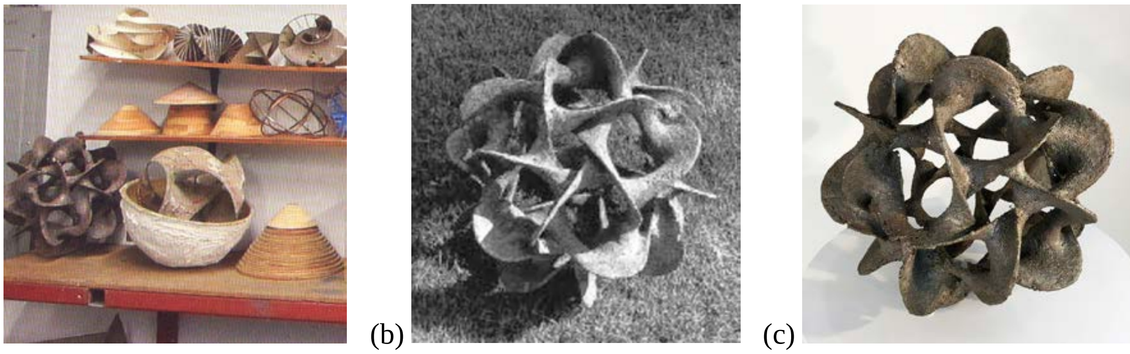
(a) (b) (c)

Figure 1: Minimal surfaces bounded by simple knots and links: (a) Seifert surface on trefoil knot; (b) non-orientable surface spanned by trefoil knot; (c) Seifert surface on Borromean rings [14].

Seifert surfaces can be constructed for any linkage of mathematical knots, but the geometry of the border curves must typically be changed dramatically, e.g., be flattened with just enough elevation changes for the crossings [15]. I am interested in designing highly symmetrical, aesthetically pleasing sculptures; so geometry is important, and the shape of the border curves must not change significantly during surface construction. On the other hand, whether the surface is orientable or not is unimportant in this context.

Introduction: "Star Cinders"

In Charles Perry's studio (Fig.2a) there is an 18 inch sand-cast bronze sculpture representing an intricate 2-manifold spanned among ten triangular loops arranged with icosahedral symmetry. A picture of this model has been published in Emmer's book [6], labeled as "*Star Cinder*" (Fig.2b). A better picture appears in Perry's book [7] on page 168; and a high-quality photo (Fig.2c) has been sent to me by Charlie's son, Paul Perry. These images inspired me to create more models of such surfaces with a high degree of "spherical" symmetries. This implies starting with "Orderly Tangles" [5] of simple knots based on the symmetries of the Platonic solids and then constructing a highly symmetrical 2-manifold bordered by these rim contours.



(a) (b) (c)

Figure 2: *"Star Cinder" (a) model in Perry's studio (p.92) [7]; (b) figure in Emmer's book (p.251) [6]; (c) photo supplied by Paul Perry.*

At Bridges 2018, I took a first step in this direction with my sculpture "*Tetrahedral Trefoil Tangle*" [9]. This sculpture is a 2-manifold suspended by four interlinked trefoil knots placed on the faces of a tetrahedron (Fig.3a). Figures 3b and 3c show two ways of suspending a soap film in this set of four border curves. These surfaces are not too difficult to construct. If the border curves are projected radially from the center of the tetrahedron onto a circumsphere, one obtains an almost 2-manifold depiction with distinct over/under crossings. The surface of the sphere is partitioned into regions with two or more sides, forming a pattern of projected edges that cross in valence-4 vertices. Such a pattern can readily be colored with two colors so that adjacent regions are always of different colors (Fig.3d). This then leads directly to two different solutions to form soap-film-like spanning surfaces. I call these "color-complements."

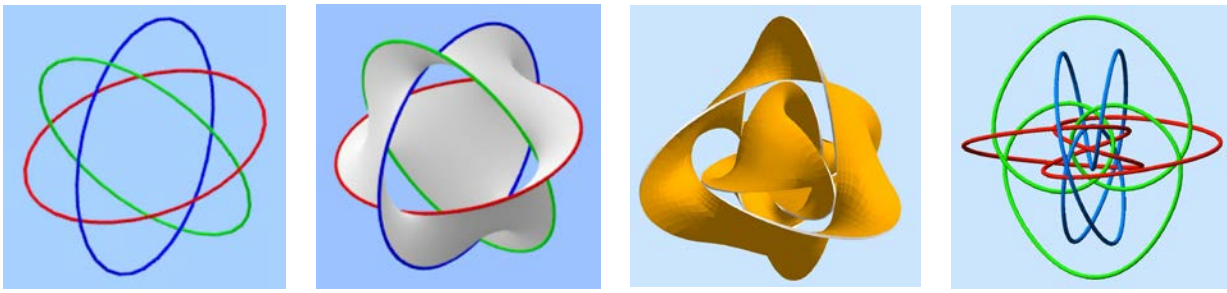
(a) (b) (c) (d)

Figure 3: *(a) Link of four trefoils; (b) a first soap film; (c) color-complement; (d) 2-color background.*

Filling the Void at the Center

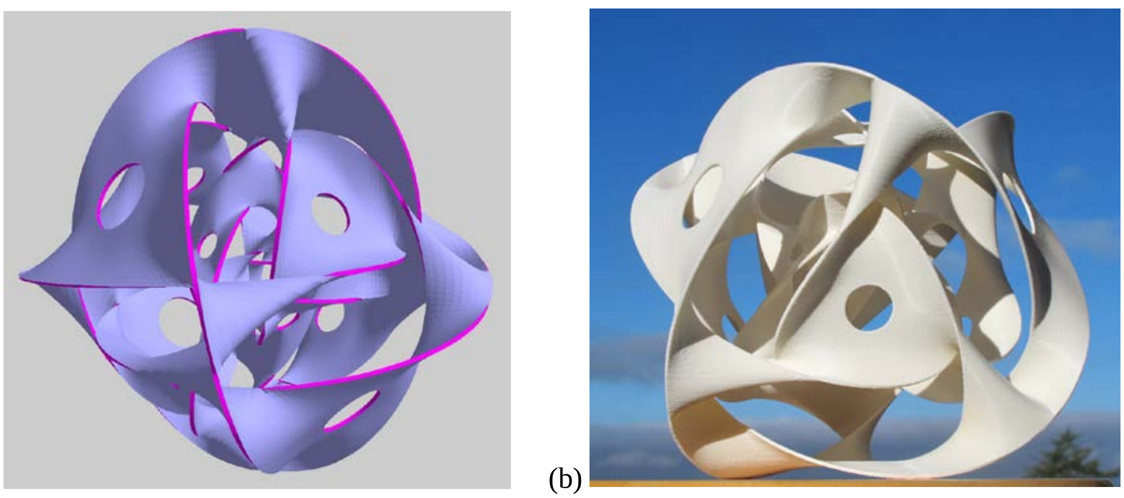
The above surfaces are spherical shells confined to about the outer 20 percent of their spherical volumes. I gave myself an additional goal to create 2-manifolds that extend further into the inner part of the sphere. To make this happen, I start by placing the border loops or knots into intersecting equatorial planes passing through the center of the sphere, rather than on the faces of a Platonic solid.

The simplest way to make such an "equatorial" link with the symmetry of a Platonic solid is to arrange three oval loops placed in the three main coordinate planes in a Borromean configuration (Fig.4a). This link has the symmetry of the oriented double tetrahedron (T_D, order 24). There are many possible ways to suspend a soap-film surface in this tangle. Ken Brakke identifies 15 topologically different possibilities [1]. But, only three of them are true, intersection-free 2-manifolds. Moreover, I am looking for maximal symmetry. The true Seifert surface (Fig.1c) has only D_3 symmetry (order 6), whereas the single-sided soap-film, shown in Figure 4b, has the symmetry of the oriented tetrahedron (order 12). The radial projection of the Borromean link onto the circumscribed sphere divides the sphere surface into eight identical 3-sided regions. My preferred soap-film surface (Fig.4b) fills in one set of four of these triangular regions that do not share any edges between them, i.e., they exhibit tetrahedral symmetry. All edge segments of the given link must be used exactly once by a single adjacent surface region, so that all edges maintain their roles as border curves of a true 2-manifold. The resulting surface breaks the mirror symmetry of the original border configuration; now it has the symmetry of the oriented tetrahedron. Filling in the other set of four triangular regions, thus forming its color-complement, will result in a mirror image of the first solution.



(a) (b) (c) (d)

Figure 4: *(a) The Borromean rings; (b) a symmetrical, single-sided soap-film surface; (c) two nested Boromean soap-films; (d) an orderly tangle of three (2,3)-torus knots.*



(a) (b)

Figure 5: *3-level Borromean surface based on three (2,3)-torus-knots: (a) CAD model, (b) 3D-print.*

However, the emerging soap-film surface still remains close to the circumsphere and leaves a big void in the center. One way to fill this void is to place a properly scaled-down copy on the inside. Figure 4c shows two such nested Borromean soap-films. But, I want the final *Star Cinder* to be a single 2-manifold with smooth border curves. For JMM 2019, I have created such nested sculptures exhibiting two and with three concentric shells [11]. It turns out that for three levels of nesting, integrating the three soap-films and joining and smoothing their border curves results in three interlinked (2,3)-torus knots clinging mostly to the three coordinate planes (Fig.4d) [10]. Figure 5a shows a complete CAD model of this surface, and Figure 5b is a photo of a corresponding 3D-print. The small holes in the centers of the triangular surface regions have been added to provide more transparency towards the inner parts of the sculpture.

Single-Shell Sculptures Formed by Equatorial Loops

Thus, one way of successfully filling in the central region of the sphere is by not just using simple equatorial loops, but by replacing those loops with suitable knots. I wanted to explore this approach for other tangles based on the symmetries of the Platonic solids. Beyond the Borromean link, there are three other "Orderly Tangles" [5] of regular *n*-gonal equatorial loops that nicely interlink with one another. I will briefly review these three tangles and the soap-films that can be supported by them. In the following section, I will then see what happens when these *n*-gonal loops are replaced with simple torus knots.

The next more complicated symmetrical linkage beyond the *Borromean Link* uses four triangles in a tetrahedral arrangement, where each triangle connects three midpoints of the edges of a cube. In 1983, I created a corresponding sculpture out of 4-inch diameter cardboard tubes (Fig.6a). The two-sided Seifert surface on this "*Triangle-Tangle*" consist of eight triangular surface regions (Fig.6b). The color-complement surface consists of six quadrilateral regions (Fig.6c); this is a non-orientable 2-manifold.

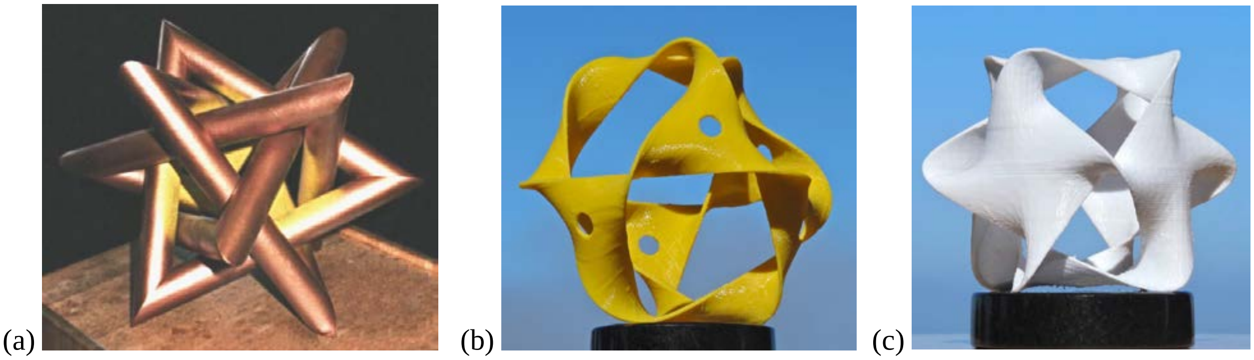


Figure 6: *(a) Tangle of four equatorial triangles. (b) Seifert surface; (c) its single-sided complement.*

Next follows an equatorial linkage of six pentagons in a dodecahedral configuration (Fig.7a). Figures 7b and 7c show the two ways of suspending a soap-film on this set of border curves. Both are single-sided because of the circuits of twisted connections with five or with three 180° flips, respectively.

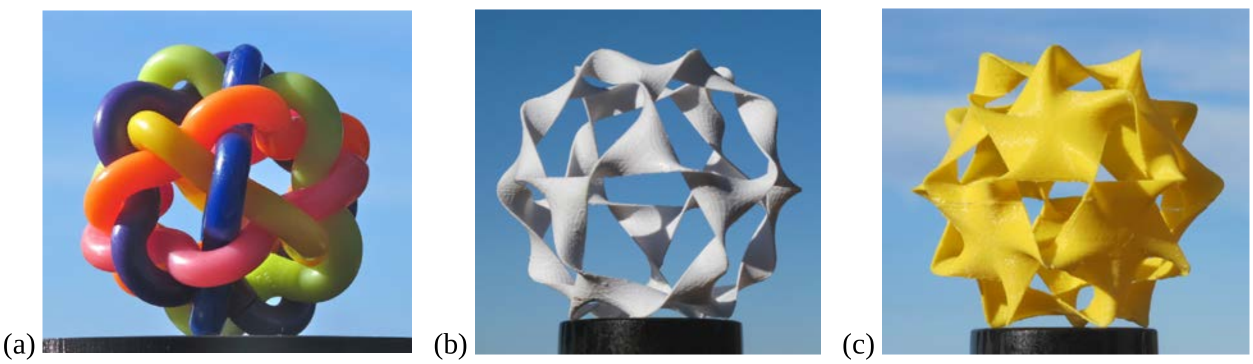


Figure 7: *(a) Tangle of six equatorial pentagons; (b) a soap-film surface; (c) its color complement.*

Finally, there is the orderly tangle of ten triangles in icosahedral symmetry (Fig.8a). This is the link of border curves for Perry's "*Star Cinder*" [6] [7], and Figure 8b shows a corresponding 2-manifold soap-film. Figure 8c shows the color complement. Both of them are non-orientable

Figure 8: *(a) Tangle of ten equatorial triangles; (b) a soap-film surface; (c) its color-complement.*

All the above 2-manifolds form a single "perforated" spherical shell or "crust," leaving the interior of the sphere mostly free. To construct 2-manifolds that more densely fill a spherical volume, I will now replace the simple loops in these three orderly tangles with torus knots of appropriate symmetry. The result will be a surface forming multiple nested shells with many radial connections between them.

Multi-Shell Sculptures Formed by Equatorial Knots

First, I start with the tetrahedral triangle tangle (Fig.6a) and replace each triangular loop with a simple trefoil knot. This results in a network of border curves that gets quite busy in the central part of the sphere (Fig.9a). It becomes quite challenging to actually create a highly symmetrical, proper 2-manifold in this region densely populated with border curves. The 2-colored background scheme, which was a helpful guide for the single-shell sculptures, no longer applies. Now the 2-manifold must branches through 3D space. The modeling environment NOME (Non-Orientable Manifold Editor) [4][16] is a big help in defining the connectivity of this surface. NOME is an emerging programming framework constructed with the help of several students. It allows the user to start with a text-based, procedural description of some key defining features, such as the knotted border curves, in a highly parameterized form. It then offers an interactive graphical interface to add some representative surface elements spanned by some segments of the border curves. Groups of manually placed facets can then be saved and subsequently iterated and symmetrically placed with a few additional statements in the original high-level, procedural description. Even after surface smoothing and offsetting, the original geometric parameters remain fully functional and allow the final sculptural shape to be fine-tuned for optimal results, while one is viewing a detailed CAD model.

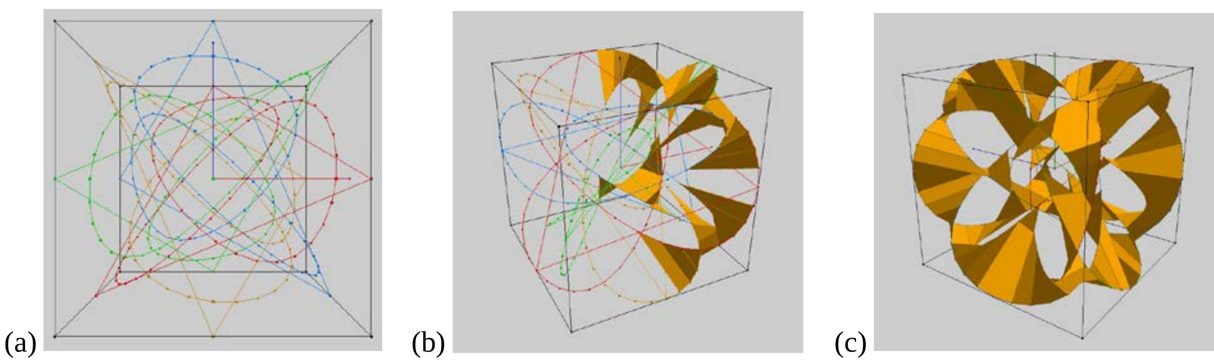


Figure 9: *(a) Trefoil border curves; (b) initial surface construction; (c) outer surface completed;*

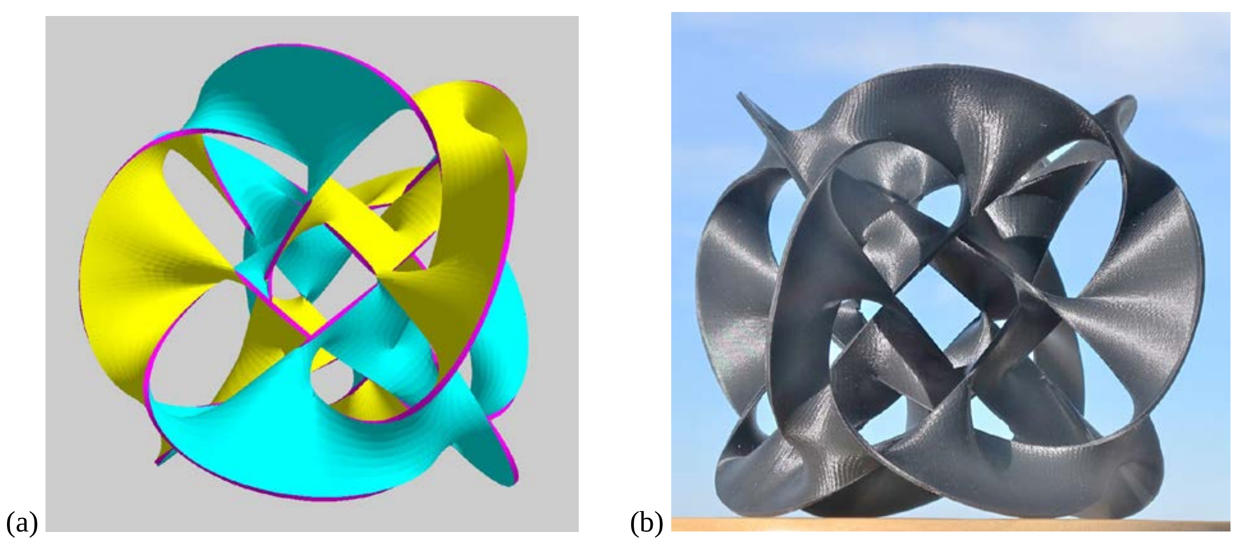
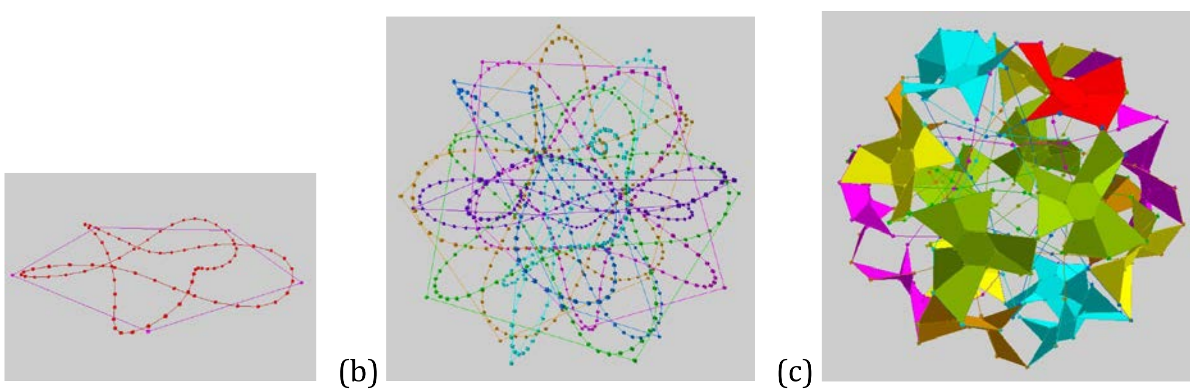


Figure 10: *"Tetra Star Cinder": (a) complete, orientable CAD model; (b) resulting 3D print.*

In the particular task at hand, I defined a relatively flat trefoil knot and then placed four copies perpendicular to the four space diagonals of a cube. Each knot curve is defined by 60 sample points (Fig.9a). This provides enough clickable vertices that allow me to build portions of the desired surface one quad-face at a time. The high symmetry of the figure lets me construct just a small portion of the overall 2-manifold, i.e., the flanges below the twelve outermost ribs (Fig.9b). These can then be replicated twelve times with the known symmetry transformations of the oriented cube to yield the complete surface. In Figure 9c the outer portions of the manifold have been completed; note the structural similarity with Figure 6b. This surface is also an orientable 2-manifold (Fig.10a).

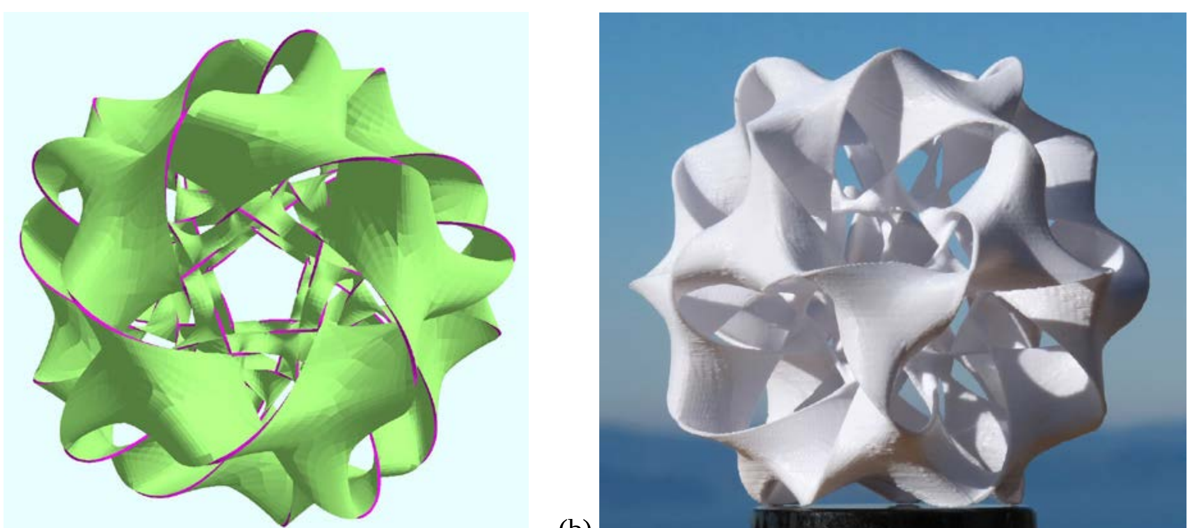
Once the inner portions of the 2-manifold have also been completed, the surface is smoothed with four steps of Catmull-Clark subdivision [3]. After the generation of two offset-surfaces (Fig.10a), the resulting boundary description can be saved as a .STL-file and sent to a 3D printer. Printing this model in black makes it look like this "*Star Cinder*" might indeed have come from outer space (Fig.10b).

Next, I apply the same approach to the dodecahedral tangle of six pentagons. In this case, I need to replace each pentagon with a knot with 5-fold symmetry. The (5,2)-torus knot, or cinquefoil, is an obvious choice, and I prepared a highly parameterized description of it (Fig.11a). Figure 11b shows the tangle of the six interlinked cinquefoils. To construct the complete 2-manifold has become even more daunting! Again, I start construction from the outside, taking Figure 7b as a guide. With this outer framework solidly in place (Fig.11c), it was then not too difficult to extend the 2-manifold towards the center in a consistent and symmetrical manner. Figure 12a shows the complete CAD model, and Figure 12b shows a 3D-print.



(a) (b) (c)

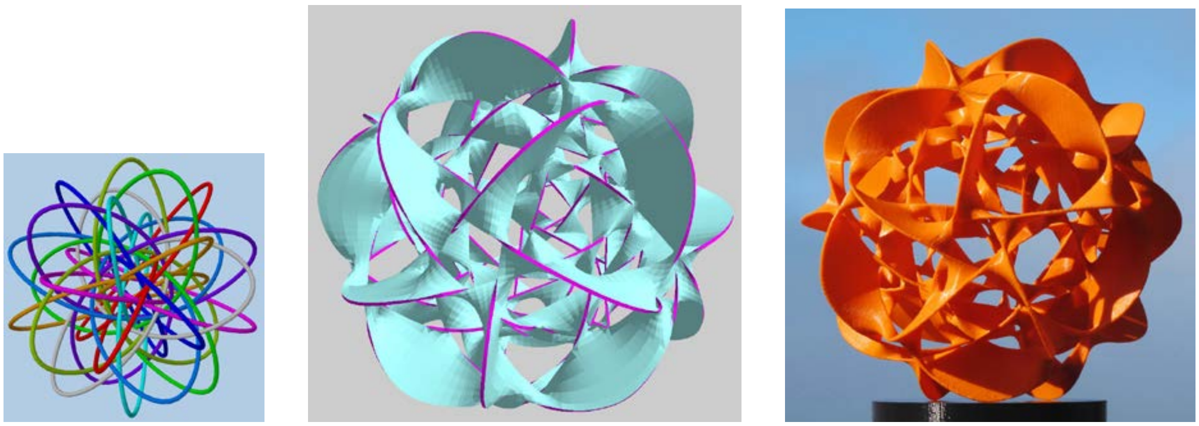
Figure 11: *(a) Parameterized cinquefoil; (b) six linked cinquefoils; (c) initial surface construction.*



(a) (b)

Figure 12: *"Dodeca Star Cinder": (a) complete CAD model; (b) 3D-print.*

Next, I tackle the tangle of ten triangles. In this case, I need to replace each triangle again with a trefoil knot. Figure 13a shows the tangle of the ten interlinked trefoils. Figure 13b shows the completed CAD model, and Figure 13c depicts the resulting 3D-print. The surfaces in Figures 12 and 13 are single-sided, since they all contain circuits with an odd number of 180° flips between adjacent faces.



(a) (b) (c)

Figure 13: *(a) Ten trefoils; (b) "2-Level Icosa Star Cinder" CAD model; (c) 3D-print.*

This is only the beginning of many more possible "Star Cinders." The trefoil knots could be replaced with a (3,3)-torus knot, which corresponds to three linked circles (Fig.15a). This then transforms Figure 9a into an orderly tangle of 12 circles (Fig.14a) and results in a 3-level shell structure with octahedral symmetry (Fig.14b). Doing the same thing for Figure 13, results in a 3-level icosahedral structure (Fig.14c).

(a) (b) (c)

Figure 14: *(a) Link of 12 circles; (b) cubic 3-level shell structure; (c) icosahedral 3-level structure.*

Discussion and Conclusions

There is no clear end to this line of investigation. I can readily choose torus knots with more turns around the center, such as the (3,4)-torus knot (Fig.15b) or even use an alternating Turk's Head knot (Fig.15c). But the constructed tangles are already quite dense in the center, and making this area even busier seems counterproductive. Alternatively, one could also use the even simpler trivial (n,1)-torus knots. These are basically just undulating n-gonal loops (Fig.15d). By enlarging the minor radius of such a knot, one may also bring sufficiently many border segments near the center of the sphere to obtain the stated goal of creating "volume-filling" "*Star Cinders*." For all of these border curve configurations, there is more than one way to suspend a 2-manifold that respects the given symmetry. As one example, in the 2019 Bridges Art Show I exhibit a 5-level shell structure suspended by a cluster of 30 circular border curves.

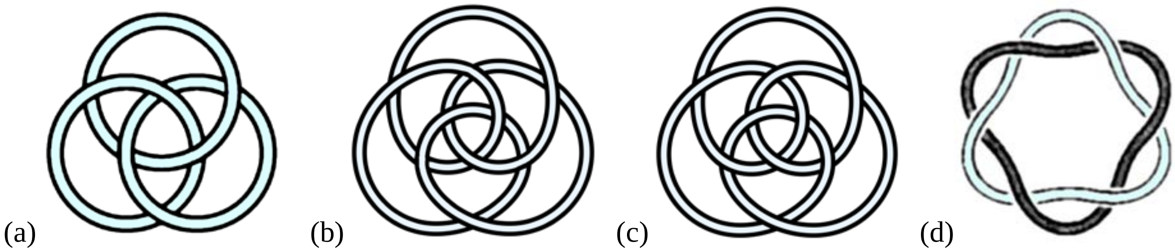


Figure 15: *Alternative border curves : (a) TK(3,3) = three linked circles; (b) TK(3,4); (c) alternating THK(3,4); (c) TK(3,1) = undulating triangle.*

The shown surfaces are only B-spline approximations to minimal surfaces. Future work will aim to integrate Brakke's Surface Evolver [2] with our NOME design environment.

Acknowledgements

I would like to thank Paul Perry for providing me with good photos and information about "*Star Cinder*." I am grateful to Andy Yu Wang [16], Gauthier Dieppedalle [4], and Toby Chen for their efforts towards the construction of an interactive computer-aided design tool (NOME: Non-Orientable Manifold Editor), which makes the modeling of these intricate free-form surfaces possible and enjoyable. I also would like to thank the staff of the Jacobs Institute for Design Innovation at UC Berkeley for their help in fabricating many of the sculptural models presented.

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