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Rendering Pacioli's Rhombicuboctahedron

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Abstract

We analyze the glass rhombicuboctahedron (RCO) appearing in a famous painting of Pacioli (1495), considering the extent to which it might agree with a physically correct rendering of a corresponding glass container half-filled with water. This investigation shows that it is unlikely that the painter of the RCO was looking at such a physical object. We then ask what a proper rendering of such an object might look like. Our computer renderings, which take into account multiple internal and external reflections and refractions, yield visual effects that differ strongly from their depictions in the painting. Nevertheless, the painter of the RCO has clearly succeeded in providing a rendering that appears plausible and awe-inspiring to almost all observers.

1. Introduction

The focus of attention in this article is the rhombicuboctahedron (RCO) (Figure 1) that appears in the painting "Ritratto di Fra' Luca Pacioli" (1495) exhibited in the Museo e Gallerie di Capodimonte [15] in Naples, Italy. The RCO is one of the 13 Archimedean solids. These are semi-regular convex polyhedra composed of two or more types of regular polygons meeting in identical vertices; they are distinct from the Platonic solids, which are composed from only a single type of regular n-gons. The central character in Figure 1a is Fra' Luca Pacioli, a famous mathematician of the Renaissance period, most likely lecturing on some topic concerning the Platonic or Archimedean polyhedra. There is some speculation that the second person in the painting might be Albrecht Dürer who also had an interest in symmetrical polyhedral objects. The great fascination with such objects at that time eventually culminated with "De Divina Proportione" written by Pacioli around 1497. This book on mathematical and artistic proportions was illustrated by Leonardo da Vinci.

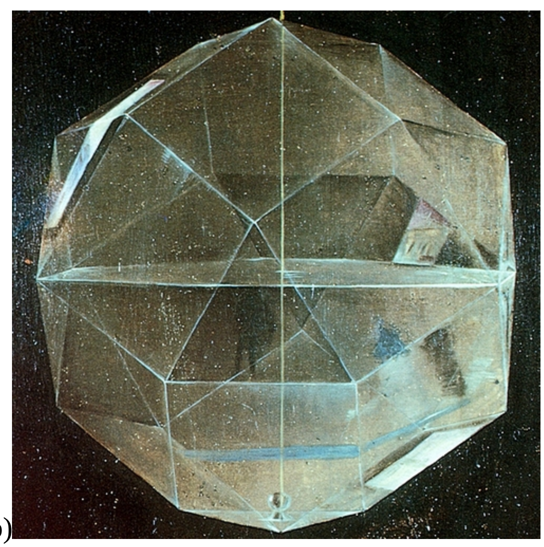
(a)  (b) 

Figure 1: *(a) Pacioli painting; (b) enlarged and enhanced view of the rhombicuboctahedron (RCO).*

The suspended RCO, shown in the top left of the painting, is composed of 18 squares and 8 equilateral triangles, and each vertex is shared by three squares and one triangle. The painting implies that this object has been realized with 26 glass plates fused together well enough that this shell can be filled with some liquid up to its centroid.

Mackinnon suggests [12,13] that the RCO, with its square and triangular faces, represents four of the five regular Platonic solids, and that by filling it partially with water it might evoke associations with the four corresponding elements according to the chemical theory of Plato's Timaeus: *Earth* (cube) by the physical glass shell, *Water* (icosahedron) and *Air* (octahedron) by the media contained in this shell, and *Fire* (tetrahedron) by the depicted bright reflections. The fifth regular solid, the dodecahedron, which for Plato represented the *Universe*, is shown in a model on the right hand side of the painting.

This painting is now generally attributed to Jacopo de' Barbari. However, there is some speculation that the depiction of the transparent RCO was added by some other artist, since in style and detail it looks quite different from the other objects in this room. Many admirers of this painting have issued glowing comments about how wonderfully the painter seems to have captured the reflections and refractions in this object (Figure 1b). Mackinnon [13] attributes this part of the painting to Leonardo da Vinci himself. Even though Pacioli had not yet met Leonardo when this panel was first painted, and he started to collaborate with him only after 1495, there is a possibility that the RCO was added later to the portrait of Pacioli when they were both in Milan [12]. Many conjectures and some speculation has originated from this painting, and the discussion of its creation and history is still on-going [2,5,8,19].

Clearly the RCO in this painting stands out in a special way. On his web-page devoted to Luca Pacioli's Polyhedra Hart states [10]:

> *The polyhedron in the painting is a masterpiece of reflection, refraction, and perspective. (Davis states that the bright region on its surface reflects a view out an open window, showing the Palazzo Ducale in Urbino.) Certainly an actual glass polyhedron was used as a model. (Pacioli states in his books that he constructed several sets of glass polyhedra, but I know of no other information about them.) The polyhedron in the painting is beautifully positioned, suspended with a 3-fold axis vertical, out of physical contact with the other objects in the scene. I suspect that Pacioli chose it for the portrait because he discovered this form and was quite proud of it. (Presumably Archimedes first discovered it, but that wasn't known in Pacioli's time.) The painting is the earliest known image of the rhombicuboctahedron.*

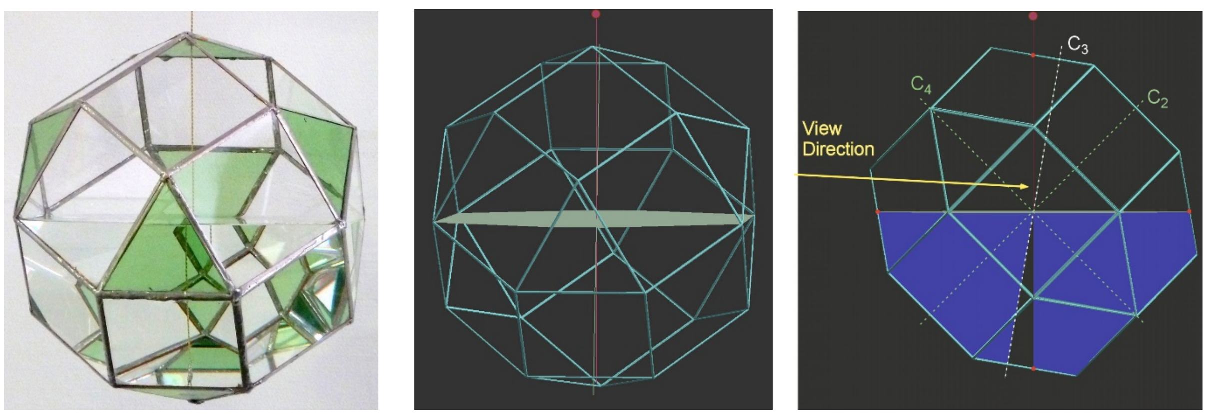
But, on a closer, more critical inspection, some things seem not quite right: The reflection of the window (not seen in the painting) on the upper left of the RCO seems more like a pasted-on sticker image that bends around one of the polyhedron edges than two separate reflections in the two differently angled RCO facets. In 2007, Rekveld raised doubts in a publication [17] and on his website [18]:

> *Last week in Napoli I revisited the Capodimonte museum and its amazing collection of paintings. At some point I found myself face to face with this canvas, attributed to Jacobo de Barbari, a portrait of the mathematician Luca Pacioli painted in 1495.*
> *... In said painting I suddenly noticed the mysterious reflections in the gorgeous mathematical shape at the top left, a rhombicuboctahedron, made of glass and half filled with water. I took a picture, and when I zoom in on these painted reflections we see buildings and sky, as if to suggest that an open window in the room is being reflected in the glass facets. The direction doesn't seem right though; am I wrong or do the reflections show that the window is high up towards the left? Or rather towards the bottom right? Both do not really make sense. Also in either case the light in the painting does not seem to be much affected by the source of these reflections.*
> *... To me the way the same window seems to be reflected three times within this shape doesn't seem terribly true to the laws of optics. Or more precisely: they seem true to a textbook notion of reflection and refraction of light in water, but the way the images are formed doesn't seem very realistic at all.*

... Its depiction doesn't seem based too much on observation or the tracing of reality through a camera obscura, but then again, I've never actually seen a glass rhombicuboctahedron half filled with water, so who am I to judge?

Thus, it seems worthwhile to investigate these issues. Just as disturbing as the "pasted-on" reflection of the window discussed above is the fact that there are no visible effects of refraction in the water body in the lower half! The RCO edges on the back surface appear in the painting in exactly the places where the computer rendering of a thin wire-frame object (Figure 2b) shows them.

In May 2011, Claude Boehringer [4], an artist working with various materials, produced a physical glass model of an RCO shell held together with lead, which was strong enough to be half filled with water. Under the guidance of Herman Serras [22], the model was suspended in the proper way to obtain the same orientation as the polyhedron in the painting. Serras then took the photograph shown in Figure 2a. This image looks quite different from the one in the painting, and so it is difficult to draw conclusions about the realism of the depiction of 1495. The surroundings where this model was photographed are entirely different, and there are no dominant reflections of a single, brightly lit window. Nevertheless, the photo of the Boehringer model (Figure 2a) confirms our intuition that those edges seen through the water body would be seriously altered in their rendered positions.



(a) (b) (c)

Figure 2: *(a) Model by Claude Boehringer photographed by Herman Serras; (b,c) viewing geometry.*

In 2011 a discussion arose concerning perceived geometrical flaws in some of Leonardo's drawings [11]. Huylebrouk's note also made a reference to the RCO in the portrait of Pacioli, and this caused one of the authors to take a closer look at the depiction of this object [20]. While there were no geometrical errors in the projection of this object, some of the visual effects due to reflection and refraction seemed clearly wrong. This the raised the question what an actual RCO glass shell half-filled with water really would look like. Several students at Berkeley and elsewhere responded to the challenge, using computer graphics programs to model such an object with its various reflections and refractive effects. However, the results offered looked quite different, driving home the point that modeling a complex object with multiple volumes with different refractive indices abutting one another is not a trivial task. When making a model of the object to be rendered, the idiosyncrasies of the rendering program to be used need to be taken into account carefully. One model does not fit all possible renderers! Even for people experienced in using computer graphics rendering programs, it is advisable to run through a series of progressively more complex test models when switching to a new renderer.

In most of our efforts we have used Autodesk *Maya* [1] as our modeling tool and *Mental Ray* [14] as our rendering engine using a basic ray-tracing algorithm [9,16]. First we constructed simple geometrical test models for which the correctness of the renderings could readily be verified, and made sure that the renderer properly interpreted the desired geometry in the various refraction events and reflections at external and internal boundaries. In the end we repeated this process once more for the open-source rendering program *Blender* [3]. Overall, these computer simulations confirm that it is highly unlikely that the painter was observing a physical glass container half-filled with a clear liquid when painting the RCO.

In the following we briefly discuss the geometrical set-up of the RCO for a geometrical analysis in the context of the paper as well as for our computer modeling. A detailed discussion of our series of test renderings, which we recommend as a preparatory debugging step for challenging rendering tasks, can be found in a Technical Report [21].

Our computer simulations look quite different from the painted RCO, proving that the latter is not a physically correct rendering. We thus conclude that the artist's most likely objective was to create the most "plausible" and "convincing" depiction of such an object for a broader public.

2. Viewing Geometry

To create a realistic rendering that can be compared with the painted RCO, we first have to figure out how this RCO has been suspended, and where to place the observer's eye. If the RCO were indeed suspended along one of its 3-fold symmetry axes piercing the centers of the top and bottom triangles, as stated by Hart [10], then these two faces would be truly horizontal. In the painting we see both of these triangles from below, which would then imply that the eye of the observer must lie below the bottom of the RCO. On the other hand, the horizontal water surface clearly is seen from above: The more strongly depicted edges of the RCO belong to its front facets, and the foreshortened left and right edges of the polygon depicting the water surface have a vanishing point that lies somewhere in the upper right. This view geometry requires an eye point above the center of the RCO.

Careful inspection of the enlarged view of the painted RCO (Figure 1b) shows that the water surface passes through 4 of the 24 vertices of the RCO and cuts the vertical, square facets on the left and right sides along their horizontal diagonals (Figures 2b, 2c). This implies that the suspension line must form an angle of 45° with the plane of the square immediately in the back of the top triangle, as well as with the square in front of the top of the RCO. The C_3 symmetry axis deviates by 54.74°–45° = 9.74° from this suspension line. A few trigonometric calculations then reveal that this suspension line cuts the altitude of the top triangle in the ratio 1:√2, i.e., a fraction of 0.4142 away from the top vertex in the painting. Serras [22] also had calculated the distance of the suspension line penetration from the closest triangle vertex as √3 *(√2–1)/2 = 0.3587 times the RCO edge length. Fortunately these two calculations agree. In addition, the water boundary intersects two more pairs of triangle edges at a fraction of √2–1 = 0.4142 away from the shared vertex (red dots in Figure 2c).

With the angle of suspension unambiguously resolved based on the boundary of the water level, we now can try to locate the eye point of the observer. The viewer must be looking towards the center of the RCO with a slight downward angle that must lie between 0° and 9.74°. Used an interactive computer graphics program that let us readily adjust the viewing parameters, we found a best match between an appropriately tilted computer model and the rendering in the painting for a downward angle of about 3° (Figure 2c). Comparing the resulting view of a wire-frame RCO (Figure 2b) with the polygon edges depicted in the painting (Figure 1b) confirms that we have found a suitable approximation of the correct viewing geometry.

Now let's review the above findings in the context of the complete painting. First we try to find the viewer's eye level with respect to the painting. Our intuition tells us that the observer's eyes are about at the height of Pacioli's nose or eyes, and that the person on the right is looking slightly down on the viewer. This would place the view center properly above the water level. We can also try to find a horizon compatible with the objects on the table (assumed to be horizontal), and this is what we found:

1.) a vanishing point on top of Pacioli's head for the slate tablet;

2.) a vanishing point somewhat above the middle of the RCO (but way out to the left) for the open book (with substantial error margins);

3.) a vanishing point slightly above the top of Pacioli's head for the red box (also with substantial error margins).

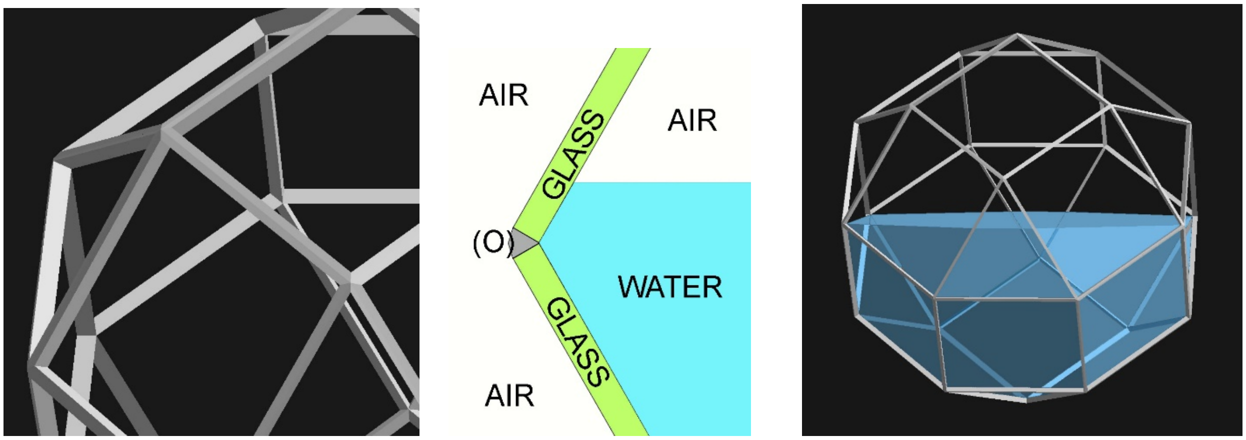
All of these vanishing points are higher than the center of the RCO, and this is compatible with our view of the water surface from above. But then, the fact that we can see the lower side of the bottom triangle implies that it must be slanted upwards towards the viewer. This confirms the tilted suspension of the RCO established above.

Now let's take a closer look at the projection used in the painting: Clearly there is some perspective involved; the back-face triangle (pointing down) is a few percent smaller than the size of the front triangle (pointing up). The same interactive computer rendering program that lets us find the best match for the view angles also lets us find the parameters for the perspective projection in which the rendered edge crossings of the wire frame model match the locations in the painting as closely as possible; this occurs when the eye is about 14 RCO diameters away from its center. Now, based on its position and comparing it to the hands and head of Pacioli, we may estimate that the RCO is about 8 to 10 inches in diameter. Thus, based on its own perspective, it must have been drawn as it would look from about 10-11 feet away. But Pacioli in the painting seems only about 5 to 6 feet away from the viewer, and the object, if it is indeed above the table, would then be only 4 to 5 feet away. Thus the perspective of the rhombicuboctahedron is not compatible with the rest of the scene. The RCO should exhibit much stronger foreshortening, or it should have been depicted at only about half its current scale. This raises a strong suspicion that this RCO was sketched out quite carefully, but separately – perhaps from a smaller model or from a geometrical construction – before it was copied into the Pacioli painting.

3. Computer Modeling and Rendering of the RCO

Let us assume that the RCO was painted in a separate sitting, possibly in a room with an open window in the left wall, offering a view of a palazzo and some bright sky, and perhaps with some more local illumination coming from the lower right. Can we find an environment and some suitable material constants and illumination levels that will produce an image closely resembling the depiction in the painting? What would a physically more realistic rendering of such a glass container half-filled with water look like?

The first task is to create an appropriate geometrical model of the object to be rendered. To have some control over the appearance of the edges of the glass container, we flesh out the wire-frame shown in Figure 2b into properly mitered prismatic beams that are composed into a polyhedral object of genus 25 (i.e., a shell with 26 facet openings). Figure 3(a) shows an enlarged partial view of this framework; to make its geometry more clearly visible, all the cross sections have been enlarged by a factor of 2. In our model, the thickness of these struts is parameterized and set equal to the thickness of the glass plates shown in green in Figure 3(b); this allows us to emphasize or de-emphasize the visibility of these seams. In most of our renderings, we have set the material of the struts to be some grey, diffusely reflective material. However, there is the option to assign them the same material parameters as are used for the glass plates, in order to simulate a completely fused contiguous glass container. Finally, the water volume is modeled as a separate polyhedron that fits snugly against the inside surfaces of the glass plates in the lower half of the RCO (Figure 3c).

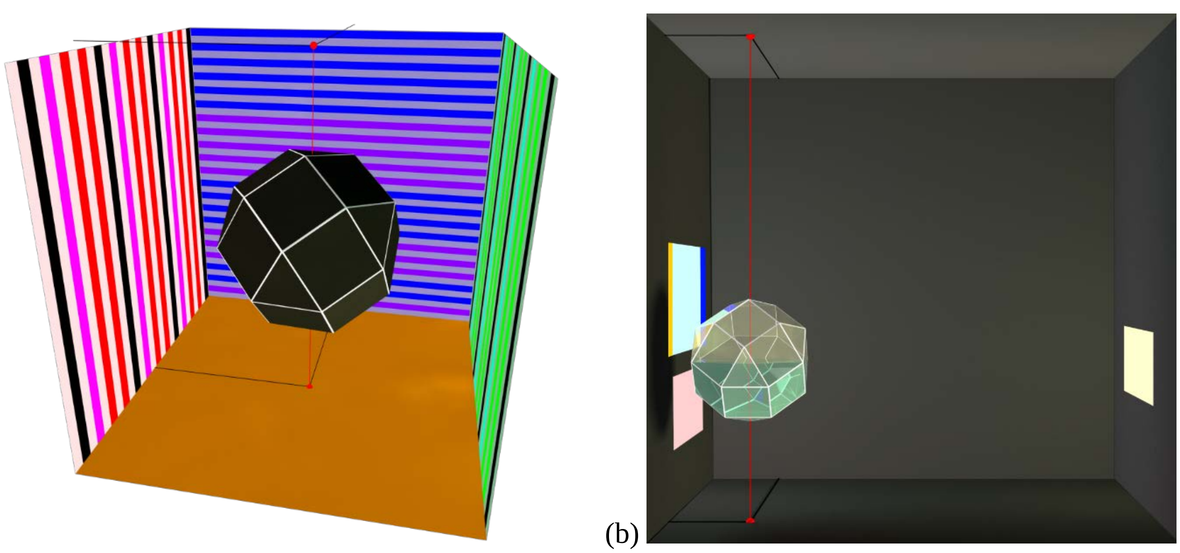


(a) (b) (c)

Figure 3: *Modeling the complete RCO: (a) Filler framework; (b) cross section showing the types of interfaces; (c) a transparent, non-refractive water polyhedron inside the opaque filler framework.*

Once we have gained confidence that the chosen modeling technique and the adopted rendering environment produce physically justifiable results [21], we can model the RCO as described in Figure 3 and render it in various environments using a basic ray-tracing program [3,14]. Overall we have three different transparent materials (Figure 3b): air (A), water (W), and glass G), plus the opaque surfaces (O) associated with the filler framework located around the edges of the glass plates. Any ray that hits the latter type of surface is terminated and returns the illumination value found at this location.

The interfaces between two transparent materials are modeled as smooth surfaces that permit refraction as well as reflection processes to occur. In general they will divide an incoming ray of generation *n* into two sub-rays of generation *n*+1; the relative strengths of the two sub-rays are determined by the Fresnel equations [7]. Special care must be taken to create the properly merged interfaces where the body of water is in contact with the glass plates forming the container. For 8 of the 26 glass plates, their inner surfaces had to be split into two parts – one interfacing with water, and the other interfacing with air. In the complete model there is one interface facet of water/air, 26 + 17 instances of glass/air, and 17 instances of glass/water.



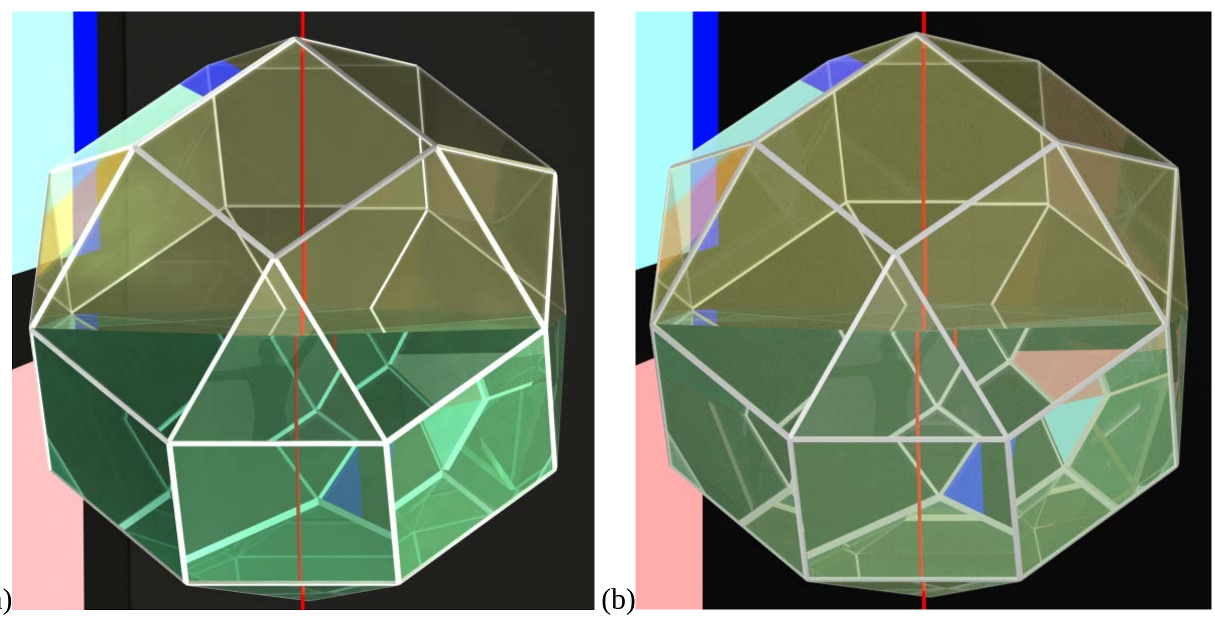
(a) (b)

Figure 4: *Placement of the RCO inside virtual Cornell Boxes: (a) for detailed studies of refractions and internal reflections [21], and (b) for final studies of external window reflections.*

We started with some extensive testing of the RCO model [21] in the synthetic environment of a virtual Cornell Box [6] with distinctly striped walls (Figure 4a), where the final stopping point of any ray being reflected on or refracted through the RCO can be identified more easily. Those tests gave us the confidence that our rendering process will give us a depiction that is close to physical truth with respect to the placement of the various visual elements resulting from reflection and refraction.

To produce the final renderings (Figure 5), we have suspended the RCO in a darkened room with three bright, colored “windows” (Figure 4b); these are really just bright, flat “paintings” on the walls. The RCO is placed close enough to the large cyan “window” on the left wall, so that portions of it appear in both the upper left panels of the RCO, where the original painting shows the primary window reflections. This “window” is enhanced with blue and yellow vertical edges to make it easier to understand what is being reflected in those two facets of the RCO. One can see the blue back-edge of that window in the upper panel; the yellow front-edge appears in the lower facet. This disparity in reflected window portions illustrates the physical impossibility of a single continuous image of the window appearing across both faces, as depicted in the Pacioli painting.

There are other features on the surface of the painted RCO that can be interpreted as depicting reflections. The lower right of the RCO has some pronounced greenish tint that can be seen as a reflection of the green table cloth, and a fuzzy dark feature might represent one of the bodies standing behind the table. More distinctly drawn, a small black reflection in the front triangular face of the RCO (Figure 1b) was probably meant to represent the artist who drew the image of the RCO; however, it is unrealistically small. A human observer standing some distance *d* in front of a plane mirror will see his reflection as being located at that same distance *d* behind the mirror, and the perceived “image” at the reflecting surface will be half the size of the observer. Thus the black figure outline is much too small to be a reflection of the artist; to a first approximation, it should be about the same size as Pacioli or his companion. In our computer rendering, we created a comparable black reflection by placing a 6-inch tall puppet cut-out in the plane of the camera. Moreover, since the triangular front face of the RCO is not perpendicular to the line of sight, the puppet had to be displaced about ten body lengths downwards and to the left of the position of the camera lens – supposedly the eye of the artist.



(a) (b)

Figure 5: *RCO in a dark room with windows: (a) rendered with Maya to a recursion depth of 10; (c) rendered with Blender to a depth of 40.*

The computer renderings (Figure 5) also show internal reflections in quite different locations compared with the painted RCO. For instance, we see portions of the light blue window and its dark blue back edge in two facets in the bottom right quadrant of the ray-traced RCO, but none of these have a similar appearance in the painted RCO, even accounting for the murkiness of the painted water. Conversely, the painted RCO shows some internal reflections that are not seen anywhere in the ray-traced RCO: in the back/right center quad facet above the water surface, and along the bottom/right of the RCO. In attempting to replicate these extra internal reflections seen in the painted RCO, we expanded the windows to fill the entire left wall, but still we could not obtain the desired internal reflection in the upper half of the RCO.

As mentioned before, the refracted locations of the string and the filler framework also appear in inappropriate locations. In the painting, the string appears as a continuous line above and below the water surface, whereas in the computer rendering the string is discontinuous at the water surface. The displacement is caused by the different indices of refraction of air and water. The same applies to the appearance of the filler framework; our ray-traced RCO renders the refracted framework struts in very similar locations to the physical model shown in Figure 2(a) – in contrast to the painted RCO.

In order to check the reproducibility of our rendering results and the robustness of the step-by-step validation approach recommended above, we also experimented with a different rendering program: Blender [3] is an open source program readily available to everybody, and may therefore be a candidate that many JMA readers might use for quick experiments. Once our model data had been adjusted to be compatible with the Blender environment, the rendered geometrical features looked the same as they look with the Maya renderer. We could also confirm that increasing the depth of the recursive ray-tree from 10 to 40 did not result in any meaningful differences; any changes that might be attributable to increased ray-tree depth are small compared to the noticeable variations in the displayed intensities of reflected and refracted components between the two rendering programs. The major differences between the two renderings occurred with respect to hue changes and brightness variations. Different rendering programs use different representations for the relative importances of the specularly reflected, refracted, and diffusely reflected photons, resulting in different relative intensities of some of the struts as seen through multiple layers of reflection and refraction.

More troublesome for our particular rendering comparison was the fact that Blender permits volume-tinting only for domains surrounded by a closed manifold surface, but not for volumes that have some individually merged interfaces as part of their boundary representation. Thus the tinting of the glass and water components had to be achieved entirely via some approximate changes in surface parameters. This explains why the reflections of the pink window are much more vivid in the Blender rendering: the rays experience less tinting while traveling a substantial distance through water. This emphasizes yet again the need to carefully evaluate the capabilities and idiosyncrasies of any rendering environment, using an incremental approach that starts with some simple, well-understood test geometries.

There are still many parameters that could be fine-tuned: the refractive index of glass; the amount of tinting imparted by the glass plates and the water body; and the amount of diffuse scattering on the outer surfaces, perhaps due to the presence of dust. We chose some of these parameters to make our rendering look somewhat similar to the depiction in the Pacioli painting, as depicted in Figure 1b. However, our main focus was to show the geometrical issues related to the placement of the reflected and refracted geometrical features, because these effects are most relevant to the question of whether the artist was actually observing an RCO half-filled with water.

4. Discussion and Conclusions

The RCO plays an important role in the Capodimonte canvas. It depicts an important accomplishment of the famous mathematician Fra' Luca Pacioli and was clearly meant to impress viewers of this painting. Baldasso [2] even argues that the RCO plays the role of a third figure in this composition.

Our analysis indicates that it is highly unlikely that the artist who painted the rhombicuboctahedron in the Pacioli painting was looking at a physical glass RCO partly filled with water of the size implied in this painting. One of the anonymous reviewers contributed the following valuable information:

> *In 1495 it was possible to make a small glass model of a polyhedron, and several are mentioned in two sixteenth century inventories of the Ducal palace at Urbino where Pacioli had worked; for technical reasons to do with the available glass it is not possible that these models were as large as the Pacioli RCO appears, nor could they have been strong enough to hold several liters of water.*

This provides strong support for our claim that the artist did not directly observe the depicted object. However, it seems quite likely that the artist used an empty RCO model made of triangular and square glass plates for generating an original sketch, because the depicted geometry of the RCO edges is in excellent agreement (including perspective distortion) with the computer generated rendering. When this initial depiction was later copied into the Pacioli painting, the discrepancies in the RCO's scale and its perspective projections, discussed at the beginning of this article, were introduced.

The primary window reflection across the top left facets is so far off from any possible reality that it cannot have been drawn based on an actual observation. It may, however, be possible that the painter observed the general nature of some window reflections on various polyhedral glass models and decided that reality was much too confusing for most observers to yield a pleasing painting. The artist may then have made a conscious decision to render a window reflection that would be more plausible and would help to identify the location of the space portrayed. A contiguous picture of the 'Duomo in Urbino' can readily be understood by most viewers, whereas physically realistic reflections that break this image into disconnected pieces would be puzzling to most observers.

Moreover, to emphasize the transparency and 3-dimensionality of the RCO, two other internal reflections of the same window were added in two facets on the right hand side, one just above the water surface, and one on the bottom right in the water. The first one occurs where one would expect to see a secondary image, if this were the result of an intersection of a parallel beam of light coming through the assumed window. Furthermore, the contrast and saturation of these imagined reflections appear unnaturally strong compared to the other depicted features in the RCO. In addition, the apparent luminosity of the three reflections has been enhanced by the artist, by surrounding them with a slightly darkened halo.

One reason why this rendering of the RCO looks so good to an uncritical observer is that our visual system is much less sensitive to the precise effects of refractions and reflections, compared to shadows or perspective. It is plausible to assume that this is a consequence of our evolution during the last million years, when humans were trying to survive in the natural world, where reflection and refraction occur only in a rather limited way.

Though we have focused in this article on the defects with respect to a physically realistic rendering of the RCO, we should remember that physical realism was not the primary objective of the painter. A more likely goal was to present this intriguing mathematical object in the best possible way that would bring about a sense of wonder and awe in most viewers. We agree with many art historians that this goal has been achieved extremely well despite the discussed rendering flaws.

Acknowledgements

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References

[1] Autodesk Maya. -- http://www.autodesk.com/products/autodesk-maya/overview (May 2015)

[2] R. Baldasso, *Portrait of Luca Pacioli and Disciple: A New, Mathematical Look*. The Art Bulletin, (March-June 2010)

[3] Blender Foundation, *Blender*. -- http://www.blender.org/ (May 2015)

[4] C. Boehringer, *Physical model of the RCO*. Private communication. -- claude.boehringer@mac.com

[5] A. Ciocci, *Il Doppio Ritratto del Poliedrico Luca Pacioli*. De computis. Revista Española de Historia de la Contabilidad 15 (2011), pp 107-130.

[6] Cornell University, *The Cornell Box*. -- http://www.graphics.cornell.edu/online/box/ (May 2015)

[7] *Fresnel's Equations*. -- http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/freseq.html (May 2015)

[8] E. Gamba, L'Umanesimo matematico a Urbino. A. Marchi and M. R. Valazzi, eds., La città ideale: l'utopia del Rinascimento a Urbino tra Piero della Francesca e Raffaello. (Milan: Electa, 2012), pp 233-247.

[9] A. S. Glassner, *An Introduction to Ray Tracing*. Morgan Kaufmann, 1989.

[10] G. Hart, *Luca Pacioli's Polyhedra.* -- http://www.georgehart.com/virtual-polyhedra/pacioli.html (May 2015)

[11] D. Huylebrouck, *Lost in Triangulation: Leonardo da Vinci's Mathematical Slip-Up*. -- http://www.scientificamerican.com/article/davinci-mathematical-slip-up/ (March 29, 2011)

[12] J. Logan, *An Analysis of Certain Mathematical Cyphers in the Portrait of Fra Luca Pacioli*. Unpublished draft; private communication, Jan. 19, 2015.

[13] N. Mackinnon, *The portrait of Fra Luca Pacioli*. The Mathematical Gazette, 77 (1993) pp 130 - 219.

[14] Mental Ray documentation. -- http://docs.autodesk.com/MENTALRAY/2012/CHS/mental%20ray%203.9%20Help/files/tutorials/architectural-library.pdf (May 2015)

[15] Museo e Gallerie di Capodimonte, *Ritratto di Fra' Luca Pacioli*. -- http://cir.campania.beniculturali.it/museodicapodimonte/itinerari-tematici/galleria-di-immagini/OA900154 and specifically: -- http://www.polomusealenapoli.beniculturali.it/museo_cp/cp_scheda.asp?ID=16 (May 2015)

[16] G. S. Owen, *Ray Tracing*. -- https://www.siggraph.org/education/materials/HyperGraph/raytrace/rtrace0.htm (May 2015)

[17] J. Rekveld, *ghosts of luca pacioli*. umwelt, observations (2007)

[18] J. Rekveld, *light matters: ghosts of luca pacioli*. -- http://www.joostrekveld.net/?p=615 (May 2015)

[19] J. Sander, ed., *Albrecht Dürer: His Art in Context.* (Munich: Prestel, 2013), pp 190-191.

[20] C. H. Séquin, *Misinterpretations and Mistakes in Pacioli's Rhombicuboctahedron*. -- http://www.cs.berkeley.edu/~sequin/X/Leonardo/pacioli_rco.html (May 2015)

[21] C. H. Séquin and R. Shiau, *Rendering Issues in Pacioli's Rhombicuboctahedron*. EECS Tech Report.

[22] H. Serras, *Mathematics on the Ritratto di Fra' Luca Pacioli.* -- http://cage.ugent.be/~hs/pacioli/pacioli.html (May 2015)