Knotted Twistors – Twistor-Knots

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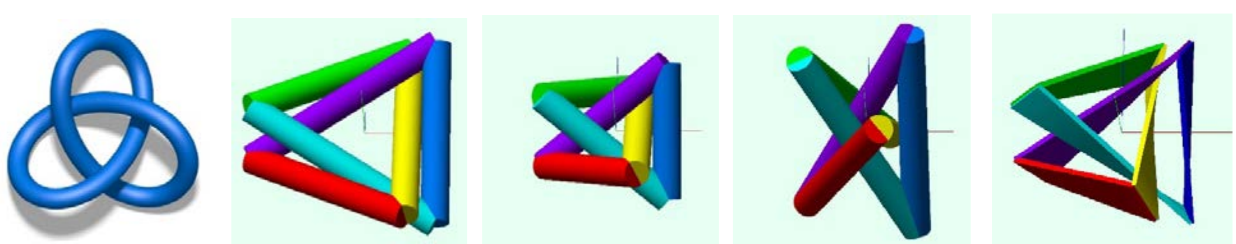
(a) (b) (c) (d) (e)

Figure 1: "Twistors" real and virtual by Nathaniel Friedman and Robert Krawczyk.

The creations discussed below were inspired by the Twistor sculpture (Fig.1a) that Nathaniel Friedman submitted to the Art Exhibit of the Joint Mathematics Meeting to be held in San Antonio, Texas, in January 2015, and by the ongoing joint development of many related forms (Fig.1b–e) together with Robert Krawczyk [2]. The idea behind this extension was simple: Rather than extruding the twisted bands along the edges of one or more triangles, let's try to fit them along the various beams that make up a sticks representation of some simple and symmetrical mathematical knots. Then try to find out how densely the various twisted beams can be packed when the twisting is optimally adjusted so that the various helical forms nicely interlock without creating any beam intersections.

Trefoil Twistor-Knot based on Knot 3_1

The first candidate is the knot 3_1, the trefoil knot (Fig.2a). Figures 2b-d show three different tight sticks versions of this knot, where every tube just touches its neighbors. Among the three configurations shown, Figure 2c is the one with the minimal overall tube length. I use this geometry to form a first Twistor-Knot by running a twisted band inside every tube (Fig.2e).

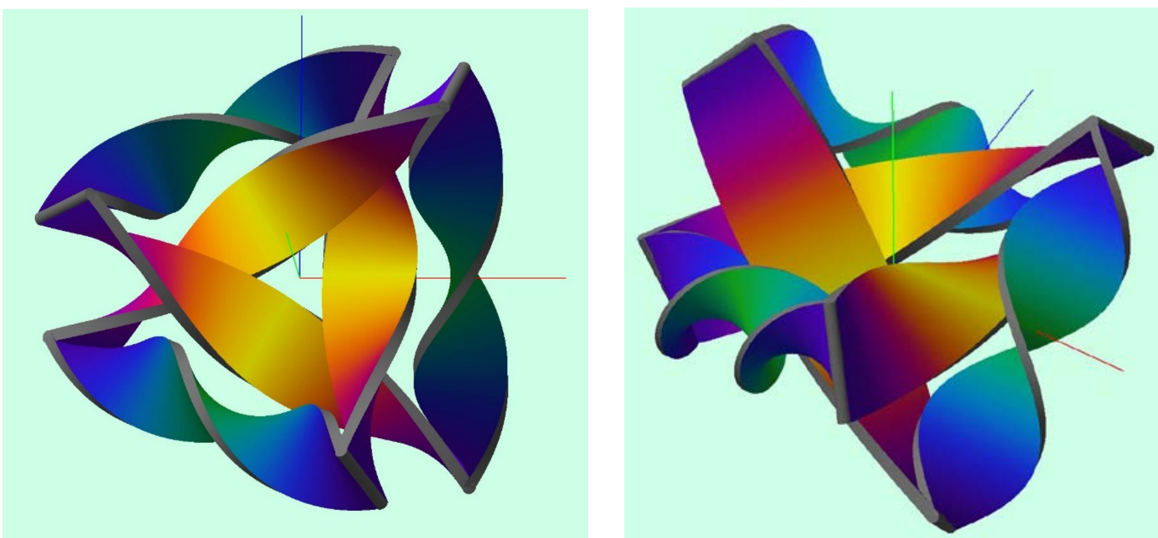


(a) (b) (c) (d) (e)

Figure 2: (a) Smooth trefoil knot; (b–d) various symmetrical, tight sticks realizations, (c) has the overall shortest tube length; (e) a twisted band run along the path of (c).

Starting with the tightest sticks configurations (Fig.2c), twisted bands of width w=2 will readily fit into the normalized tightly packed tubes of radius one. Thus we just need to adjust the azimuth and twist angles for the two types of beams (the inner three and the outer three). The cleanest results are obtained if the six outer seams, where a pair of twisted bands meet, are oriented perpendicularly to the plane defined by the two beams ending in that seam. For each type of beam, we adjust its azimuth angle (rotation around the axis of the beam), so that the start of the beam lines up with the orientation of that normal vector; next we adjust the beam's twist so that the end of the beam lines up with the plane normal at the far end.

However, we still can make some choices about these two twist values and change them in increments of 180°. Figure 3a, shows a configuration based on the tight trefoil with beams of a nominal width *w*=2, where the inner beams are experiencing a half twist, and the outer bands execute a full twist. This yields a graceful Twistor-Knot that is not packed too tightly. In Figure 3b the direction of twisting has been reversed. This allows to broaden the width of the bands by a factor of 1.4, without occurring any band intersections – which in the arrangement of Figure 3a, would first occur near the center. The double half-flip of the outer beams also comes in handy: they arch neatly over the maximal radial extension of the inner bands at their mid-points. However, this result looks somewhat too crowded. In addition the difference of the actual amount of twisting of the inner bands as compared to those of the outer bands has increased, because of the way that the normal vectors at the ends of the beams are tilted with respect to one another.



(a) (b)

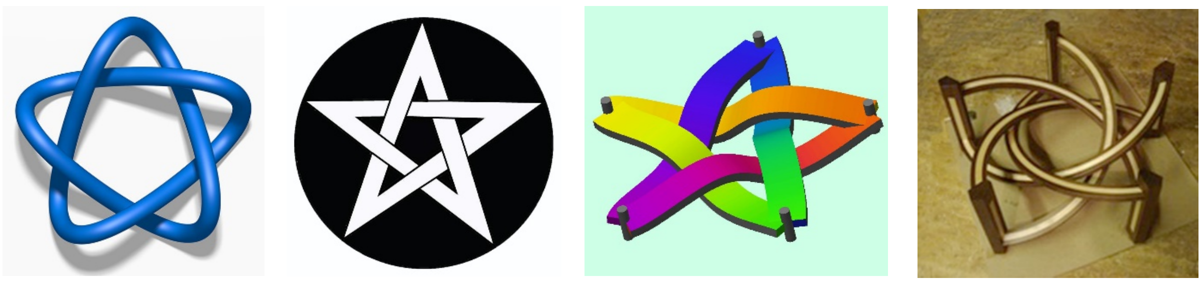
Figure 3: Twistor-Knots 3_1: (a) Tight configuration with a half twist on the inner beams and a full twist on the outer beams; (b) a denser configuration with twists in the opposite orientation.

The reader may wonder whether one could combine the inner twisting of Figure 3b, which is optimal to prevent intersections of those bands near the center, with some other more gentle twist on the outer beams. It turns out that using opposite twisting in the two types of beams is not advisable: the pairs of beam joining in the outer seams would jam into one another in an unattractive way. On the other hand, giving the outer band a minimal amount of twist, just enough to make up for the different orientations of the seams at the two ends, results in a simple, but pleasing configuration. (Even the configuration with no effective twist in any of the bands looks somewhat attractive! – see Figure 2e.)

For this Twistor Knot I designed two special textures to be applied to the beams in the above virtual renderings. They are both of dark blue color at the ends of the individual beams and get brighter towards the middle of the beam. This symbolically represents the way that beams get fabricated for a metal sculpture like Figure 1a: The ends are held in heavy clamps or in a vice; the exposed middle is heated to make the metal soft, and then the desired twist is introduced into the beam.

Pentafoil Twistor-Knot based on Knot 5_1

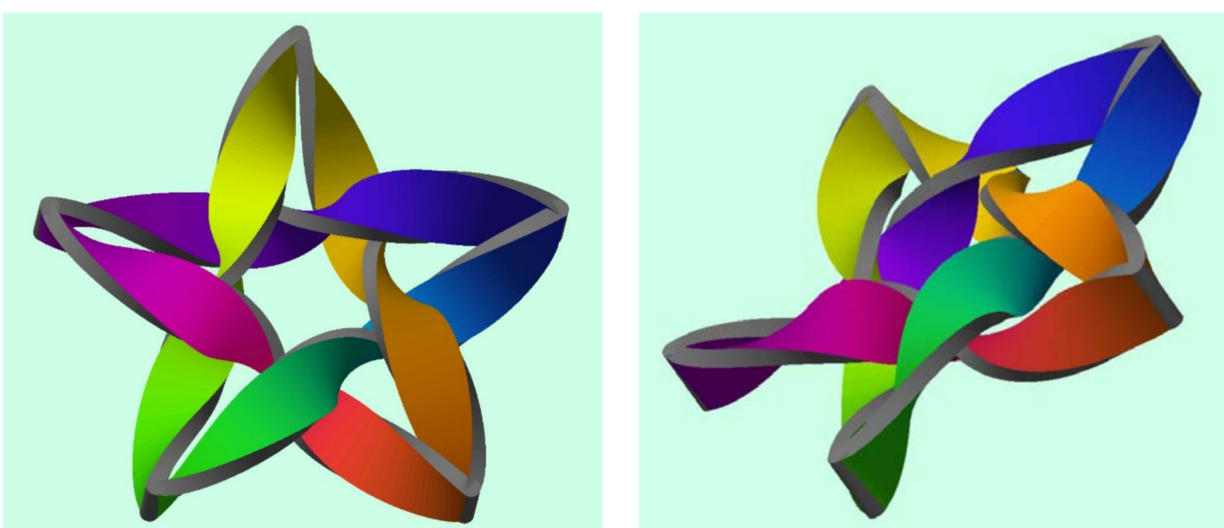
For the Twistor-Knot_5-1 I took a different approach. Rather than starting with a sticks configuration of knot 5_1 (Fig.4a), I started from the planar Pentagram shown in Figure 4b. To avoid creating intersections, I had to introduce some curvature into the five beams. They are bending out of the symmetry plane to form the needed over/under-crossings; they also bend laterally in this plane to place these crossings into optimal locations (Fig.4c). Each ribbon makes a double-flip, so that is in a mostly horizontal orientation at the locations of the crossings (Fig.5). The sweep path of each beam is specified with a quintic (6^{th}-order) Bézier curve. The four inner control-points allow me to fix the tangent direction in which the beams take off from the five outer junctions and to control the elevation at the over/under-passes. They also allow me to adjust the rate of twisting in a non-linear manner; I pushed the highest twist-rate to the center of the beam, so that the ribbons have the optimal orientation where they cross.



(a) (b) (c) (d)

Figure 4: (a) Smooth pentafoil knot; (b) 2D pentagram; (c) pentagram with curved bands; (d) legs for "Pentafoil Knot Table" [1].

Again the issue arises how to orient in space the five seams where two ribbons join. For simplicity, I have placed them normal to the symmetry plane, as indicated by the five legs of the "Pentafoil Knot Table" [1] shown in Figure 4d. This seems justified, since I can keep the initial tangent directions of all the sweep paths within this plane. Figure 5 shows the resulting Pentafoil Twistor-Knot from "the top", as well as from the side, to give a better understanding of the 3-dimensional bending of the beams. In this model I used a continuous rainbow texture to make it easy to track the flow of the ribbon around the pentagram loop.

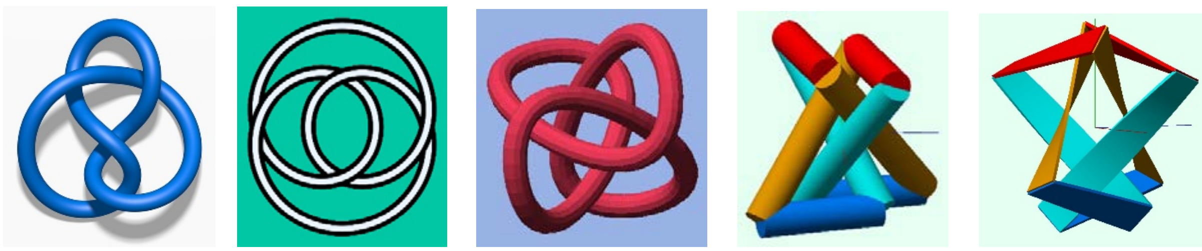


(a) (b)

Figure 5: Twistor-Knot 5_1: (a) view from the top, and (b) view from the side.

Figure-8 Twistor-Knot based on Knot 4_1

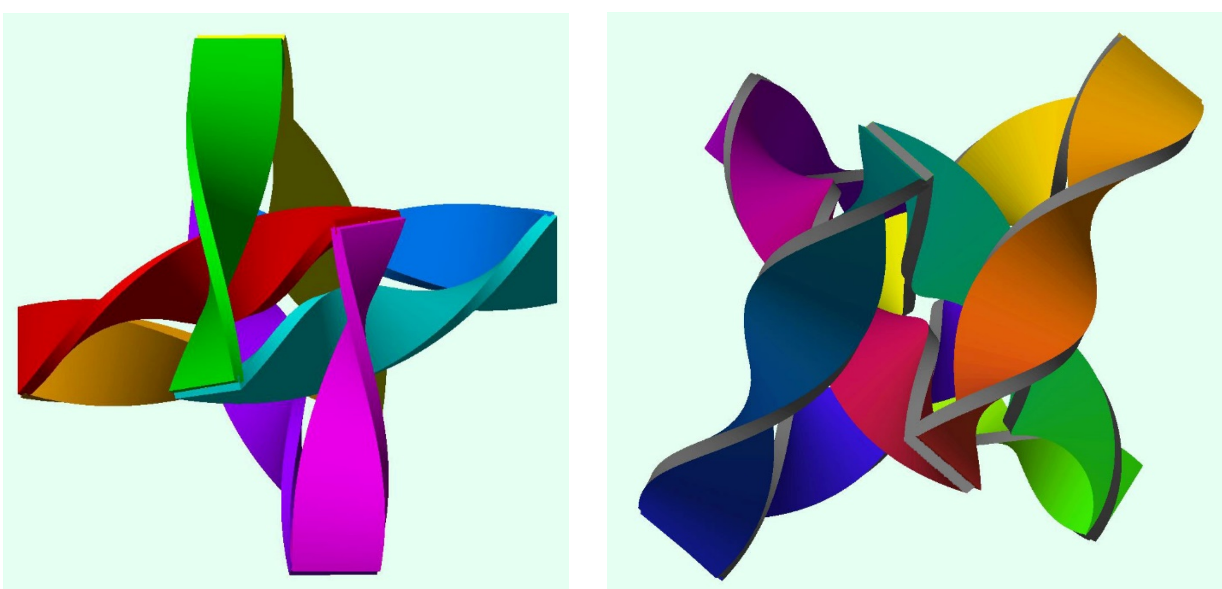
The third knot studied offers a new challenge. The previous two knots have some obvious chirality, and they exhibit D3 and D5-symmetry, respectively. Knot 4_1 is amphichiral, i.e., it can be deformed into its own mirror image. In some renderings this is more obvious than in the standard representation (Fig.6a). Figure 6b exhibits D2-symmetry but does not hint at the amphichiral quality of this knot. On the other hand, the configuration shown in Figures 6c–e all have 4-fold glide symmetry (S4), where a rotation by 90° around the z-axis and simultaneous mirroring of the *z*-coordinates will bring the transformed copy into coincidence with its original shape. Figure 6d shows a reasonably tight sticks configuration, and this serves as a starting point for the minimally twisted ribbon version shown in Figure 6e, in which the red and blue beams have been given some slant to make the loops more balanced and the overall shape less stiff.



(a) (b) (c) (d) (e)

Figure 6: Figure-8 Knot: (a) Smooth rendering; (b) D2-symmetric diagram; (c) S4-symmetric 3D model, (d) 8-sticks model, (e) minimally-twisting ribbon knot, – also with S4-symmetry.

Starting with the configuration of Figure 6e we can select two different amounts of twist in the “horizontal” (red, blue) and “vertical” (orange, cyan) beams. Because of the S4 symmetry the twists are equal and opposite for both color pairs; thus the net overall twist must be zero, as is required for an amphichiral knot. Two resulting designs are shown in Figures 7a,b; their twist values are listed in the figure caption.



(a) (b)

Figure 7: Twistor-Knots 4_1: (a) Twist: hor:±155°, ver: ±251°; (b) Twist: hor: ±338°, ver: ±430°.

References: [1] *Pentafoil Knot Table*. (2006), -- http://www.kosticks.com/furniture-and-cabinetry.html
[2] N. Friedman and R. Krawczyk: *Twistors*. Hyperseeing, Winter 2014 (this issue).