

A 10-Dimensional Jewel

Carlo H. Séquin

CS Division, University of California, Berkeley

E-mail: sequin@cs.berkeley.edu

Abstract

In 3 dimensions there are five regular polyhedra – the Platonic solids. In four dimensions there are six such polytopes. If we allow the polygonal faces to intersect then we also obtain the four Kepler-Poinsot polyhedra in 3 dimensions and 10 such objects in 4 dimensions. If we also allow these constructions to be single-sided, then we can form some non-orientable cells such as the hemi-cube, hemi-icosahedron and hemi-dodecahedron in 4D space. These building blocks in turn allow the construction of two truly amazing regular polytopes, the 11-Cell and the 57-Cell. The full 660-fold symmetry of the 11-Cell, composed of 11 hemi-icosahedra, can only be seen if it is immersed in 10-dimensional space. This highly symmetrical, 10-dimensional jewel seems like a worthy object to be discussed at G4G10.

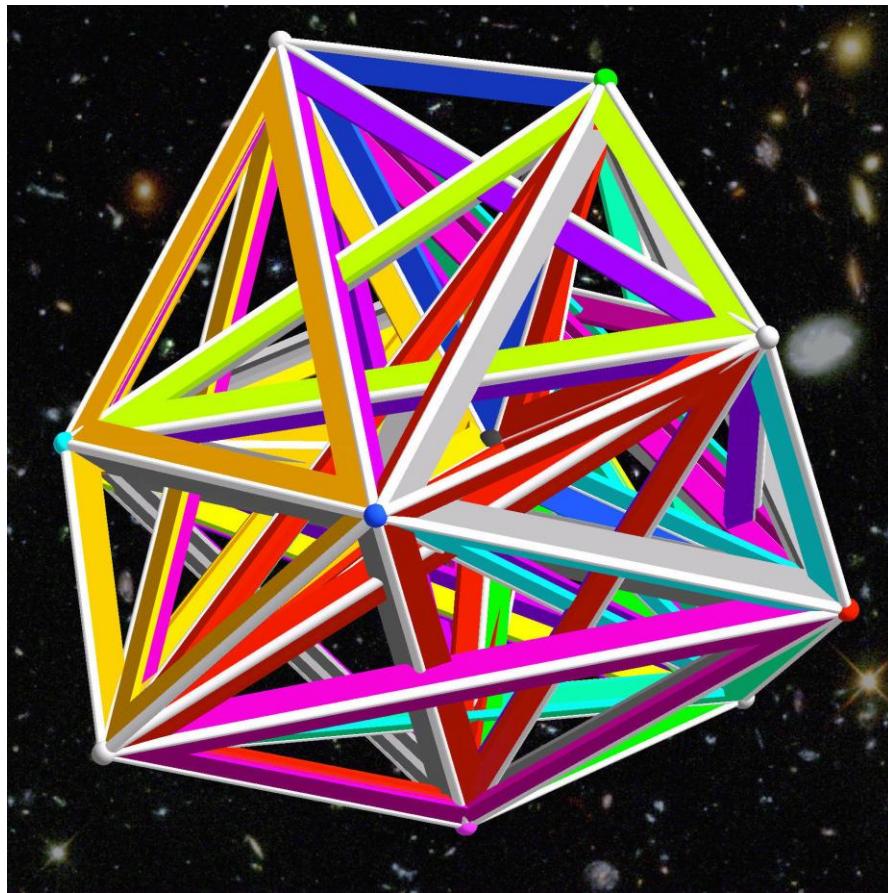


Figure 1: The regular hendecachoron (11-Cell) is a self-dual 4-dimensional polytope composed from 11 non-orientable self-intersecting hemi-icosahedra. But only when immersed in 10-dimensional space can its full symmetry of 660 automorphisms be seen. This artistic construction shows its connectivity.

1. Regular Polytopes in N Dimensions

In this paper we discuss regular polytopes: geometrical objects in which all vertices, edges, faces, cells ... look the same, respectively. In 2 dimensions there are infinitely many regular polygons. In 3 dimensions there are five regular polyhedra, which use regular polygons on their surface: the Platonic solids (Fig.2).



Figure 2: The five Platonic solids in 3D space, bounded by regular polygons.

It is easy to see that there are exactly five of them: When starting to construct a regular polyhedron from equilateral triangles, we need at least three around each vertex to form true 3D solid corners; this leads to the tetrahedron. If we place four equilateral triangles around each vertex, we obtain the octahedron; and five such triangular facets at each corner form a dodecahedron. Six equilateral triangles no longer form a 3D corner, but result in a flat planar tiling. Using square facets, only three give a viable corner, and this produces a cube. Using regular pentagonal facets, three per vertex form a viable corner and produce the dodecahedron; four or more of them around a vertex would result in a non-convex saddle formation. Hexagons, or any larger regular n -gons, are unable to form any convex polyhedral corners. This is why there are only five Platonic solids.

Just like the Platonic solids are formed from regular (2-dimensional) n -gons, regular 4-dimensional polychora are formed by using the Platonic solids as cellular building blocks. An analysis very similar to the one described above [6] reveals that there are exactly six regular polychora in 4D space (Fig.3).

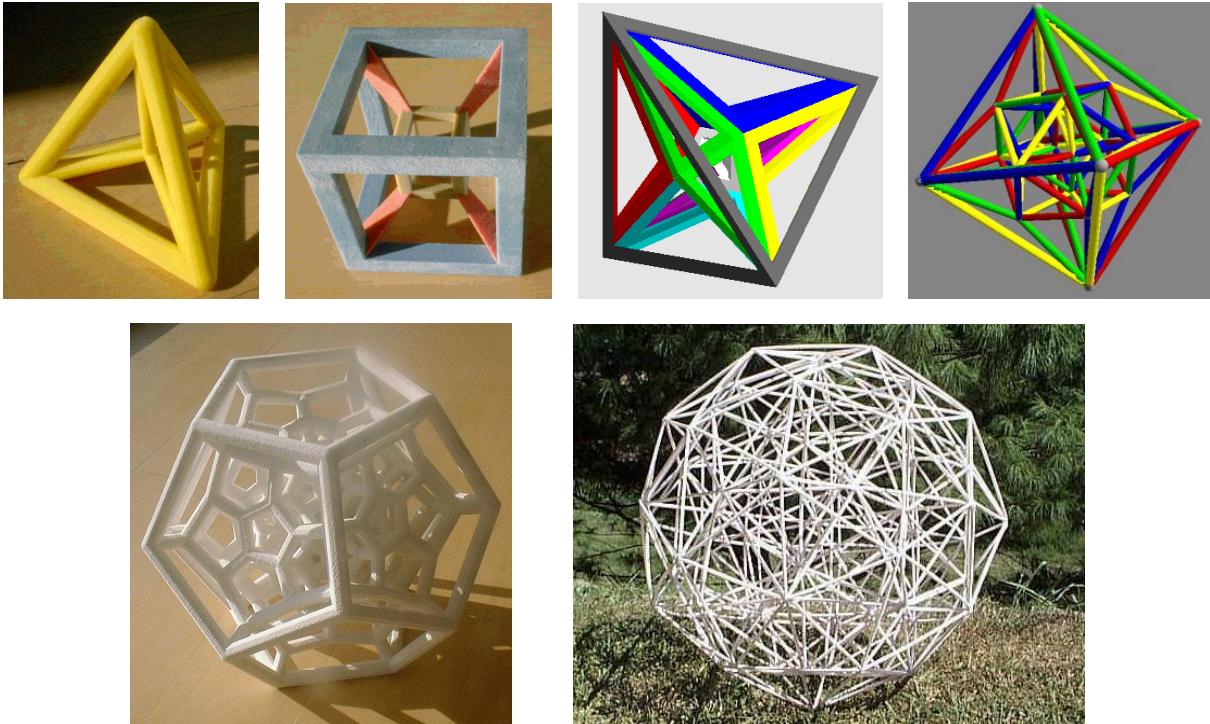


Figure 3: The six 4D polychora: simplex (5-cell), hypercube, cross-polytope (16-cell), the 24-cell, the 120-cell, and the 600-cell.

For all the five Platonic solids, we check how many can be fit around a single shared edge, based on their dihedral angles. As above, we need at least three per edge to form a viable polychoron, but the sum of all the dihedral angles clustered around that edge must remain below 360° to yield a convex polytope.

The regular 4-dimensional polychora may now serve as potential building blocks for regular polytopes in 5D space. As it turns out, most of them are too “round”, i.e., their dihedral angles are too large, to be useful. Only three constructions succeed: The simplex can be used to build the n -simplex in the next higher dimension; the measure polytope (or n -cube) is useful to create a corresponding higher-dimensional object; and the dual of that object, the so-called n -orthoplex or cross-polytope, where the roles of vertices and hyper-cells are exchanged, also exists. This is true for all dimensions larger than 4; those three types of regular polytopes are the only ones that exist in spaces of dimensions 5 or higher [8].

2. Self-Intersecting Polyhedra

We can take a broader perspective of what we consider to be a regular polytope. We may allow faces to intersect one another – or even to be self-intersecting star-polygons; then we obtain an additional four regular polyhedral objects in 3D space: The Kepler-Poinsot polyhedra (Fig.4). Equivalent constructions can also be done in 4D space and result in ten additional self-intersecting 4D polytopes [8].

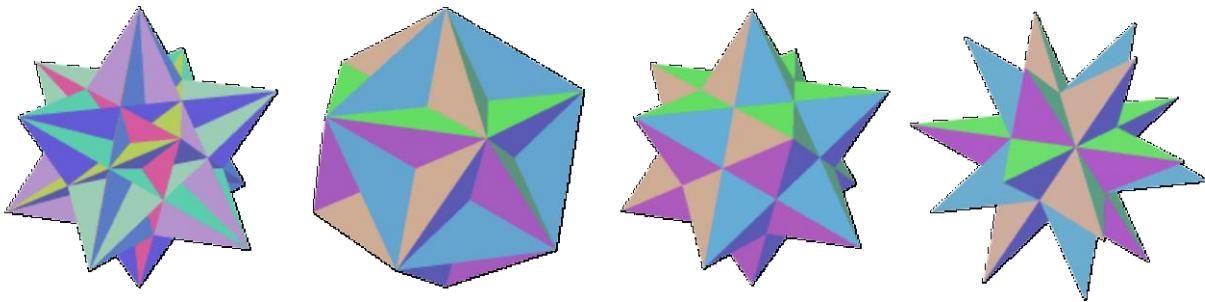


Figure 4: The four self-intersecting Kepler-Poinsot polyhedral in 3D space [9].

3. Single-Sided Polychora

As we move into 4D space, there is one more way in which we can expand our definition of what we accept as a valid regular polytope: We can construct non-orientable, single-sided “cells.” The simplest one of them is the hemi-cube. It is constructed by starting with only half a cube (Fig.5a), in particular the three faces clustered around one of its vertices (D), and joining the six open edges of this 2-manifold so that opposite edges and vertices fall onto one another, as indicated in Figure 5a by the three pairs of arrows with matching colors. Thus we obtain an object with 4 vertices, 6 edges, and 3 quadrilateral faces. The skeleton of this abstract polytope, composed of its vertices and edges, corresponds to the graph K_4 , the complete graph with four nodes, in which every node is connected to every other node (Fig.5b). In 3 dimensions this can be represented as a tetrahedral frame, i.e. the 3D simplex (Fig.5c,d).

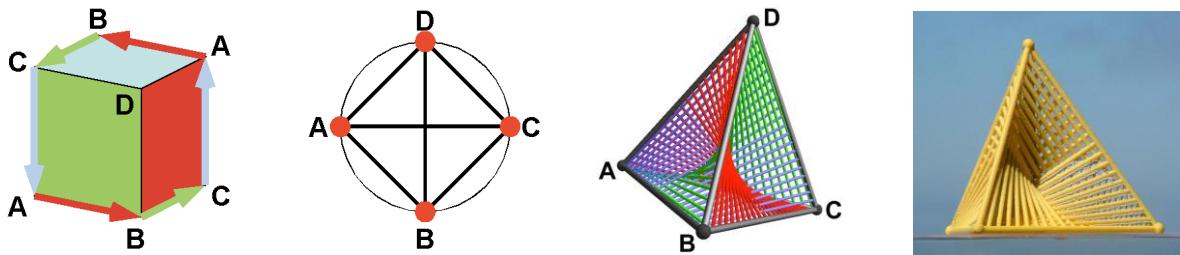


Figure 5: (a) Hemicube; (b) complete graph K_4 ; (c) tetrahedral frame with three saddle faces; (d) rapid-prototyping model of a hemicube.

If we label the 4 vertices as A, B, C, and D, then the three quadrilaterals have the contours: ABCD, ABDC, and ADBC. Clearly these quadrilaterals are not planar, but they can be realized as bilinear Coons patches, and they will all intersect with one another. Topologically, this object has the same connectivity as the non-orientable projective plane (on which you can walk north, pass through infinity, and return from the South – but upside down).

In a way analogous to the above, we can construct an abstract polytope from half an icosahedron (Fig.6a). Again we glue the open edges along the boundary onto each other, so that opposite edges and vertices are being joined. This yields a structure with 6 vertices, 15 edges, and 10 triangular faces. The skeleton of this abstract polytope forms the complete graph K_6 (Fig.6b). This is equivalent to the 5-dimensional simplex. If the 6 vertices are labeled: A, B, C, D, E, and F, then the 10 faces will be the triangles: ABC, ACE, ADF, AED, AFB, BDC, BED, BFE, CDF, and CFE; these are half of all the faces that would appear in the complete 5-dimensional simplex (Fig.6c).

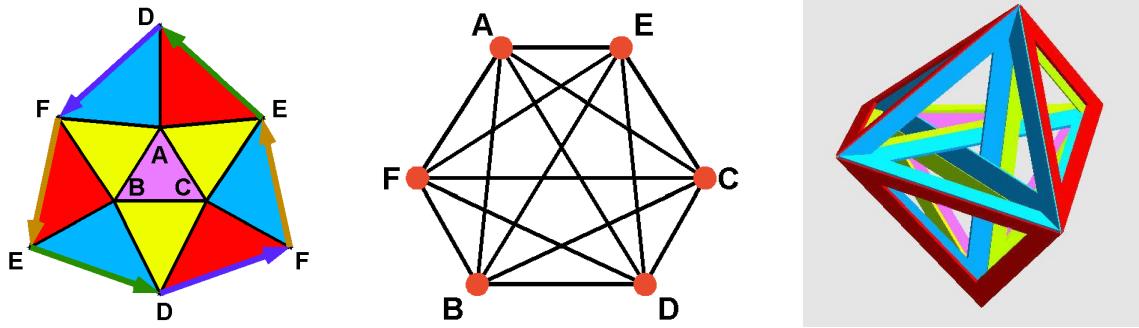


Figure 6: (a) Hemi-icosahedron; (b) complete graph K_6 ; (c) asymmetrical octahedral frame.

Finally, we can do the same construction also with half of a dodecahedron and thereby obtain a self-intersecting polytope with 10 vertices and 6 pentagonal faces.

Both the hemi-icosahedron and the hemi-dodecahedron can be used as building blocks to construct two additional intriguing and difficult-to-visualize self-intersecting polytopes: the 11-Cell [3][2] and the 57-Cell [1]. This paper is focused on the simpler 11-Cell, because to exhibit its full 660-dimensional symmetry, it needs to be immersed in 10-dimensional space. The 10th Gathering for Gardner thus seems like a fitting occasion to take a closer look at this object.

Since our senses have not evolved to experience objects in higher-dimensional spaces, the best we can do, is to project this 10-dimensional object down into 3-dimensional space and study its “shadow” in this more accessible domain. However, simply looking at such a projection would not give us an understanding of this complicated polytope. We will thus take a more constructive approach. First we will take a closer look at the hemi-icosahedron and construct a 3D model of it. Then we discuss how – conceptually – eleven of these cells need to be interconnected to form the regular hendecachoron, and how the result can best be displayed.

Constructing a Hemi-Icosahedron Model

The hemi-icosahedron needs at least a 5-dimensional space to show its full symmetry; in that space all 6 vertices can be placed pair-wise at mutual distances of the same unit length. The most symmetrical way in which we can project this configuration into 3D space is to place the 6 vertices at the corners of an octahedron; however, in this case, three of the edges of this object would coincide at the centroid. To avoid this 3-way intersection, we move the 3 pairs of opposite vertices slightly away from their respective coordinate axes to separate the 3 central edges by a corresponding amount [6]. In Figure 6c, the middle part of all triangles has been cut out, so that only a narrow frame is left; this lets us see inside this self-intersecting polyhedron.

There is more than one way to choose a subset of faces of this 3D graph to form a single-sided surface that is topologically equivalent to the projective plane. I prefer a configuration that is closely related to Steiner's Roman surface (Fig.7a). This representation retains four tetrahedral faces on the basic octahedral shape and adds to those the three medial squares that connect the triangles into a non-orientable 2-manifold. These three quadrilaterals are partitioned into two triangles each to yield a total of ten triangles. Such a hemi-icosahedron is shown in Figure 7b – again with cut-outs in the triangular faces; but now with the edges shown as thin white cylinders and a color assignment that assigns each of the 10 faces a unique color. Figure 7c shows the unfolded net of this object.

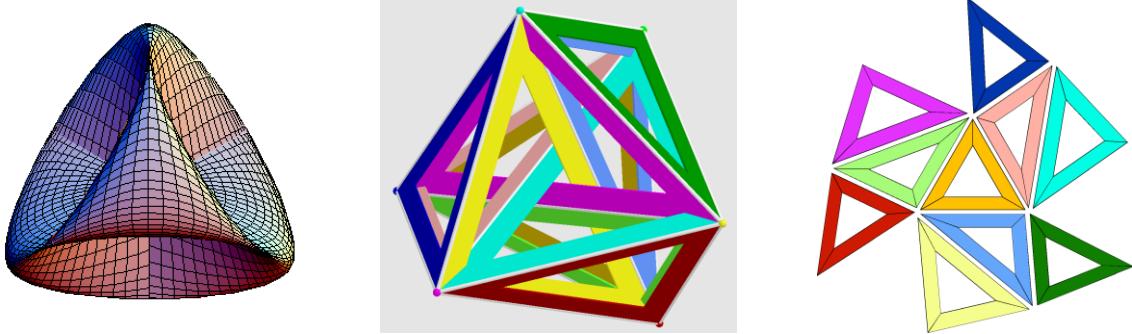


Figure 7: (a) Steiner surface; (b) “Steiner” hemi-icosahedron; (c) the unfolded net of this object.

Here are some instructions for building a paper model of this crucial building block. Print a copy of the template shown in Figure 8a as well as a mirrored copy. Glue the two prints back-to-back so that the shapes are properly aligned. Then cut out the 15 struts with two-sided coloring. Assemble the four outer triangles first (Fig.8b). Then join three of them at their corners as shown in Figure 8c. At this stage, insert the 3 central diagonal struts; and finally close the object by inserting the last of the outer triangles.

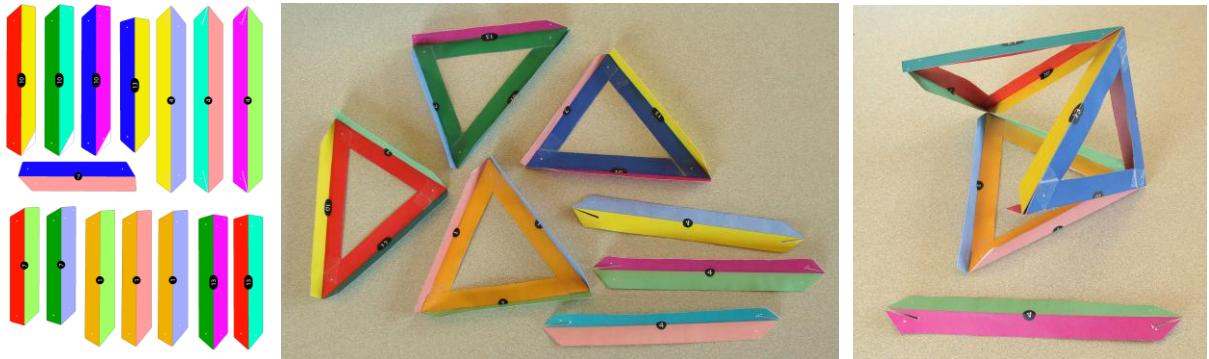


Figure 8: Construction of a paper model of the hemi-icosahedron: (a) template for the 16 struts; (b) octahedral triangles assembled; (c) 3 triangles joined, waiting for a central strut to be inserted.

4. Bottom-up Construction of the Hendecachoron

By now we should have a robust understanding of the crucial building block of our 10-dimensional jewel. Now the question arises, how do 11 of these single-sided cells fit together? Conceptually we start with an initial hemi-icosahedron and glue a copy onto each one of its 10 triangular faces. In 5-dimensional space the hemi-icosahedron, just as the skeleton of the simplex, is totally regular; all edges are of equal length, and all faces are equilateral. Thus two of these cells can readily be glued together by joining two triangular faces. Now we add a third hemi-icosahedral cell to this assembly so that it also shares one of the edges that the first two polytopes already share. Since the dihedral angle of the hemi-icosahedron is 70.53° (the same as the tetrahedron), there is a large wedge of empty space left along that edge between

the third cell and the first cell in this assembly. If we now try to forcefully close this wedge of free space, then the assembly of three hemi-icosahedra will have to bend into the next higher dimension, where the three polytopes form a 3-fold symmetric constellation around the shared edge. This process must now be repeated on all the edges of the hemi-icosahedron. Clearly, this is a conceptual view that cannot be carried out in 3D Euclidean space!

Overall, the assembly of eleven hemi-icosahedra is thus warped into a higher-dimensional space so that all the free-space wedges along all the edges can be closed, and every cell shares a triangle face with every one of the other 10 cells. The enforced bending is so strong that the whole assembly curls up through itself, with opposite faces falling onto one another, so that in the end there is no face left unmatched, and every edge in this structure has exactly 3 hemi-icosahedra around it. The resulting abstract polytope is the regular hendecachoron. It has 11 vertices, 55 edges, 55 triangular faces, and 11 hemi-icosahedral cells; considered as a 4-dimensional object, it is thus self-dual. Of course all those cells wildly penetrate one another, and one would have to go to 10-dimensional space to exhibit this object without any self-intersections. In 3D space it is difficult to make sense of this cluster of mutually intersecting edges and faces. However, if the beginning of this assembly is shown step by step, one hemi-icosahedron at a time, then one can get an idea how all of this fits together.

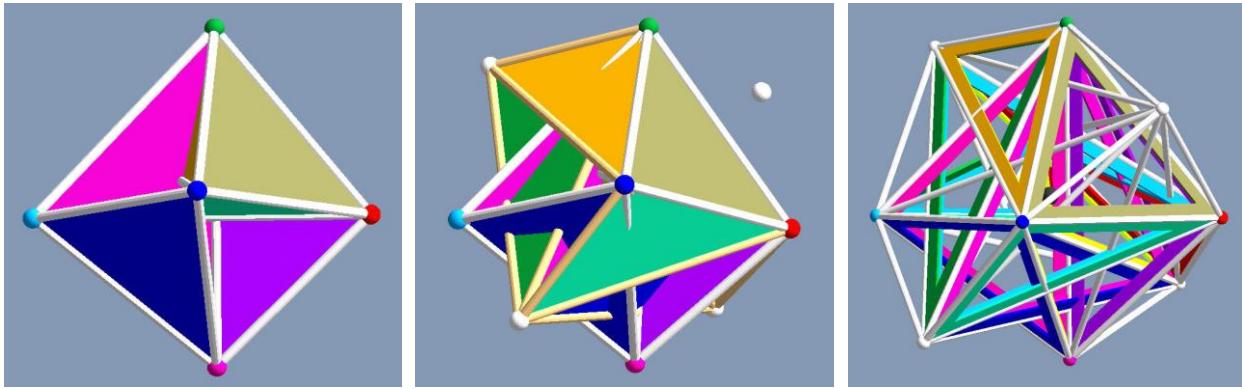


Figure 9: (a) One Steiner hemi-icosahedron; (b) a second penetrating cell added at the tan triangle; (c) a view with cut-out faces to reveal the inside and the new edges needed for the next three cells.

Figure 9 shows the start of such a process. In Figure 9b a second hemi-icosahedron has been added to inner side of the tan surface of a first Steiner cell. In this first phase we will add symmetrically four such mutually interpenetrating cells on the inside of the four octahedral triangles. However, since we already have used six vertices to start with, and can add only five more, these four additional cells will have to share many vertices. The most symmetrical way to add 5 more vertices to the original set of 6 is to add a (white) vertex above each tetrahedral face, and one more (black) vertex in the center of the first Steiner cell (Fig.9c). The four additional cells thus need to penetrate the original one to reach the three vertices on the “other” side from the face that they share with the original cell. Figures 9b and 9c show the state after a first such cell has been added to the original one – attached to the inside of the tan triangle face. Figure 9c also shows the additional edges that will be used by the other three cells added in this second phase.

After this phase, we need to add six more hemi-icosahedra, and these are exactly those six cells that share the (black) vertex in the center of the structure. But again, the structure gets so cluttered that a small-scale rendering does not make a useful contribution to the understanding of the 11-cell.

Figure 10 shows in diagrammatic form how the 11 hemi-icosahedral cells connect with one another. The little colored square in the lower left of each unfolded cell diagram shows the color of that cell, and the facet colors in each diagram indicate to which other cell that facet is connected. The 11 vertices are labeled “0-9”, and “t”. Tom Ruen has added a corresponding coloring [5] to the hemi-icosahedron faces visible in Coxeter’s diagram (Fig.1 in [2]) showing the connectivity of these 11 cells.

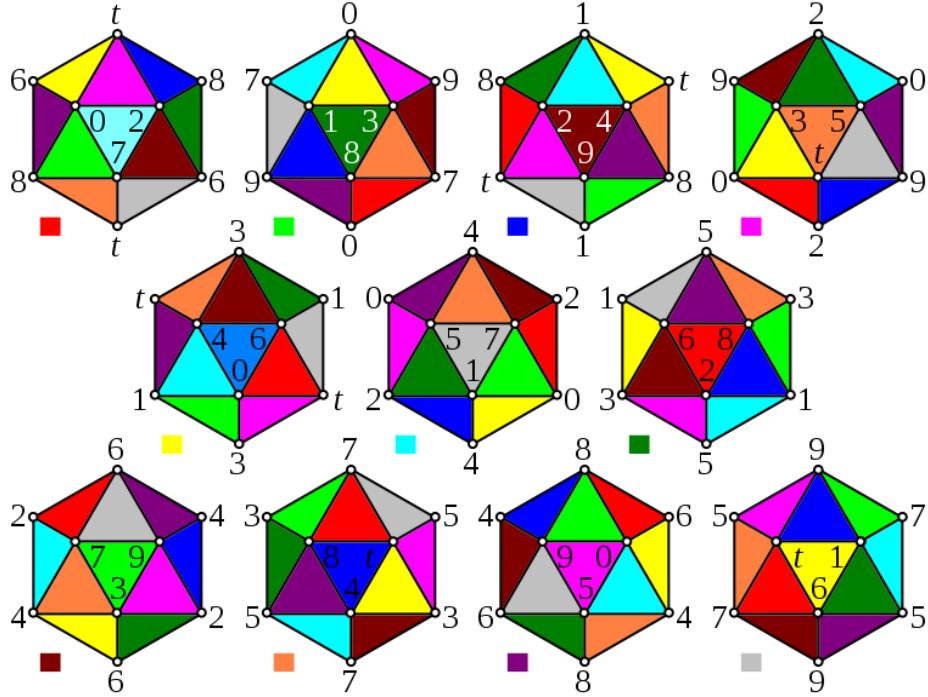


Figure 10: Schematic connectivity diagram of the eleven cells of the hendecachoron [5].

5. Top-down Construction of the Regular Hendecachoron

The above discussion completely defines the connectivity of the regular hendecachoron. But the real challenge still is to find a way to make a 3-dimensional visualization model of the whole thing. Clearly, the placement of the eleven vertices in 3D space is a crucial choice for a good visualization. One would like to preserve as much symmetry as possible, yet at the same time avoid too many coincidences that mask some edges or vertices. Putting all eleven vertices on a circle in a plane would make all 55 edges easily visible, but it would yield a pretty useless visualization of the hendecachoron, since its triangular facets would all lie in the same plane. The key is to find a 3-dimensional arrangement of the eleven points that yields a low variation on the individual edge lengths. The best arrangement that I have found is shown in Figure 11.

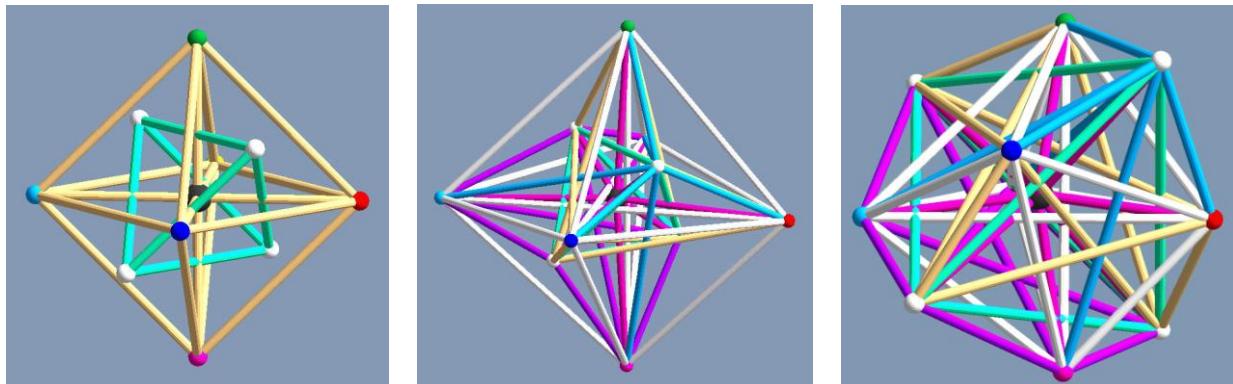


Figure 11: Spherically symmetrical arrangement of eleven points in 3D space; (a) the Plato shells, (b) the complete edge graph among all eleven nodes, (c) ten of the nodes lie on a common sphere.

This vertex configuration starts from an arrangement that locates the eleven vertices on three concentric shells [7]: six points in an octahedral configuration, four points in a tetrahedral arrangement, and the last point at the center of this assembly (Fig.11a). In Figure 11b I have added all the edges of the complete graph K_{11} among the eleven vertices, and a slight asymmetry has again been introduced into the octahedral frame to prevent the edges between opposite poles to pass through the center vertex. Finally I moved the four vertices of the tetrahedral frame outwards to fall also onto the circum-sphere of the octahedron (Fig.11c).

In this framework of points and edges we can now display all 55 faces as narrow triangular frames, and this leads to the display shown in Figure 1. The ten faces belonging to the same hemi-icosahedron have been given the same color – corresponding to the little colored squares in Figure 10. However, since each face is shared by two cells and the hemi-icosahedra are single-sided (non-orientable) surfaces, the resulting visible coloring is somewhat arbitrary, and each face would have two different colors on its two sides. An attempt at an artistic visualization of this object has resulted in Figure 1. The “starry” background filled with galaxies photographed by the Hubble space telescope is another whimsical reference to the fact that this object really wants to live in 10D space: Many models in String theory postulate that our universe is 10-dimensional, but with 6 of its dimensions curled up smaller than even the sizes of the known elementary particles. This raises the capricious question, whether this 10-dimensional polytope might be a building block at the Planck scale of such a 10-dimensional universe.

Acknowledgements

I am grateful to Jaron Lanier for focusing my attention onto this object, when he asked me to create a visualization of the strange and hard-to-visualize “hendecatope” for his column *Jaron’s World* in Discover magazine [4]. I am also indebted to many other people, who helped me deepen my understanding of this object and convinced me that a better name for it is: “Regular Hendecachoron.” This work is supported in part by the National Science Foundation: NSF award #CMMI-1029662 (EDI).

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