Prototyping Dissection Puzzles with Layered Manufacturing

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Abstract

Dissections of simple geometrical forms can be used to train students' spatial understanding and to teach geometrical modeling as well as some of the practical aspects of rapid prototyping by layered manufacturing. Two types of dissection puzzles have been used as class exercises in a graduate course on computer-aided solid modeling: Helicoidal sectioning of simple geometrical shapes and a burr puzzle based on a cubic grid.

1. Introduction

Dissection puzzles [1] are excellent vehicles to study geometric shape design and to train one's understanding of 3D space. For this reason we have introduced dissection puzzles in many offerings of a graduate course at U.C. Berkeley concerned with modeling of solid shapes (CS285). Before we give the students their own assignments, we typically show them some Hamiltonian dissections of the Platonic solids (Fig.1). In these Hamiltonian dissections, a particular solid may be cut by a sweep path anchored at the center of the shape, while the other end of the cut line is swept along a Hamiltonian circuit formed by the edges of the polyhedron. These dissections don't form any mysterious "puzzles" – except perhaps for the case of the icosahedron; this shape typically resists being taken apart for a while, because one must grip each part properly with three fingers to slide the two halves apart. However, such dissections are still satisfying, since they lead to nice geometrical forms that are of potential sculptural interest.

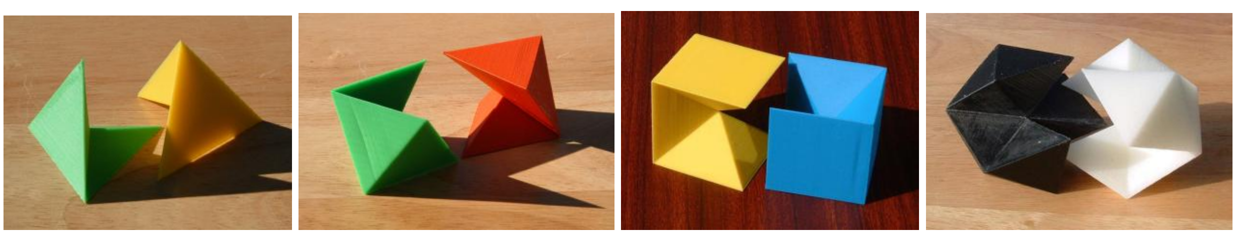

Figure 1: *Hamiltonian dissections of Platonic solids.*

We also discuss in what other ways one might cut these solids to make puzzles that are more challenging. A particularly intriguing dissection of the cube is shown in Figure 2, where each part is of genus 1. Again, it is mostly an issue of gripping the two halves properly to take this puzzle apart easily.

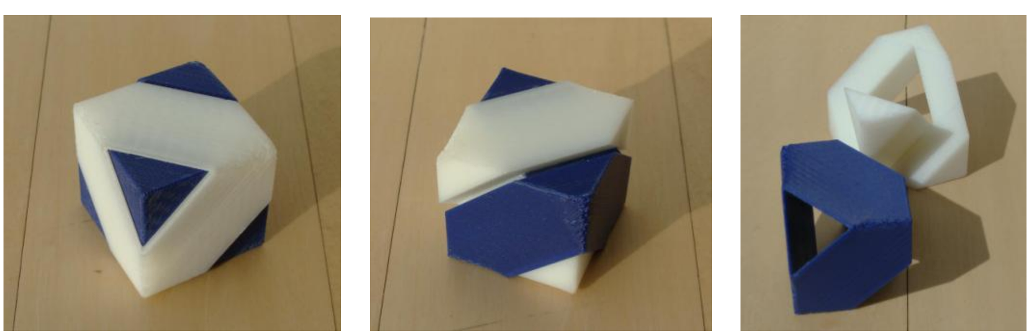

Figure 2: *Dissection of a cube into two congruent parts of genus 1.*

In this paper we discuss two kinds of puzzles: Helicoidal sectioning of some simple shapes (Sections 2 and 3) and realizations of some burr puzzles (Section 4). In the first case, the challenge for the students consists in conceiving and visualizing an appropriate shape and then finding a way of describing that geometry with some simple CAD tools. In the second case the geometry is mostly given, and the focus is on realizing that shape in the most cost-effective manner via rapid prototyping using layered manufacturing on a Fused Deposition Modeling (FDM) machine.

2. Helicoidal Dissections

In the fall of 2011, I decided to experiment with curved surfaces, so I gave the students the following assignment:

- *Design a two- or three-piece geometrical puzzle in which a simple shape is partitioned into all congruent parts via a helical screw motion.*

The assignment was done with teams of 3-4 students. They first discussed the range of possibilities and then picked their own creative designs. All teams quickly figured out that the parting surfaces would have to be one or more helicoids winding around a common straight line, *e.g*. the *z*-axis. It was then their choice to select a suitable overall shape, positioned symmetrically with respect to the chosen system of helicoids, so that the resulting dissection (or tri-section) pieces can be congruent. Different teams picked quite different shapes. The students who were relying solely on the default modeling software offered with this course, the SLIDE [10] system designed in the 1990s, were typically using rotationally symmetrical shapes, because SLIDE offers powerful and easy-to-use sweep constructs, but has no Boolean CSG (constructive solids geometry) operations. Figure 3a shows one of the resulting shapes.

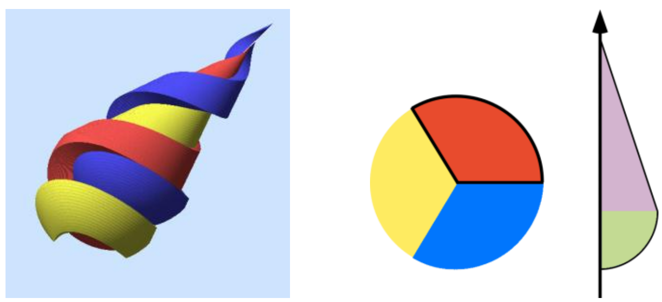

Figure 3: *Rotationally symmetrical helicoidal tri-section puzzle: (a) 3-part CAD model; (b) generic scalable cross-section; (c) scaling function profile along the z-axis.*

All the students had to do in this case was to choose a disk segment spanning 120° as the generic cross-sectional profile for each of the three parts (Fig.3b) and then sweep that profile along the z-axis. During that sweep, an arbitrary scaling function (Fig.3c) can be applied, which will generate the desired rotationally symmetrical form.

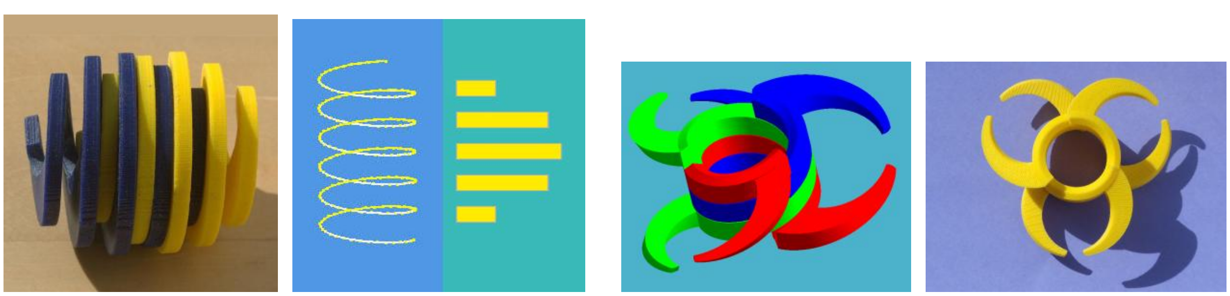

Figure 4: *Dissection using helicoidal sweep: (a) fabricated puzzle; (b) sweep path with 5 some sample cross sections. – Helicoidal dissection of Bio-Hazard symbol: (c) CAD model; (d) physical artifact.*

Other students, who created more heavily intertwined screw-shapes (Fig.4a), started by defining a simple helical sweep path (Fig.4b) and then swept a suitably parameterized cross section along this path. In this particular case it was a rectangle of constant height determined by the pitch of the helical screw (Fig.4b), but which was varied in width (along a hemi-circle) so as to give the desired overall spherical shape (Fig.4a). As a variation on this basic approach, other students started from a ring-shaped base, but added some "decorations." One team created a 3D solid rendering of the Bio-Hazard symbol (Fig.4c and 4d).

Another team of students who had access to SolidWorks [12], which provides Boolean Constructive Solids Geometry (CSG) operations, decided to partition a cube along a space diagonal into three congruent pieces, which by themselves also exhibited 2-fold rotational symmetry. To accomplish this, they adjusted the pitch of the helicoids so that each cutting surface makes a 780° turn around the z-axis while sweeping from the top to the bottom corner of the cube (Fig.5a and 5b). To create this shape within SLIDE is more challenging, since SLIDE does not offer CSG operations. Thus the external point of a sweep line cutting through the cube – and simultaneously defining the curved surface of the resulting trisection parts – must be programmed explicitly to follow the surface of a suitably oriented cube.

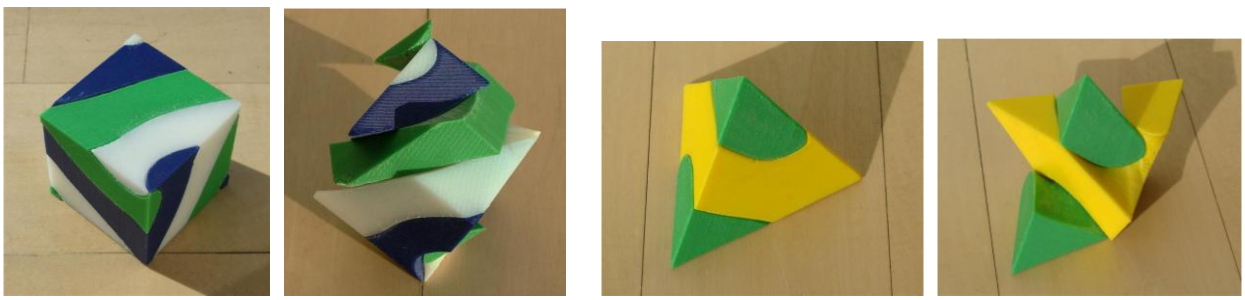

Figure 5: *Helicoidal tri-section of a cube: (a) aligned, and (b) with the green part slightly twisted apart. Helicoidal dissection of a tetrahedron: (c) aligned, and (d) the yellow part slightly twisted apart.*

As a demonstration that this task is not "impossible" I created the helicoidal dissection of the tetrahedron shown in Figures 5c and 5d. In this case the cutting path on the surface of the polyhedron had to be traced through only four faces from one edge center to the opposite edge. Since the chosen geometry exhibits D_2 symmetry, the path had to be calculated through only two faces. Six path points were calculated explicitly on each surface and connected with a piece-wise linear polyline.

In all these puzzles, issues of geometrical design, numerical accuracy, and suitable tolerancing had to be addressed. The primary question to be decided is how many turns the helicoids should make while passing through the overall shape to be dissected. In the case of the Bio-Hazard puzzle, having each part execute a full 360° turn was barely enough to hold this loosely structured puzzle together. A similar problem plagues the elegant quadrisection of the rhombic dodecahedron posted by George Hart on his website [6]. The four pieces slip easily together (Fig.6b), but they also tend to slip apart under the force of gravity, if the puzzle is set down with its helicoidal axis positioned horizontally (Fig.6b).

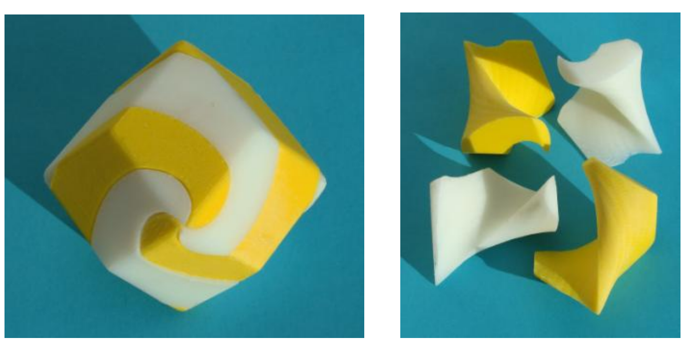

Figure 6: *Helicoidal quadrisection of a rhombic dodecahedron (design by George Hart [6]).*

For other designs like the trisected teardrop – or "upside-down ice-cream cone" (Fig.3) the geometry was so tight and the friction was so high that the three pieces could not be fully screwed together until the helicoidal parting surfaces had been sanded thoroughly. The designers of the trisected cube (Fig.5a) struck a good compromise and hollowed-out a central portion of the cube with a diameter equal to about half the cube edge-length. This reduced friction dramatically, and the puzzle slipped together with only minimal sanding. The central cavity can then also be used to hide a small surprise trinket.

3. Generalizing Helicoidal Dissections

A general way to create helicoidal dissections is to start from any dissection that is based on a linear sliding motion, and then twist space in a helical manner around an axis parallel to the original sliding action. For instance, we could start with a structure like the cube dissection shown in Figure 2 and create a similar prong and sleeve structure that twists through some reasonable helical angle. Alternatively, we can give each part more than one prong, as is the case in the Hamiltonian dissection of the icosahedron (Fig.1d). This concept is illustrated in Figure 7. In this case the central region of space occupied by the prongs has been partitioned into six segments of 60° degrees each. Every other one of the six prongs is connected to one of the two end-caps (Fig.7a). Two congruent parts now can engage via a linear sliding motion. If the whole geometry is twisted around the z-axis in a helical manner with constant pitch (Fig.7b), then two identical parts slide together with the same helical screw motion (Fig.7c). Figure 7d shows this geometry realized on our FDM machine. Since no explicit gap had been programmed into this geometry, quite a bit of sanding was required to make the parts fit together smoothly (Fig.7e).

Figure 7: *Multi-prong helicoidal dissections: CAD models: (a) linear slide-apart geometry; (b) one part twisted; (c) two parts intertwined. Realization of this concept: (d) the two pieces shown individually; (e) the two pieces partly intertwined.*

Figure 8: *Helicoidal dissections with unequal prongs: (a) two pieces with sector'ed prongs, and (b) the two pieces partly intertwined. More diverse prong geometries: (c) the two pieces as designed with linear slide-apart prongs, and (d) the two pieces twisted and partly intertwined.*

Of course, there is no need to give all the prongs the same shape. Figure 8a shows two congruent parts with four prongs that all span different angles, and Figure 8b shows that they still fit together. The prongs need not necessarily "touch" the helical axis.

A more varied arrangement that still guarantees congruence of the two parts is shown in Figures 8c and 8d. Each prong fits into a corresponding sleeve tunnel in the "end-cap" of the other part. These end-caps have been shortened to expose the geometry of the prongs and sleeves (Fig.8c). Once the basic arrangement has been designed, the whole geometry can again be twisted in a helical manner around a screw axis that runs parallel to the original prismatic prongs (Fig.8d). The prongs may pass all way through the end-caps to create puzzle pieces of higher genus.

In a previous offering of the course, Matthias Goerner designed an intriguing isohedral helicoidal tile that can tessellate all of 3D Euclidean space. His solid tile consists of a two-story composite of two heavily serrated, helical pinwheels (Fig.9a). Each tile slides into the collection of its nearest neighbors with a helical screw motion, thus leading to a layered tessellation of 3D space (Fig.9b). This can be seen as the ultimate generalization of helicoidal dissections.

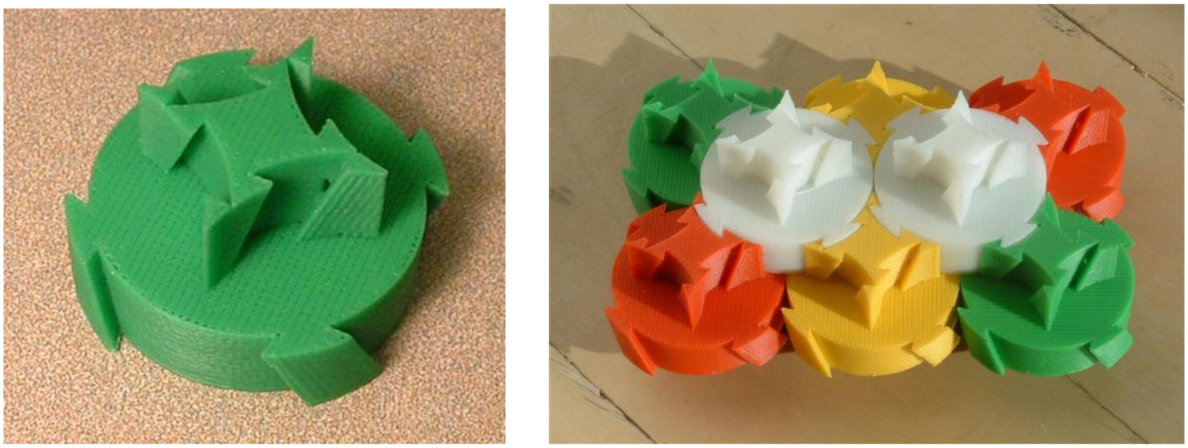

Figure 9: *A modular helicoidal tile and the way it tiles 3D Euclidean space.*

4. Interlocking Cubic Burr Puzzle

Another design exercise for this class was based on an interlocking burr puzzle, dissecting a 4×4×4 cube into four different pieces, each composed of 13, 14, 16, and 20 cubelets, respectively. A small version of the puzzle, measuring only 16mm on a side (Fig.10), was bought from ShapeWays [9] for less than US$10.--. This object was fabricated on demand by selective laser sintering (SLS). Since the price of such a part is more or less proportional to the build volume of the object, I posed the following problem to my graduate class:

- *Figure 15 shows an appealing cube-dissection puzzle. I would like to have one like it – but at a much bigger scale (say, scaled up by a factor of 8 or 10)! Find a design that can be built at this larger scale in an economical way with a layered manufacturing process.*

A consensus evolved during a couple of in-class discussions that it would be advantageous to design just one cubelet module and then instantiate this cubelet as needed to form the different parts of such a burr puzzle. Since our class had ready access to an old Fused-Deposition-Modeling (FDM) machine from Stratasys [13], and to make the design task more specific (and also somewhat more difficult), the students were asked to design such a cubelet module specifically for our FDM machine (type 1605). The main challenge here is that overhanging segments of the part, constructed sequentially layer by layer, need to be supported with some "scaffolding" material, which is later removed by manually breaking it away. The goal is to minimize the total build time as well as the total amount of build- and support-material used. (These two requirements are essentially identical for this machine).

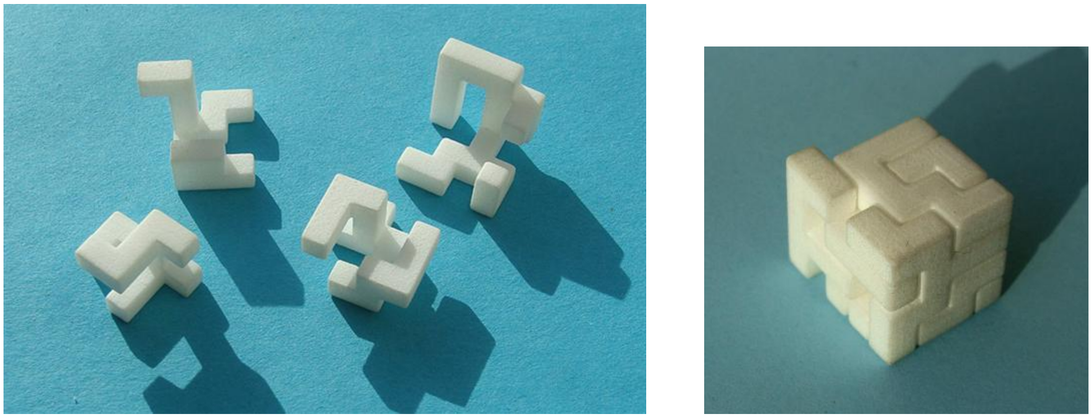

Figure 10: *Small cubic burr puzzle from Shapeways [9].*

One of the plausible solutions is to build each cubelet as a thin hollow shell of a cube. Each such cube would then be filled loosely with some scaffolding material, which, however, cannot be removed (Fig.11a). It would fill each cube completely but in a much looser way than the density of the deposited material in the cube shell itself. It would add mechanical strength to each cubelet, and thus might allow us to make its shell rather thin (perhaps only about 1mm), which would correspond to four layers (or four bead-widths) of deposited material. – The question is: *Can we do better?*

Another approach is to build only a thickened edge-frame of the cube (Fig.11b) – akin to the style used by Leonardo DaVinci to depict some to the regular and semi-regular polyhedra. This approach can save build material as well as support material, if the scaffolding material can be restricted to the vertical walls of each cubelet, and if the central portions of the cubelets can be kept free of any type of build material (Fig.11c).

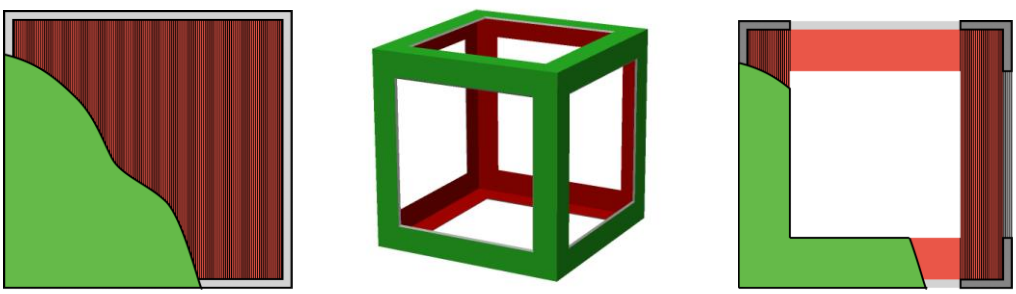

Figure 11: *Possible cubelet designs: (a) cross section through a hollow cube shell; (b) Leonardo-style cube frame; (c) cross section through this cube frame with support structure (dark vertical lines).*

This approach makes the puzzle more transparent and thus makes visible its internal geometry; which may be seen as a plus or as a minus. In a good burr puzzle the first few pieces should not be able to slide out immediately from the overall cube. In a first step they should be able to move only by a small amount, which then allows some other piece to make some limited movement, until some other piece can be freed completely after a carefully orchestrated sequence of moves of several different pieces. This makes the puzzle rather hard to solve if there is a lot of friction and if one cannot see which pieces are adjacent to internal voids that would allow them to move at all.

When trying to limit the scaffolding material to the vertical walls, we had to contend with some idiosyncrasies of *QuickSlice*, the software that slices the geometrical part into 10mil (0.25mm) thick layers and then drives the FDM machine to "paint-in" each such layer with a back-and-forth motion (in the x-y-plane) of the nozzle that dispenses the hot, semi-liquid ABS plastic. In this program, the user has the option to specify what kind of supports the machine is supposed to build. First we tell the machine to simply build straight vertical support structures. Since we expect all support structures to be small and locally confined to the vertical walls of the cubelets, there is no need to use any tapered lateral growth for stability. Furthermore we specify that overhanging faces of 45° or steeper need no support structures, but can be built by relying on cantilevering outwards the beads in subsequent layers by half their diameter.

Unfortunately, in the first few trial runs, QuickSlice still wanted to construct scaffolding throughout the whole volume of the cubelets. – There seems to be some routine that tries to fill in small areas (of less than about 0.5 inch^2) surrounded by scaffolding. We had to scale up individual cubelets to an edge-length of 1.2 inches, and narrow down the frame width to 0.18 inches to create large enough openings in the horizontal cube faces that will not get filled in by support material.

To minimize the support structure within each vertical cube face window, we set the flange thickness of the cube frame to 0.06 inches; this will become visible as the width of the "window sill." Moreover, all the struts of the cube frame are beveled at 45° on the inside, so that no support material is needed below them (Fig.12a). Thus, theoretically, it is only a volume of 0.84×0.84×0.06 inch^3 for each vertical cubelet face window that needs to be filled with support material. QuickSlice thickens this support pad to 0.12 inches (yellow/black regions in Fig.12a). This is still a very small volume of support material, and the resulting pads can be removed easily in the clean-up phase.

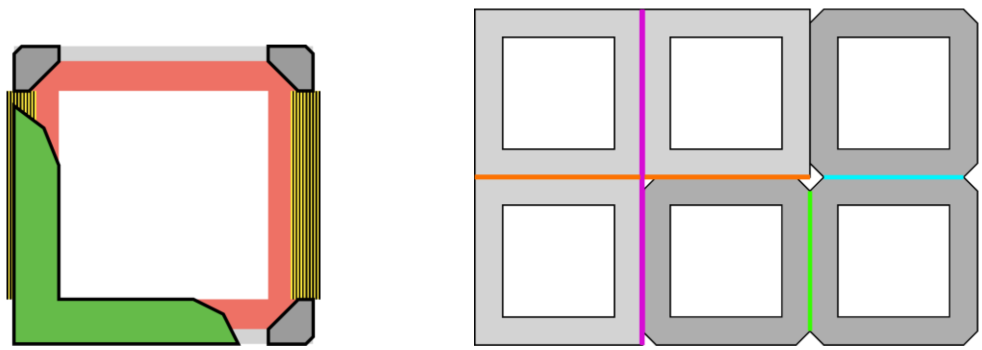

Figure 12: *Design details for cubelet frame: (a) cross-section through one cubelet, showing internal and external bevels on edge beams. (b) Arrangement of six abutting cubelet frames: Without the external bevels, double-length edges (orange and purple) would be formed by QuickSlice.*

Another problem was that the file we sent to QuickSlice was not a true 2-manifold structure, but rather a union of abutting boundary-representations for individual cube frames (Fig.12b). Thus in all the contact planes, the STL file sent to QuickSlice had coplanar polygons with reversed polarity (opposing face-normals). We hoped that QuickSlice would remove the resulting coinciding edges with opposite directionality – and in about half the contact faces this actually happened. It took me quite some time to figure out why this was not happening everywhere.

First we suspected numerical inaccuracies due to rounding errors in the transformations applied to individual cubelets and to the individual struts within their cube frames; – but the STL files had perfectly good numerical values in them. Eventually we concluded that QuickSlice must merge edges produced in the slicing operation as it goes along, rather than delaying the merging process until after potentially redundant edges have been removed. In this process QuickSlice combines collinear edges that share one end-point (corresponding to collinear cube-edges in neighboring cubes) into a resulting edge of double length (orange and purple edges in Fig.12b). Single-length edges that might later coincide with only half of this combined edge then have no redundant partner with which they could be annihilated. To get around this problem, each cubelet was provided with a tiny bevel along all its 12 outer edges (right-most elements in Fig.12b). This prevented collinear edges in adjacent cubes from sharing a vertex and from being merged (blue and green edges in Fig.12b). In this way it was possible to prevent the generation of undesirable boundaries between abutting cube faces, which in turn assured that all cubelet frames were solidly fused to one another.

The reader may wonder why I am spending so much time discussing ways to get around the idiosyncrasies of a 17 year old rapid prototyping machine. New machines and their software are not necessarily better; they have their own quirks! Moreover, as systems become more automated and more encapsulated, it becomes even harder to predict what exactly a machine will be doing when it calculates the detailed instructions for the implementation of a particular part. It is thus highly advisable to always run a few informative test parts when switching to a new machine or implementation service.

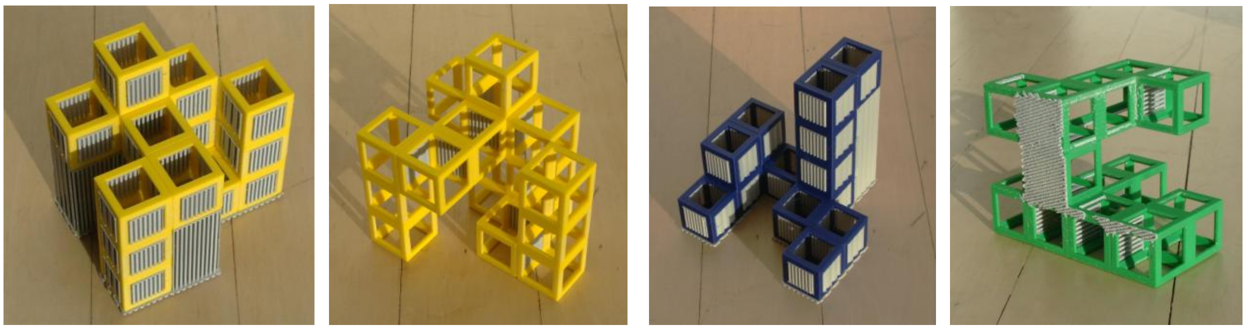

Figure 13: *Grid-frame parts of the cubic burr puzzle: Yellow part (a) with, and (b) without support material (from a different view angle). (c) Blue piece as it comes out of the FDM machine and (d) green piece with its support partly removed.*

After all the above design adjustments, the Stratasys 1650 FDM machine built the various composites of cubelets as we had intended. For the yellow part with 16 cubelets (Fig.13a) the build process took about 37 hours. Manually removing the support structure (Fig.13b) took about 30 minutes. Two additional parts are shown in Figures 13c and 13d. Minor sanding on the outside edges was required to remove some of the rough spots and to make sure that the puzzle pieces would slide together easily (Fig.14a) and eventually form the complete 4×4×4 cube (Fig.14b).

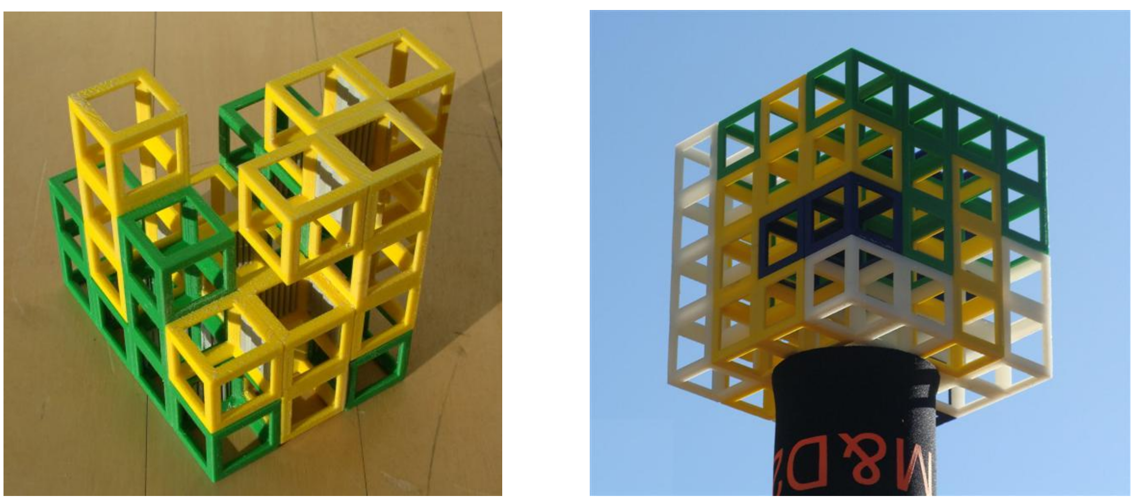

Figure 14: *(a) Partly assembled cubic burr puzzle, showing two of the four pieces in their final, desired positions. (b) All four pieces assembled into a 4×4×4 cube.*

Inspired by the above design exercise, Frederick Doering, a student in the 2011 class, decided that for his final course project he would develop a program that helps in the analysis and design of such interlocking burr puzzles based on cubes. Relying heavily on the pioneering work by W. H. Cutler [2][3][4][7], he developed a program that determines the movability of individual parts and then uses an exhaustive search to find non-trivial combinations of moves that will take the puzzle apart. He found a 2-piece, 3×3×3 cube puzzle that takes three moves to come apart completely.

Since this conference track is also concerned with sculpting, it should be noted that these grid-frame puzzle pieces are rather intriguing and attractive shapes by themselves. I could readily see them enlarged to a 30-foot scale and then installed in some public plaza as a monumental constructivist sculpture (Fig.15).

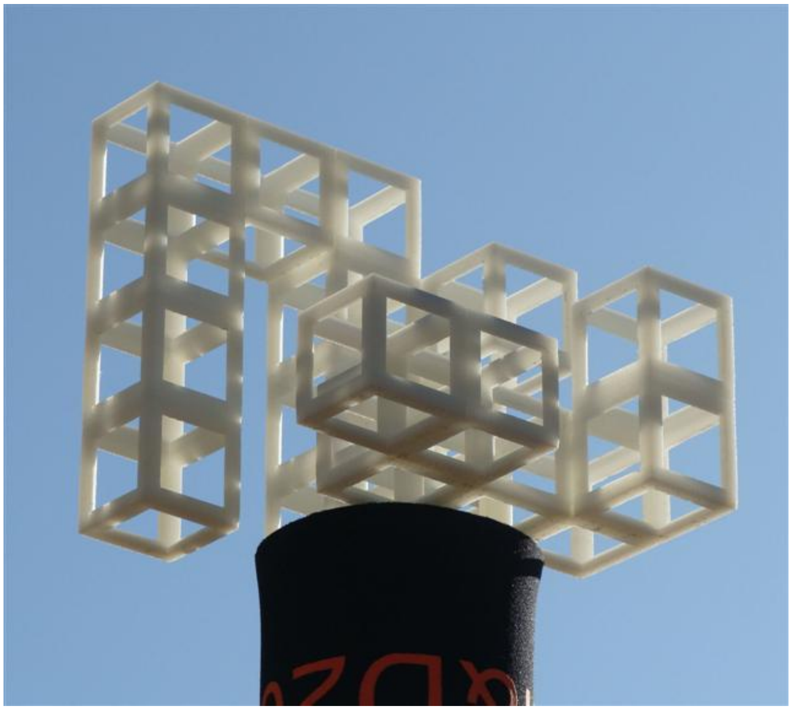

Figure 15: *A single puzzle piece seen as a sculpture model.*

5. "Multi-hand" Dissections

All puzzles described so far allow the individual movement of one puzzle piece at a time. But there are other puzzles that "cannot be taken apart with two hands [11]." They require a simultaneous, coordinated motion of several pieces to get the puzzle apart. A simple example of such a puzzle, which relies on linear disassembly motions, is shown in Figure 16. In this compound of three angled parts, each having an angle of 120° between the prong and the tunnel, all three parts have to move at the same rate in a star-shaped manner for this compound to be able to separate. The question then arose whether a puzzle like this can be designed so that all the movements are helical screw motions. I leave this question open as a challenge for the reader.

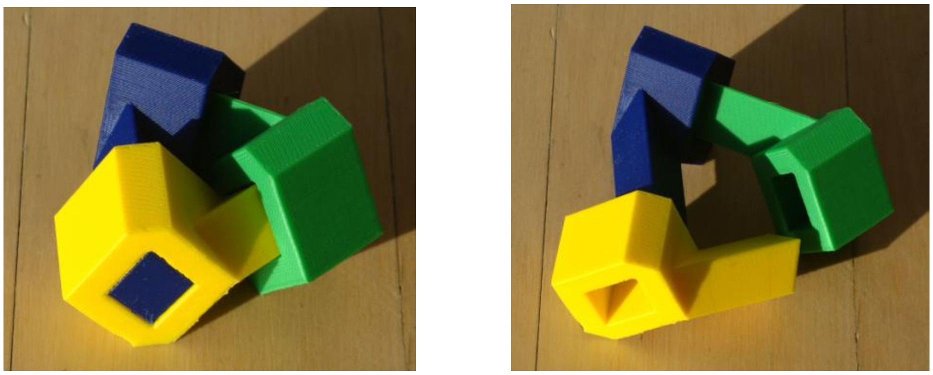

Figure 16: *"Three-handed" dissection puzzle: (a) assembled; (b) partly disassembled.*

6. Conclusions

Over the years, the design of simple dissection puzzles and the fabrication of some of them on a Fused Deposition Modeling (FDM) machine have been highly valuable experiences for the students in a graduate course devoted to computer-aided solid modeling and rapid prototyping. Since that course focuses on procedural design, the generation of helicoidal surfaces, along which pairs of parts that separate with a helical screw motion, is particularly suitable. The individual parts of these puzzles are supposed to abut tightly against one another; thus problems of accuracy and tolerances become a primary issue.

The possibility to hold in their hands a smoothly working puzzle that they can show off to their friends and relatives is a strong motivating force for the students. For the teacher in this class, this excitement is infectious. Every time this course is offered, I am surprised by the creativity of some students and I learn something new myself.

Unfortunately course time is limited; so we can only explore a small fraction of all the intriguing design aspects of the various types of puzzles. For interlocking polyhedral puzzles, Cutler's work [4] is inspirational. The IBM Burr puzzle site [7] is another good starting point. And recently some rather artistic solutions have been presented to turn arbitrary free-form shapes into appealing puzzles [8][14].

Acknowledgements

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