Tori Story

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Abstract

All possible immersions of a torus in 3D Euclidean space can be grouped into four regular homotopy classes. All possible immersions within one such class can be transfigured into one another through continuous, smooth, homotopy-preserving transformations that will put no tears, creases, or other regions of infinite curvature into the surface. This paper introduces four simple, easy-to-understand representatives for these four homotopy classes and describes several transformations that convert a more complex immersion of some torus into one of these representatives. Among them are operations that turn a torus inside out and others that will rotate its surface parameterization by 90 degrees. Some new, aesthetically interesting torus models are also presented.

1. Introduction

Topologically, a torus is a rectangular domain folded up onto itself, so that opposite edge pairs merge with proper orientation (Fig.1a). Already the first merger of such an edge pair can be done in many different ways: It may either form a cylindrical tube (Fig.1b), or the rectangle may pass through itself to form a figure-8 shape (Fig.1c) or a cylinder with two or more layers (Fig.1d). In the second merger step, where these "tubes" are closed into a loop, this closure may be performed without any twist, or with discrete twists in increments of a full turn (360°). The sweep path of this loop may be circular, or it may form multiple loops, self-intersecting figure-8 shapes (Fig.2), or even more complex knots or tangles [11].

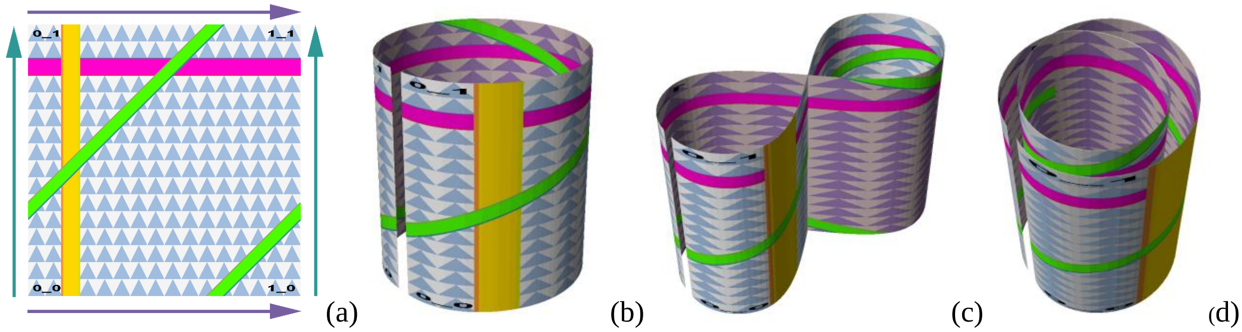


Figure 1: *Folding up a rectangular domain (a) into a torus: Merging the two vertical edges into a cylinder (b), a self-intersecting figure-8 shape (c), or a multiply-rolled tube (d).*

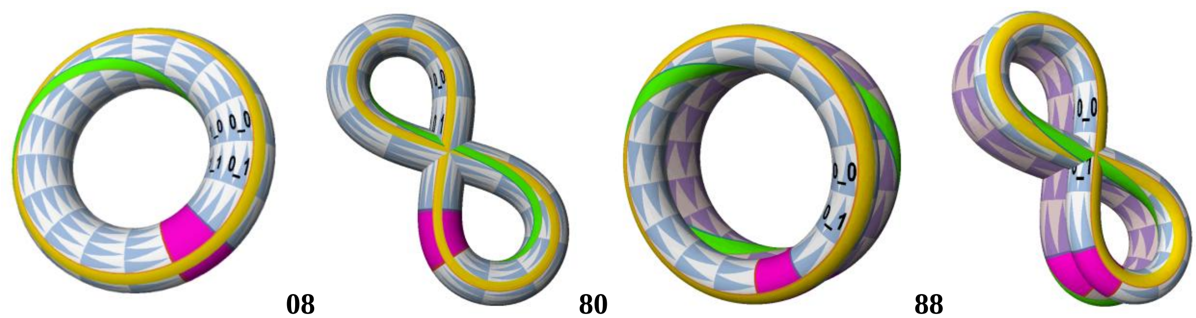
We evaluate the result not just as an unmarked shape, but as a parameterized surface that displays the parameter markings of the original rectangular domain, the front of which is shown in Figure 1. The back of this rectangle carries the same (mirrored) pattern but with a more purplish hue and darker shading. We will only allow transformations that maintain the end-to-end connectivity of the pattern placed on this rectangle. In these transformations, surface regions may pass through one another, but no tears, punctures, creases, or spots of infinitely sharp curvature are allowed. At any moment, every small localized piece of the torus surface must be homeomorphic to a disk. If this condition is fulfilled, we call this an *immersion* of the torus in 3D Euclidean space ($\mathbf{R}^3$). Now the question arises: How many different parameterized (decorated) tori can be formed that cannot be transformed smoothly into one another?

According to a paper by Hass and Hughes [5], an orientable 2-manifold of genus g, immersed in $\mathbf{R}^3$, has 4^g regular homotopy classes. Surfaces are in the same class, if they can be smoothly transformed into one another while remaining proper immersions throughout the whole process. For surfaces of genus zero there is only a single class. This implies that a sphere as well as the sphere with its parameterization mirrored must belong to the same regular homotopy class, and that therefore a sphere can be turned inside-out through a smooth continuous sequence of immersions. This was proven by Steve Smale in 1958 [12], but it took many years before a first sequence of actual moves was published [9]. A different sphere-eversion process was later conceived by Bernard Morin [8], and it led to a first computer-graphics movie of that transformation created by Nelson Max [7]. The same basic series of moves was later optimized in the film *Optiverse* [13] so as to minimize the highest level of bending energy reached during that process. Another sphere eversion process relying on *Dirac's Belt Trick* [4] has become well-known through the video *Outside-In* [6]. Both these movies also exhibit very high aesthetic quality.

Based on that same paper [5], we expect four different immersion classes for the torus (a genus-1 surface), where the representatives in one class cannot be turned into that of another class through regular homotopy-preserving moves. Unlike for the case of the sphere, much less work has been done to make visualizations of the regular homotopy classes for the torus. With the exception of the eversion process of an ordinary torus, I have not seen any good depictions of homotopy-preserving torus transformations.

2. Capturing Four Different Representatives

Following a suggestion made in a personal communication by John Sullivan [14], an easy way to generate distinct representatives of the four regular homotopy classes of the torus is to fold up the two dimensions of the rectangular fundamental domain of the torus surface in either a circular way or along a figure-8 path. This will result in the four classes of tori shown in Figure 2.



00 **08** **80** **88**

Type 00: meridians (r): untwisted, parallels (y): untwisted, (1,1)-diagonals (g): 360° twist;

Type 08: meridians (r): untwisted, parallels (y): 360° twist, (1,1)-diagonals (g): untwisted;

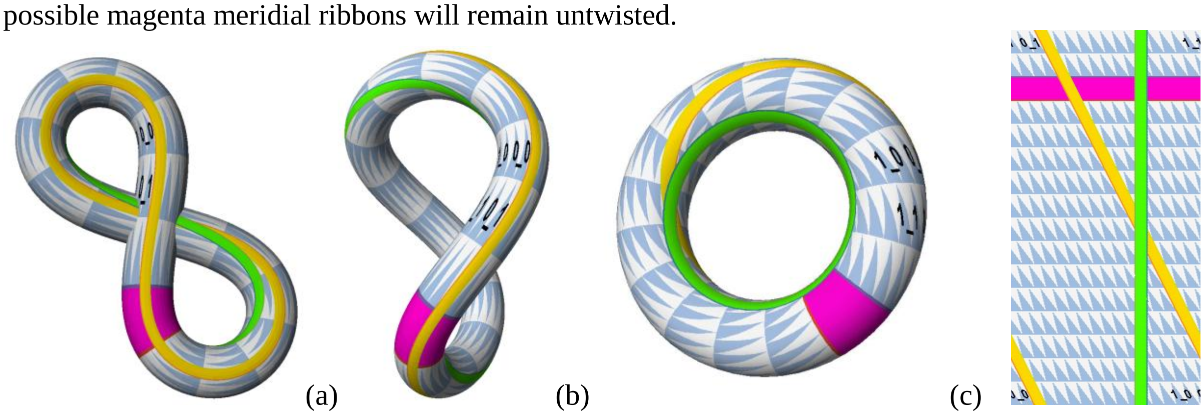
Type 80: meridians (r): 360° twist, parallels (y): untwisted, (1,1)-diagonals (g): untwisted;

Type 88: meridians (r): 360° twist, parallels (y): 360° twist, (1,1)-diagonals (g): 360° twist.

Figure 2: *Tori in four different homotopy classes and their characterization by the amount of twisting of some characteristic ribbons embedded in their surfaces.*

Figure 2 displays the parameterization of the torus surface, by relying on the texture pattern shown in Figure 1a. To make the discussions in this paper easier to understand, we introduce the following naming convention based on our envisioned construction of the torus: We take a rectangular domain and first roll up one dimension (say *x*) into a straight cylinder, either with a circular cross section (Fig.1b); or forming a figure-8 type profile (Fig.1c). These profile curves, lying in planes perpendicular to the cylinder axis, are called *meridians*. In a second step we bend the cylinder axis into a closed loop, again either along a circular or along a figure-8 sweep path. To reduce the ambiguity as to how the second set of parameter lines should be drawn, we will form a torus with a **planar sweep path** as a reference. The plane containing that sweep path is called the *equatorial plane*. All meridians then lie in planes that are perpendicular to this equatorial plane. If the cross section is constant and is swept in a torsion-minimizing manner, then the second set of parameter lines will lie in planes that are parallel to the equatorial plane, and they can thus naturally be called *parallels*. (Topologists often call these lines “longitudes”; this is unfortunate, because on the globe the lines of longitude are called meridians.) We also call the few special parallels that lie in the equatorial plane *equatorials*. There are also sets of *diagonal* lines that close on themselves. They combine integer numbers of loops along a meridian and along a parallel. On a normal torus, a *(1,1)-diagonal* will travel once around the major and minor circles, respectively. Figure 2 shows ribbons along a *meridian* in *magenta* (red, r), ribbons along *parallels* in *yellow* (y), and an additional *(1,1)-diagonal* ribbon in *green* (g). The amount of twisting exhibited by these characteristic ribbons is listed in the table within Figure 2.

Some readers may object to the self-intersecting figure-8 sweep paths used in the tori of Type 08 and Type 88 in Figure 2. We can eliminate the self-intersections at the sweep-path cross-over points and untangle the figure-8 sweeps into perfectly round tori (Fig.3), but in this process we will introduce ±360° of twist around the sweep path – where the sign depends on the directions in which we move the two crossing branches apart. The result is equivalent to a regular torus on which a Dehn twist [3] along a meridial cut line has been introduced; we thus call this an *M-twist*. In this un-warping operation all possible magenta meridial ribbons will remain untwisted.



(a) (b) (c) (d)

Figure 3: *(a → c) Untangling a torus of Type 08 starting with a* ***down****ward move of the torus branch with the yellow-green ribbon crossing; (d) unwrapped torus surface texture.*

In Figure 3c, the original green “diagonal” ribbon has been turned into a parallel ribbon, and the yellow ribbon has become a (-1,1)-diagonal. This can best be seen in the unwrapped texture pattern of the torus surface (Fig.3d). If, alternatively, we separate the crossing torus branches in the opposite directions, then the yellow ribbon becomes a (1,1)-diagonal, while the green ribbon now becomes a (2,1)-diagonal, which shows up on the torus surface as a (1,2) torus knot with a total twist of the corresponding ribbon of 720°. The reader is encouraged to take a long, thin, physical paper strip, to model it in the shape of any of the shown colored surface ribbons, and then to verify the amount of twisting found in the ribbon.

Figure-8 Cross-Over Moves

By going through a regular homotopy transformation starting from Figure 3c, going through Figure 3a, and ending up in the alternative circular torus mentioned above, we have changed the (meridial) twist in the toroidal loop by 720° while remaining in the same regular homotopy class. By repeating or reversing this *Figure-8 cross-over move*, we can readily add or subtract meridial twist in increments of 720°. Thus for our classification of torus immersions, we will always count twists modulo 720°.

The *Figure-8 cross-over move* can also be used to untangle the central crossing in tori of Type 88 (Fig.4a). Again, depending on which way the crossing strands are moved apart in order to separate them, when the figure-8 path has been unwound into a planar, circular loop, a twist of ±360° will have been introduced. For this surface the difference between a negative twist (Fig.4b) and a positive twist (Fig.4d) is much more obvious, since it shows up in the surface geometry itself, and not just in its parameterization. Again, the move from Figure 4b, through Figure 4a and 4c, into Figure 4d will result in a change of twist of 720°. Here is a preview of what will become clear later in this report: If we do **not** pay attention to the parameterization of the torus surface, the tori of Types 00, 08, and 80 are all in the same homotopy class, but the torus of Type 88 (Fig.4) is in a class of its own (Fig.9).

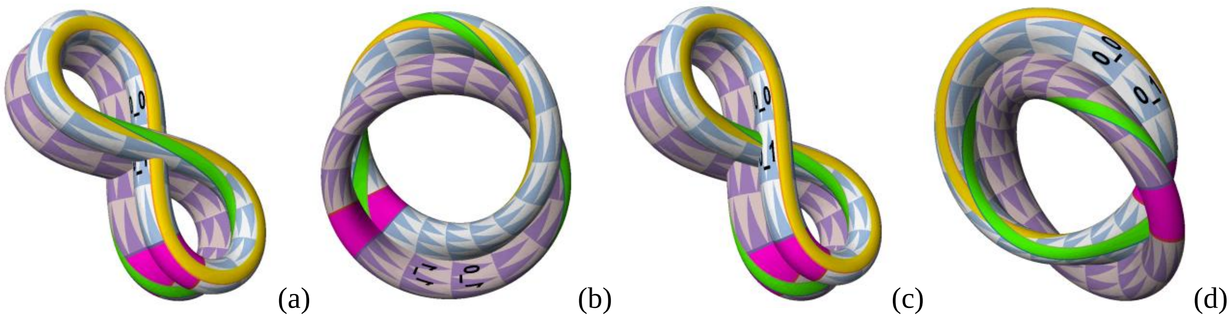


Figure 4: *Untangling the figure-8 path of Type 88: (a→ b) result of moving green-yellow crossing* ***up****wards; (c→ d) result of moving green-yellow ribbon-crossing* ***down****wards.*

3. Additional Regular Homotopy Moves

Turning a Torus Inside-Out

On the web one can find several references to the process of *Torus Eversion*. In particular Cheritat presents an elegant and easy-to-understand video that shows this process [1]. A closely related process is described in diagrammatic form by [2]. Here is my own depiction of this process (Fig.5):

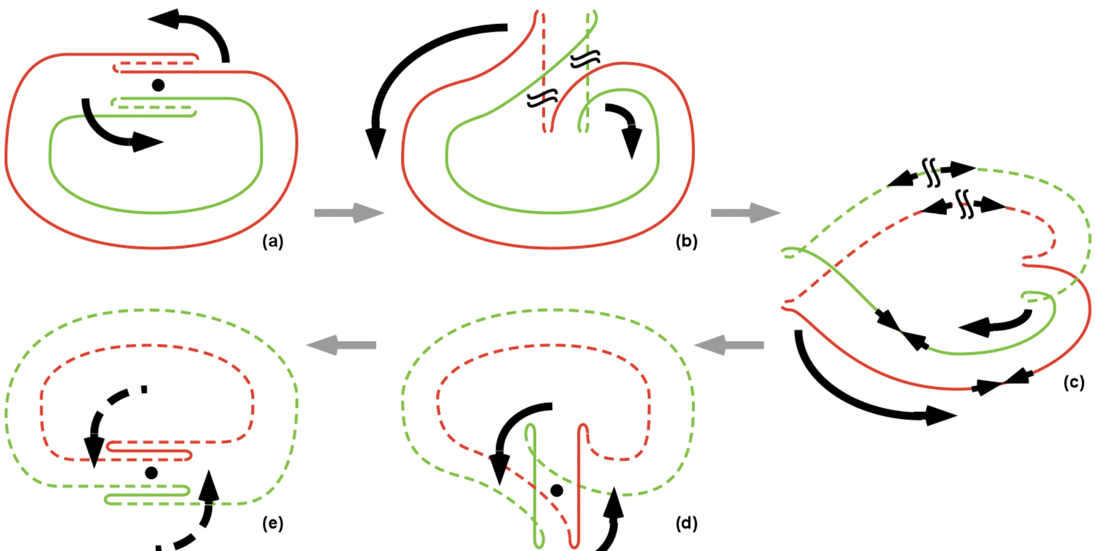


Figure 5: *Turning a torus inside out: (a)→ (e) schematic view of two parallels (equatorials).*

During the whole process, the sweep curve remains planar, and thus the whole transformation can be depicted by simply showing two opposite parallels in red and green, respectively, which are the intersection lines of the torus with the equatorial plane (two equatorials). Surface parts of the torus that were originally facing inward and are finally facing outward are shown as dashed lines; this makes it obvious that the torus gets turned inside-out. The process starts by introducing a rotationally symmetrical fold around some meridian in the torus (Fig.5a). This everted tube segment is then turned through 90° (Fig.5b) and further turned and stretched, so that the two "Klein-bottle mouths" that delimit this segment can be moved apart (Fig.5c). They are moved in opposite directions around the toroidal loop and brought together on the other side. There they are recombined into a short, straight tube segment (Fig.5d). This tube segment is then turned into alignment with the toroidal ring (Fig.5e) and unfolded in place.

All these moves are "planar" operations, and thus they introduce no twisting of any kind. The 3-layer fold introduced in Figure 5a can also readily be generated for a tube with a figure-8 cross-section. Moreover, the two Klein-bottle mouths have no problem passing through the (topologically irrelevant) cross-over intersection generated by a figure-8 sweep path. Thus this eversion process is directly applicable to tori of all four regular homotopy types.

Changing Parameterization and Profile

Another planar transformation that introduces no twist allows us to swap the parameterization and exchanging the roles of parallels into meridians (Fig.6). We introduce a triple fold into a torus of Type 08 and maintain the resulting inverted segment as the core of a future "tubular" torus. By sliding the lobes of the figure-8 loop next to the inverted segment trough one another, we obtain just a simple loop connected to the inverted segment. When we let this loop contract, we realize that the new meridians formed have a figure-8 shape and thus are twisted (Fig.6d). The result is a Type 80 torus with swapped parameterization.

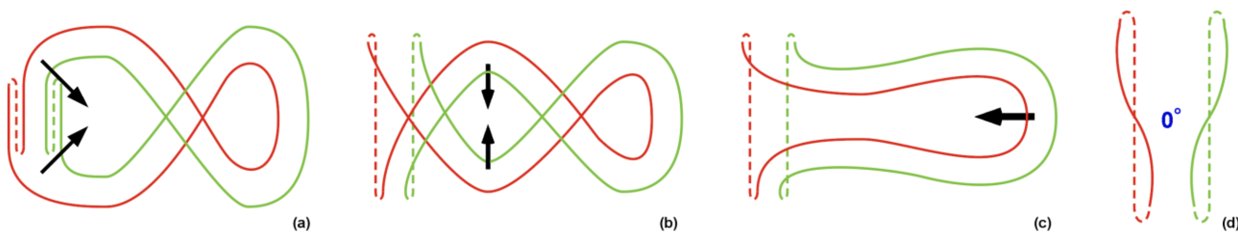


Figure 6: *Swapping the parameters is equivalent to switching between torus Type 08 and Type 80.*

If we try to do a similar operation on a torus of Type 00 (Fig.7a), we find that we cannot collapse the loop without moving its sweep path out of the equatorial plane. The loop has to be un-tangled by going through the 3rd dimension (Fig.7b). This 180° flip introduces an M-twist of 360° into the contracting tube segment (Fig.7c). The previously untwisted parallels have now become untwisted diagonals in this Type 80 torus with a 360° E-twist. We have achieved a swap of the parameterization combined with a profile change and the introduction of 360° E-twist.

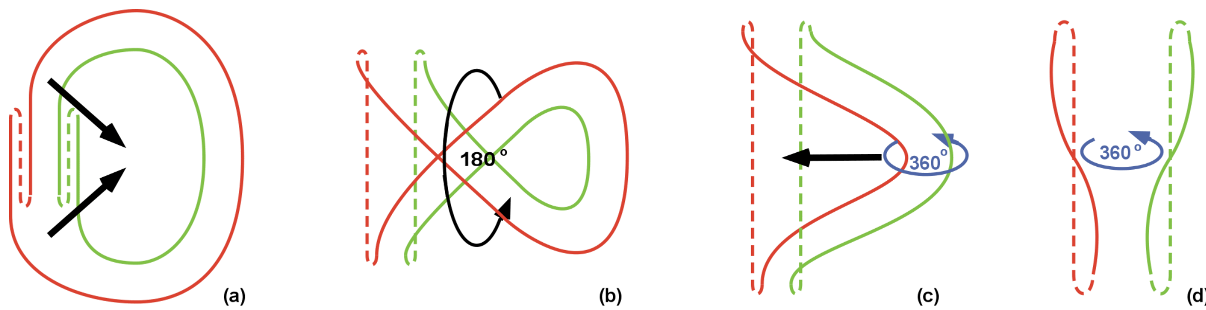


Figure 7: *Turning a torus of Type 00 into Type 80 with a twist and swapped parameterization.*

Adding Dehn Twist

At this point it is worthwhile to point out that there is also a simple operation to willfully add incremental E-twists of ±720°, which would be equivalent to a "cut-twist-reconnect" operation along an equatorial cut line. Figure 8 shows how we might grab the inner wall of a tubular torus, pull it sideways through the outer wall, and subject it to a complete *Figure-8 Cross-over Move* (Fig.8b and 8c). This will introduce an M-twists of ±720° into this tubular loop, which then results in a ±720° E-twist in the final torus (Fig.8d).

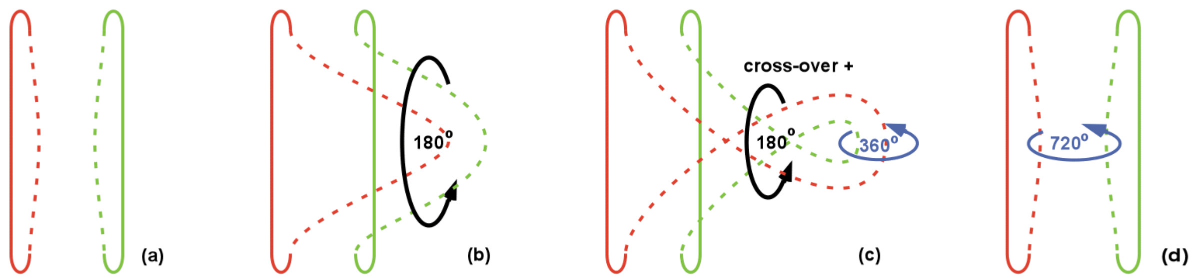


Figure 8: *Introducing an incremental E-twist of ±720° with two half-flips of a loop in the inner tube.*

4. Torus Re-Parameterizations

So far we have looked at sequences of transformations that leave the surface in the same regular homotopy class. Now we turn the question around and explore how the homotopy type changes as we modify the surface parameterization in various ways. The operations we can perform on the parameterization of a torus include:

Evert: Turning the surface of a torus inside out,

Twist: Adding E-twists or M-twists of ±360°,

Swap: Swapping the roles of the meridial and parallel parameter lines.

We have already seen (Fig.5) that the eversion process, which is equivalent to reversing (mirroring) the direction of exactly one of the two parameterization axes, keeps any of the four types of tori in the same regular homotopy class. Based on the discussion associated with Figure 3 we also know that the introduction of an M-twist of 360° switches a torus back and forth between Type 00 and Type 08; and Figure 6 tells us that the swapping operation, i.e. a 90° rotation of the parameter grid converts Type 08 into Type 80, and vice versa.

For a convenient overview over the effects of the many different re-parametrizations possible, we introduce the map in Figure 9. It shows the four types of tori symbolically in the four gray circles, including a diagram of the geometry of a key representative as shown in Figure 2 and a characterization of the twistedness (“*u*”/”*t*”) of three characteristic ribbons. The amount of twisting of two such ribbons yields an unambiguous description of the type of a torus, because the amount of twist that is built into a closed ribbon loop cannot change under regular homotopy transformations. Any untwisted circular loops, like the meridians and parallels in Type 00, are characterized by u=1. But if we connect a paper strip into a figure-8 loop without any twisting, then we find, when we open the figure-8 path into a circular loop, that the ribbon now shows a 360° twist; such a ribbon is characterized by t=1. Since we can always add or subtract twist in increments of 720° with a *Figure-8 cross-over move*, twist is counted modulo 2, and thus twist is always either 1 or 0.

An important insight is that *un-twistedness u*, the complement of twist t, defined as $u = (1–t)$ mod2, is more important than twist itself. The value of u is directly linked to the turning of the *Darboux frame*, the local coordinate system that is used in differential geometry to describe the behaviour of curves on surfaces. For the planar meridians and parallels, the turning number modulo 2 sets the value of u. This makes u an additive quantity when we concatenate multiple ribbon loops [14]. Thus the doubly-looped meridians in Figure 1d have u = 2(mod2) =0; which implies t =1; meaning they are twisted! But ribbons passing around a circular loop an odd number of times are untwisted. The u-values of two different characteristic ribbons allow us to unambiguously characterize each of the four regular homotopy classes of a torus: the four possible combinations of 1 and 0 for the u-values of two characteristic ribbons (e.g. meridians and parallels). With this knowledge we can calculated the u-values for arbitrary (m,p)-diagonal ribbons from the u-values of the meridians and parallels: $u_d = (m*u_m + p*u_p)$ mod2. With some simple matrix calculations we can then readily verify all the transformations depicted in Figure 9 [10].

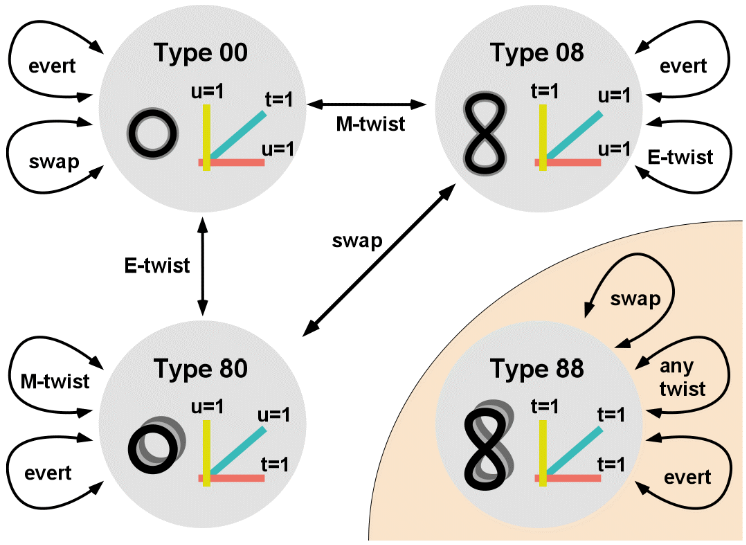


Figure 9: *Complete map of the effects of re-parameterizations on the different torus immersion classes.*

The double-headed arrows in Figure 9 indicate possible changes to the parameter grid on the surface of the torus. We can readily see that the four homotopy classes fall into two different universes: There are no arrows leading from Type 88 to any of the other three types. This means that when we ignore surface parameterization, there are still two different torus homotopy classes. The map also tells us that there are many different ways in which we can obtain a torus of Type 00; we might:

- Evert a Type 00 torus (Fig.5).
- Swap the parameterization of a Type 00 torus (Fig.10).
- Add 360° of E-twist to a Type 80 torus (Fig.7).
- Add 360° of M-twist to a Type 08 torus (Figs.6+7).

To get a more intuitive feel of how these transformations work and the crucial role that twist plays in their characterization, let's try to verify that applying an M-twist to Type 80 leaves the type unchanged. The addition of any M-twist does not change the geometry of the surface; it only slices the parameterization along one meridian and shears the parameterization grid on one side of this meridian through one or more ±360° meridial periods. For Type 80 the turning number along this loop is zero; thus all the ribbons that end on this cut (the parallels and diagonals) and are being shifted around the whole meridial loop do not gain or lose any twist. The meridial ribbons do not experience any twist changes either. Thus the type remains indeed the same.

Figure 9 gives us the full map of all transmutation operations between the four tori types. Now we would also like to know what specific homotopy-preserving transformation processes exist that correspond to the various arrows in this map. Most transformations can be synthesized by concatenating the operations that we have already discussed. In the following we are particularly interested in finding the most elegant transformations that turn a torus inside out or that swap its parameterization.

5. More on Torus Eversions

Just as there is more than one way to evert a sphere [6],[7],[8],[9], there are also several ways to do this for the torus. Here are some alternatives.

All tori with a figure-8 profile can be seen as half-way points of a torus turning inside out, since the same amounts of both surface sides are exposed to the outside. Thus we can use Figures 6 and 7 to transform tori of Type 00 and Type 08 into this convenient halfway point, then flip the figure-8 profile through 180°, and reverse the transformation back to the original shape, but now with the surface turned inside out. Using Figure 7d as a conceptual halfway point for the eversion of a torus of Type 00, leads to the streamlined process depicted in Figure 10. Here the chosen halfway point is shown with both tubular surfaces being bulged out in opposite directions (Fig.10c). For clarity we now show the torus wall as a double line; the gray side of the torus wall moves from the inside to the outside (Fig.10a → 10e).

Here is the actual eversion process: We introduce a ring-shaped triple-fold into the Type 00 torus (Fig.10a). We then pinch the main loop to the right of the triple-wall segment and flip it through 180° (b). In the step from (b) to (c) we also pull out the dashed segment towards the left. In the step from (c) to (d) we now flip the dashed loop through 180° while shrinking the solid segment into a straight tube. The directions of the two flips can be chosen so that the resulting meridial twists cancel out and the resulting toroidal loop is twist-free (Fig.10e). Now we just need to remove the ring-shaped triple-fold on the right.

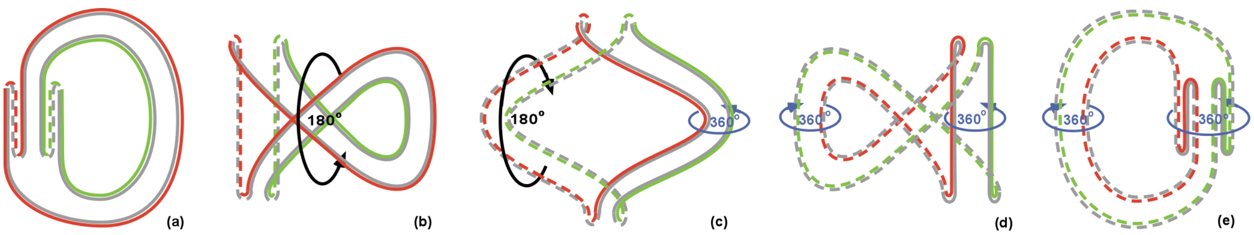


Figure 10: *Another view of the process of turning a torus inside out.*

Looking at this sequence of transformation steps, we can see that it is in principle the same as the process described in Figure 5. Rather than letting the two Klein-bottle mouths travel around the toroidal loop, we now keep them more or less stationary, pointing in opposite directions, and we let the connecting tube segments carry out the necessary deformations and flips. As a side-benefit we obtain a nice, geometrically symmetrical half-way point (Fig.11a) for the torus eversion process. But fundamentally this is just a torus of Type 80 with 360° twist! If we are trying to minimize the total bending energy contained in this surface, the shape will assume the form of a surface of revolution similar to what is shown in Figure11b.

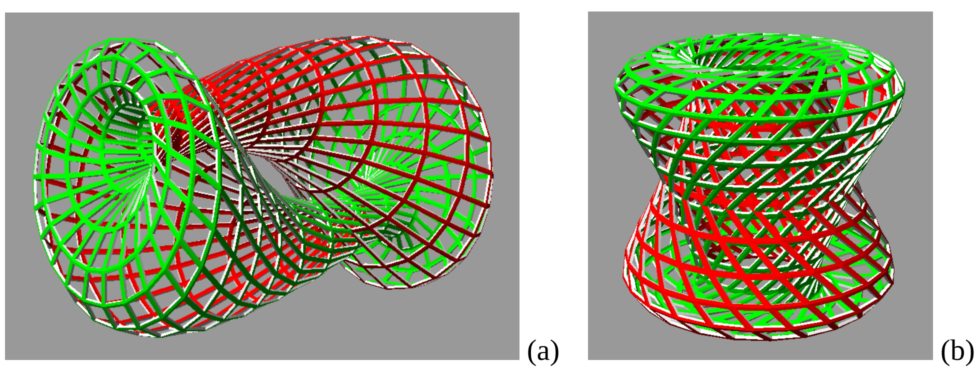


Figure 11: *Half-way points for torus eversion: (a) based on Fig.10c; (b) minimum-energy form?*

For the sphere the energetically optimal eversion process [13] started from a suitably chosen half-way point. Could a similarly optimal torus-eversion process be found starting from the shape in Figure 11b? Since the presence of parametric twist is crucial in this context, and since we expect that the desired removal of that twist would force this shape to bulge out into a toroid, we clearly need a new surface energy model with a very strong penalty term for twist or for any shear in the parameterization grid. Interesting work lies ahead to see whether a suitable energy model can be developed.

6. Parameter Swap

Finding an elegant transformation to swap the parameterization in the ordinary torus of Type 00 is another challenging task. I have not yet found a fully satisfactory solution. Perhaps this is related to the fact that it is not intuitively obvious that one can indeed turn the parameterization grid of an ordinary torus by 90° so that meridians become parallels and vice versa, and do this in a continuous smooth transformation that preserves the regular homotopy class!

We start again by looking for an appropriate half-way point. Inspiration may come from two tightly interlinked tori (Fig.12a). Each of the two circles along which they touch serves as a meridian in one torus and as an equatorial in the other. The inner part of this configuration forms a *handle/tunnel* shape that can also be obtained by bending the rectangular fundamental domain into an extreme saddle and merging the midsections of opposite edge pairs (Fig.12b). Figure 12c is a close-up view of the resulting geometry. To form a complete torus, the square boundary of the depicted geometry has to be closed with an additional surface patch. We can do this in two different ways: We can close it with a surface that passes around the viewer and thus puts the viewer *inside* this structure. In this case the sweep path of the torus would be the horizontal loop shown in white, and the corresponding parallels would also be horizontal. Alternatively, we can close the torus domain away from the viewer, behind the image plane; this would yield an *outside* view of this (deformed) torus. Now the sweep path would be a vertical circle looping away from the viewer, and horizontal cuts through this structure would produce meridians. Thus going from one type of closure to the other swaps the parameterization and also turns the torus inside out. Figure 12d gives an external view of this situation. If we use the lower *bowl*-shaped closure shown as a magenta wire-frame, we obtain one state of the torus; if instead we use the yellow *dome*-shaped closure, we obtain the other state. We can get rid of the unwanted surface eversion with the process depicted in Figure 5, and thus we could obtain the desired pure parameter swap – if we could legally switch from *bowl* closure to *dome* closure.

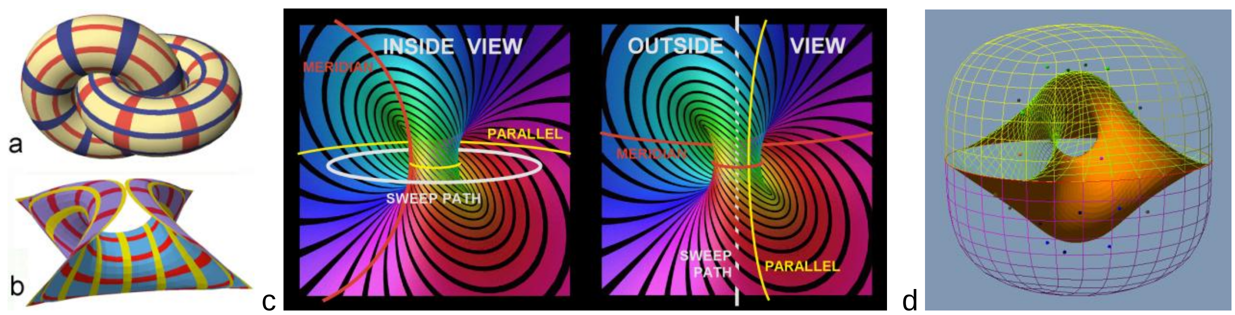


Figure 12: *(a) Two interlinked tori. (b) Domain to be glued into a partial torus. (c) Resulting handle-tunnel shape. (d) Conceptual half-way model for parameter-swap with two alternative "closures."*

However, this is not so straight-forward: Simply pushing the *bowl* surface through the equatorial plane to become the *dome* surface will produce some pinch-off points. The half-way point itself (Fig.13c) is not legal: It has four pinch-off points in the shape of Whitney umbrellas [15] associated with the corners of the square boundary of the domain that needs to be closed off. As the equatorial membrane is pushed downward (Fig.13b) or upward (Fig.13d) the Whitney umbrellas (marked by white arrows) merge pair-wise and annihilate one another. The two end-states (Fig.13a and 13e) are then clean and legal immersions. Unfortunately we are not allowed to move through this illegal half-way point associated with our simple conceptual process.

For the moment, we ignore the detailed the *handle/tunnel* geometry depicted in Figure 12c and just consider the square patch and one of the two *bowl*- or *dome*-shaped closure surfaces. We then obtain two states of a topological sphere, one being an everted version of the other. Thus we can rely on a classical sphere eversion process to accomplish the needed switch from *bowl*-closure to *dome*-closure. For a particularly nice visualization, we stitch the patch containing the *handle/tunnel* geometry into a small circular hole at the north pole of an ordinary sphere (Fig.13f) and then apply the "Outside-In" eversion process [6]. In this case the polar region is simply shifted to the other pole along the globe's axis. At the same time the closing surface moves from lying below it to lying above it and thus accomplishes the desired parameter swap combined with a surface eversion. Following this with the simple torus eversion process depicted in Figure 5, we can then obtain a pure parameter swap in a torus of Type 00.

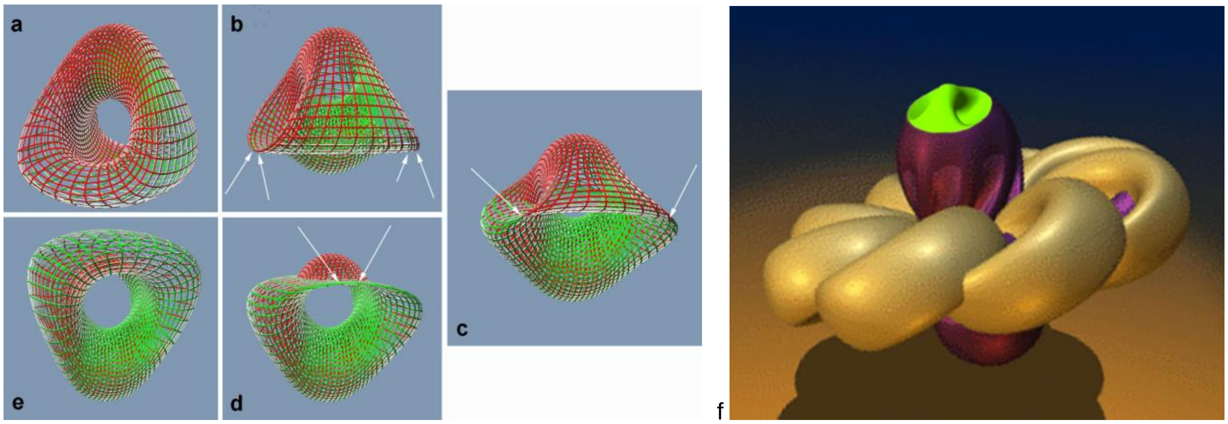


Figure 13: *Parameter swap: (a→e) Conceptual – using non-immersive intermediate steps. (f) A valid but cumbersome approach making use of the classical "Outside-In" sphere eversion process [6].*

This process seems lengthy and cumbersome. I am still hoping to find a better, more direct way! Ideally I am looking for a process, which I might call "Torus OptiSwap," that has the fewest topological events and also encounters the lowest maximum bending energy along the whole transformation.

Acknowledgements

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