# The Beauty of Knots

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### Abstract

Simple knots with just a few crossings can be made into attractive 3D geometric sculptures. Various approaches to achieve such a transformation are discussed and illustrated with examples. High-lighted are two larger-scale sculptures based on knots which were assembled in a few hours by participants of G4G9.

## 1. Knot Theory Primer

To a mathematician, knots are closed curves embedded in three-dimensional space. They are typically classified by the minimal number of crossings that will appear when they are projected onto a plane. A simple loop without any crossings is called the trivial knot or the unknot. The simplest true knot is the 3-crossing trefoil knot; it comes in a right-handed and a left-handed version. The next more complicated knot is the mirror-symmetric Figure-8 knot with 4 crossings. There are two different knot types with five crossings, three with six crossings, seven with seven crossings, and then the number of different knots with a given number of crossings rises very quickly. For the case of 9 crossings, there are already 49 different knots. Knots are typically drawn projected onto a plane, so that they exhibit the minimal number of strand crossings. This means that many of their intrinsic properties may remain hidden. Often the depictions found in the classical knot tables do not even try to reveal the maximal number of symmetries possible in this planar depiction. Moreover, knots are actually 3-dimensional structures, but their possible 3D forms and symmetries are completely lost in those tabulations.

Once we have closed a strand into a knotted loop, we can deform and squash this knot as much as we want, but its knot type cannot be changed without breaking and re-closing the strand. Thus if we start with a closed-loop rubber band, we can further warp and twist and knot this loop – it will always still be the unknot!

**Figure 1**: *Sculptures based on the unknot: (a) José de Rivera, (b) Keizo Ushio, (c) Max Bill (d) Carlo Séquin.*

## 2. Sculptures Based on Simple Knots

In 2009 I set out with a group of undergraduate students to investigate “The Beauty of Knots.” We focused on the question how the simplest knots might be turned into attractive sculptural forms. First we took a look at what artists all over the world had already done in this respect. It turns out that even the unknot can make fascinating sculptures. For example, José de Rivera has made dozens of attractive tubular metal sculptures, which are often just simple twisted loops that are not actually knotted, like his “Construction 56” (Fig.1a). Other spectacular examples of unknots are the "Oushi Zokei" sculptures by Keizo Ushio (Fig.1b), or the many celebrations of the Moebius band by Max Bill (Fig.1c). Perhaps the most complicated but still regular unknot-sculpture is the "Hilbert-Cube 512" by Carlo Séquin (Fig.1d).

The simplest true knot is the trefoil, and this one also shows up in de Rivera's sculptures (Fig.2a). A different realization of a trefoil in soap stone comes from Kenya, Africa (Fig.2b). Brent Collins has created a trefoil knot with a looping crescent-shaped ribbon (Fig.2c), and Keizo Ushio created one by splitting a triply twisted Moebius band (Fig.2d).

**Figure 2**: *Trefoil sculptures by: (a) José de Rivera, (b) anonym. Kenya, (c) Brent Collins, (d) Keizo Ushio.*

Color Plate 1a shows an artistic interpretation in bronze of the 4-crossing Figure-8 knot. It garnered the second prize at the 2009 Exhibition of Mathematical Art of the Joint Mathematics Meeting in Washington, DC [3].

## 3. Turning Simple Knots into Sculptures

So what does it take to design an attractive knot sculpture? Clearly, the result should be a truly 3-dimensional form, and it should look interesting and attractive from many different views. Below we describe some of the techniques that we explored in our working group to turn a schematic depiction in the knot tables [1] into a model for a 3D sculpture. This paper discusses what can be done with individual simple prime knots. Techniques for assembling multiple such knots into sculptural forms or generating knots of arbitrary complexity by using recursive substitution techniques are discussed elsewhere [9][10].

The first step is to create an interesting space curve that brings out the full 3D potential of a knot. This curve may then be fine-tuned in an interactive 3D modeling program to fit the taste of the artist/designer. The aesthetic qualities of the final form can be further enhanced by sweeping along that curve a non-circular cross-section and by playing with possible variations of size and orientation of that cross section.

In our working group we mostly focused on the generation of attractive space curves. First we set our goals to maximizing the possible symmetries inherent in any given knot structure. Somewhat to our surprise, the most effective way to accomplish that first step did not involve computers or mathematics at all. It simply relied on realizing the knot in question with some pliable material such a wire, aluminum foil, or pipe cleaners, and then deforming this knotted shape in 3-space, exploring different possibilities. In this phase possible symmetries are discovered and noted. For the simple knots that we studied, comprising from 5 to 8 crossings, often the only symmetry we found was a rotational $C_2$ symmetry. The most promising looking configurations were then selected for a more detailed exploration.

At this point the knot is entered into the computer as a simple poly-line connecting a sequence of control points. In order to guarantee a certain type of symmetry, corresponding coordinates of symmetrical points are tied to the same shared variables, and these variables are linked to sliders with which their values can be adjusted interactively. Thus, when the designer interactively repositions any control points of the poly-line, the locked-in symmetry is maintained automatically. The actual knot paths, displayed and judged for their aesthetic qualities, were typically smooth cubic or quartic B-spline curves, defined by those adjustable control points. Some care has to be taken, that the knot type of the smoothed curve remains that of the desired knot, and that no topological crossings are lost or created accidentally.

## 4. Computer-Assisted Techniques

Computer graphics tools can be employed right from the beginning. Any 2D-editor can be used to draw a sparse sticks diagram of the projection of the desired knot (Fig.3a). Two or three points are used to characterize each hair-pin turn, and for every strand-crossing an additional pair of control points is added to the poly line. This then allows us in the next phase to individually lift the control points of the upper branch of such a crossing to a higher $z$-value, and push down the lower crossing point to a negative $z$-value. Some manual post-processing of the generated file, introducing shared variables for symmetrical points, allows us to lock in desired symmetries. The smooth B-spline curve generated from the collection of control points can now be inspected in a 3D rendering tool, and the positions of the control points can be fine tuned to make the shape as attractive as possible. Often this is all that is needed to make an attractive looking 3D knot structure (Fig.3b).

Good quality knot curves have no sharp kinks and no unnecessary wiggles. The spacing between the various lobes of the strand should be nicely balanced, and there should be no localized dense tangles. Various computer-based optimization routines can be employed to achieve these goals. A classical technique assumes a flexible and stretchable wire to represent the knot curve; this wire is then loaded with electrical charges of the same sign [5][4]. These charges try to repel one another electrostatically, and in doing so, will push closely spaced wire segments apart and smooth out wiggly sections into nicely rounded loops [6]. Once an overall rough structure for the knot path has been defined, this procedure can then be used to fine-tune the curve into an elegant shape.

A similar smoothing of the knot curve could be achieved by minimizing the total bending energy of the curve or the arc-length integral of its curvature-variation [8]. The necessary calculations can easily be carried out in discretized form on the control polygon or on a suitable subdivision of it. The critical quantities to be summed are the squared values of either the turning angle at each junction of the polyline, or of the differences between subsequent turning angles for the case of the minimum-variation curve (MVC).

The most direct approach of forming equally spaced strands in a knot curve is to assume an infinitely flexible, but incompressible tube of fixed diameter and then trying to pull the given knot as tight as possible. Experimentally a good approximation to this procedure is realized with flexible drier ducts made of a helically wound steel wire clad with vinyl or aluminum foil. A result of such a tightening of the knot $8_{18}$ is shown in Figure 3c. This result already makes a nice constructivist sculpture. On the computer it is somewhat more complicated to compute this tightest center line. Typically a simulated annealing approach [7] is used on a finely subdivided poly-line. All line segments act like rubber bands that try to shorten themselves as much as possible; good care has to be taken to detect and avoid mutual intersections between tube segments [2]. The center path of the tube can now be used to sweep a more interesting cross section, or it can be subjected to further manipulations. When this approach is used with as few straight tube segments as possible, we can generate the tightest formations of sticks knots. Figure 3d shows the densest sticks form of the trefoil knot with a minimal number of six straight sticks segments.

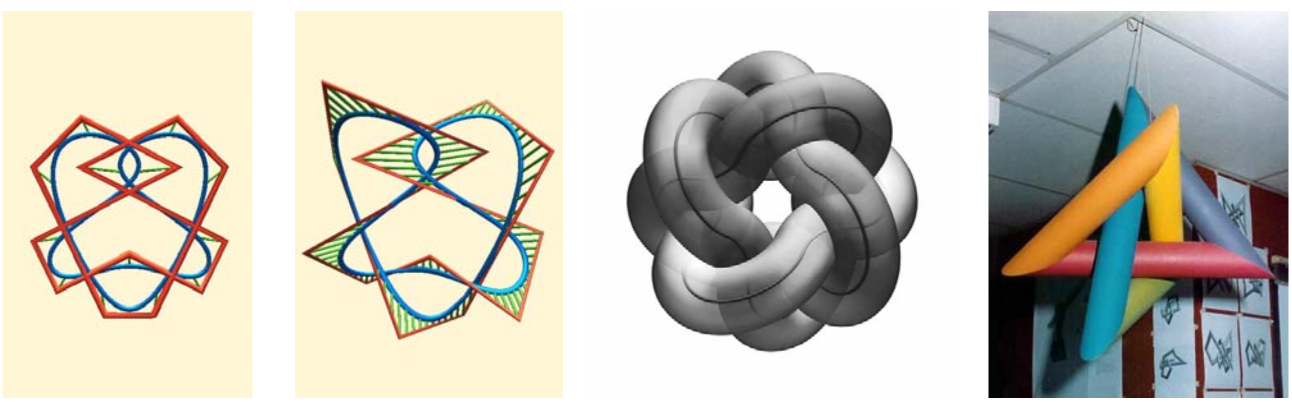


**Figure 3:** *Knot optimization: (a) editing the planar layout, (b) adding depth to it, (c) tightest tube for Knot $8_{18}$, (d) minimal sticks realization for the trefoil knot.*

A limitation of many of these optimization procedures is that they will head for the closest local optimum, and thus tend not to dramatically change the knot configuration. Because of the way knots are depicted in the knot tables, knots with more then 5 or 6 crossings will end up relatively flat, with a maximal depth complexity of 2. To explore more fundamentally different arrangements of the strands, we need procedures that force more radical overall reconfigurations. We found two procedures that were quite effective in that respect. They both carry out an overall geometrical remapping of the knot, which can be applied more than once to any given knot.

The first procedure selects a point near the centroid of the current knot, but not too close to any strand, and then makes this point the center of an inside-out sphere eversion. Every control point of the given sticks knot is mapped to a new point on the same ray through the center, but with a reciprocal distance from the center. The knot, turned inside-out in this manner, can then be further fine-tuned and optimized with a routine described in the previous paragraph. Again, some care has to be taken, that the inversion process does not change the knot type. Long sticks segment in the original knot may have to be broken into several shorter segments, to obtain a faithful re-mapping of the whole knot.

A second procedure remaps the whole knot in a different way. In this case we start from a mostly planar 2.5D representation as one finds in the knot tables. A thick slab of space containing the whole knot is now wrapped around either a cylinder or around a sphere with a large enough diameter so that this slab does not overlap with itself. This approach also yields radically new configurations, to which further refinement steps can be applied.

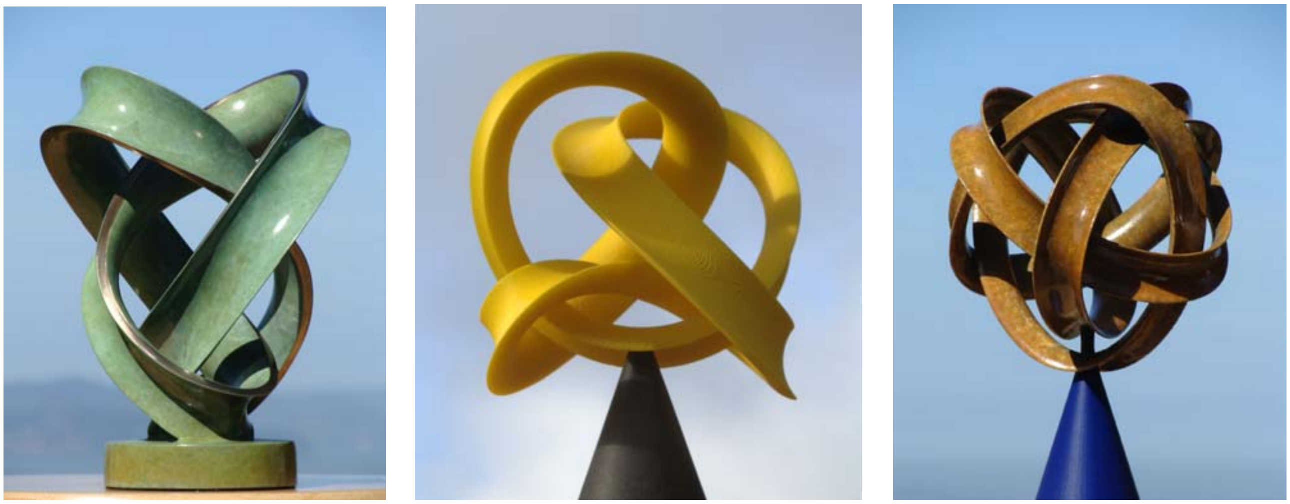
## 5. Knot Sculpture Suspension

When turning a knot into a sculpture, an additional issue comes up: How should it be mounted? This is not a concern when I simply make a small knot model to hold in my hand, such as the realization of Knot $7_7$ shown in Figure 4a. A possible suspension is to "balance" a knot on one of its larger lobes; this technique has been used for the trefoils shown in Figures 2a and 2d. For stability we may want to rest the knot on more than just one lobe. This may also be a necessity if we want to emphasize a particular symmetry of the knot and want to make the $z$-axis the symmetry axis, as in the case of the Figure-8 knot (Color Plate 1a).

Often the various lobes of a knot are its most prominent aesthetic features, and having them touch the ground may diminish their visual impact. The possibility often exists to suspend a knot from a more central point, possibly lying on the symmetry axis. This approach has been chosen for Knot $5_2$ (Color Plate 1b). If the knot is more complicated and thus consists of a much longer strand, having this strand supported at only one point may not be enough. The whole structure me become too "springy" or may even sag. If the ribbon makes a second crossing with the symmetry axis, that point could be used as a secondary suspension point. This technique has been employed in Knot $7_7$ (Fig.4b) and in Knot $9_{40}$ (Color Plate 1c). What if the knot path has no crossings with the symmetry axis at all? Knot $6_1$ can be deformed into such a case. The ribbon then just wraps around the symmetry axis. I chose to mount it clinging to a pole like a koala bear (Fig. 4c). Finally I explored whether a knot can "stand on its own feet." A first example is the trefoil knot from Kenya (Fig.2b). My own version of a trefoil knot standing solidly on two of its lobes is shown in Figure 4d.

**Figure 4**: *(a) Knot $7_7$ by itself, (b) Knot $7_7$ as a sculpture, (c) suspension of Knot $6_1$, (d) free-standing Knot $3_1$.*

**The Beauty of Knots: Color Plates**



**Color Plate 1**: *(a) Figure-8 Knot, (b) Knot $5_2$, (c) Chinese Button Knot.*

**Color Plate 2:** *Construction of the Chinese Button Knot.*

**Color Plate 3**: *Construction of the Trefoil Flower.*

## 6. Variations on a 9-Crossing Knot

Since this is the 9th Gathering for Gardner, let's focus our attention on a 9-crossing knot. As a "signature-knot" for this 9th Gathering I have chosen Knot $9_{40}$. This knot has much sculptural potential that is not visible from the typical depiction of it in the knot tables. It is typically shown with $D_2$ symmetry (Fig.5a), but it really could just as well be shown with $D_3$ symmetry (Fig.5b).

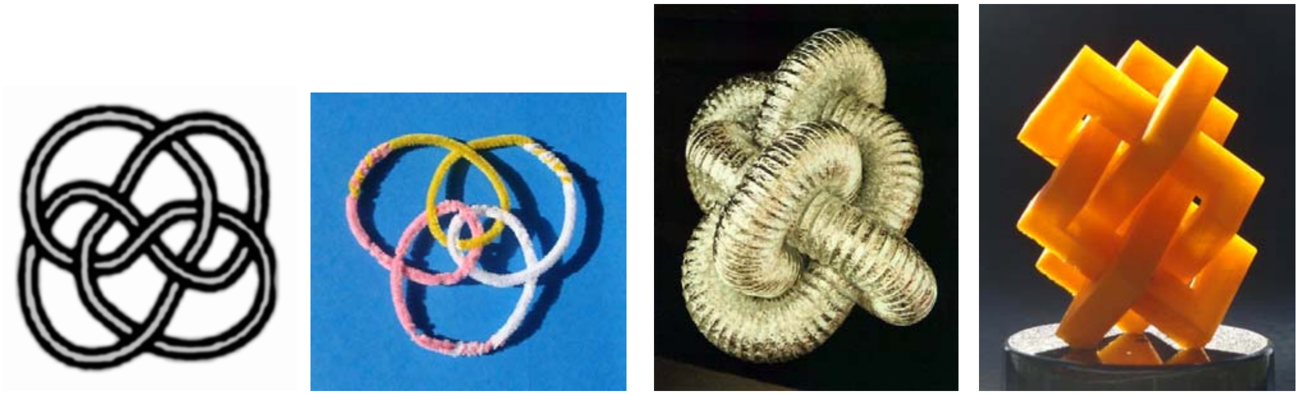


***Figure 5**: Signature knot $9_{40}$: (a) as shown in knot tables, (b) pipe-cleaner model showing 3-fold symmetry, (c) a tight configuration of the Chinese Button Knot, (d) a "cubistic constructivist" version.*

If we optimize this knot by knot-tightening [2], the structure shown in Figure 5c appears. This tight, button-like configuration has given this knot its name; it is indeed used as a button on some Chinese clothing (Fig.6b). Figure 5d shows the same basic arrangement of the lobes but in a more cubistic style using a square cross section for the strand. Once we examine this knot as a truly 3-dimensional structure, a nice spherical symmetry becomes apparent. It can be drawn nicely on the surface of a sphere with its 9 alternating over/under crossings located at latitudes zero and ±60 degrees, and with longitudes 120 degrees apart. Figure 6a depicts this structure in a knot made from copper tubing, with a smaller version inside made from copper wire. Further possibilities arise when the knot curve is swept with a non-circular cross section. A crescent-like cross-section was used in the small bronze sculpture shown in Color Plate 1c. Figure 6c shows an implementation with a flat rectangular cross-section, realized with acrylic plastic strips.

**Figure 6**: *Knot $9_{40}$ showing spherical symmetry: (a) made from copper tubing/wire, (b) as a true button knot, (c) plastic ribbon sculpture.*

## 7. Assembling the "Knot Flowers"

The high point of each Gathering for Gardner is the party at Tom Rodgers' residence. About 200 participants gather in Tom's garden and talk about mathematics, games, puzzles, art, or magic tricks, while enjoying delicious food and drinks. While all this is going on, a half-dozen new permanent sculptures are being erected by small groups of volunteers. These sculptures have been pre-conceived and pre-engineered by various artists and are designed specifically so that they can be assembled by a few conference participants in a couple of hours. For G4G9 I was invited to lead one such activity.

Keeping in line with my plans for talking about the "Beauty of Knots" at the 2010 gathering, I immediately thought of some banded knot representations as presented in Figures 6c and 7a. Since the conference organizers were looking for some self-contained, free-standing out-door sculpture, I thought that such banded knot realizations could be set on some poles and thereby turned into "Knot Flowers" (Fig.7b). Geometrically, this does not pose any big problems, and I readily generated several small scale models made from wooden cubes and paper strips.

**Figure 7**: *Other ribbon sculptures made from: (a) plastic strips, (b) paper strips, (c) metal bands.*

The real problem arises in the engineering of a robust and durable outdoor sculpture that can withstand high winds and does not deteriorate under the sun's UV radiation. Plastic strips do not seem like a viable solution. Aluminum or galvanized steel appear to be a more appropriate materials. I spent considerable time looking for suitable bands about 4 inches wide in those materials. Ideally, those bands should be perforated or gridded to give the sculptures a somewhat "airy" transparent look, and to also reduce the wind resistance of the band loops. Ideally these strips would be springy and would be of a thickness that could be bent by hand at the time of installation, and where each loop would assume a graceful minimum-energy curve due to the elasticity of the material. In this case, no pre-forming of the strips would be necessary, and shipment of the parts would also be easy: it would involve just one bundle of 8- to 10-foot long pliable strips.

In spite of a considerable amount of searching time, I could not find a suitable kind of metal bands. The readily available materials, such as aluminum flashing, or steel wire meshing turned out to be too flimsy, and have sharp edges that would not make a good project for group assembly. Also, some test structures that I built created nasty grating sounds as the edges of the bands rubbed against one another when the sculpture was shaken or exposed to strong winds (Fig.7c). Thicker bands would have to be cut from sheet metal and custom-perforated, and I had no good guess what combination of thickness and perforation pattern would result in the appropriate strength and display the desired elastic behavior. Also, due to some procrastination, I had run out of time to do the necessary series of experiments to determine the right set of parameters, before ordering a full set of parts for several flowers. This project may be resurrected for a future gathering.

Thus for this year I prepared just two medium-size "Knot Flowers" made from colorful translucent acrylic plastic bands to be hung in a somewhat sheltered environment. I have a fair amount of experience with this type of material, having built several such sculptures over the last 15 years. One of the two Knot Flowers is again a celebration of the G4G9 signature Knot $9_{40}$, also known as the Chinese Button Knot. It just took three 6-foot long bands and a few nuts and bolts (Color Plate 2a) to build a nice spherical version of this knot. Weaving the bands together is a little trickier than it first appears. Letting the volunteers in my group do their own experimenting, the first knot they realized was just a simple trefoil knot. With this structure in hand, they realized that the bands had to be curved quite a bit more tightly to form a 9-crossing knot (Color Plate 2b). Once they had the right number of crossings, they had to undo part of the knot once more, because they had not yet obtained the desired alternating over-under-over crossing pattern. But eventually everything fell nicely in place, and the very attractive result was hung from the branches of a nearby tree (Color Plate 2c).

The second knot flower is not strictly a knot, since it is not formed from a single continuous band. Rather it is composed from three separate loops that all start and end on different faces of a small central cube-shaped body with an array of bolt-holes going through it. The slight difficulty in assembly rests on the fact that one pair of bolts ties the ends of two different plastic bands to opposite faces of the central cube (Color Plate 3a). Some interesting questions about chirality were raised in this context. The finished sculpture still very much looks like a trefoil, and thus was named “Trefoil Flower.” It is best hung above eye-level underneath the ceiling (Color Plate 3b). Both these sculptures seem to give off an internal glow when touched by the rays of the setting sun.

## 8. Conclusions

It is not too difficult to transform the simplest knots into attractive sculptures. As demonstrated by many artists around the world, many possibilities exist to do this. The medium complexity knots with about 7 to 10 crossing are somewhat harder to work with. Their strands are more intricately interwoven, which makes it more difficult to find an elegant configuration that does not have too much of a “woven” look. Thus in this context less complexity seems preferable. More emphasis should then be placed on the proper selection of the materials used and on the detailed shaping of the geometry of the individual lobes, as well as the choice of cross-section and surface finish of the strand forming the knot. For small sculptural models, making maquettes on a rapid-prototyping machine is a natural and convenient way to get tangible results. But such sculptures are limited in size to about a cubic foot. For larger sculptures, a constructivist approach is more cost-effective. Specifically the Knot Flowers made from colorful translucent plastic strips are rather inexpensive and quite beautiful. Significantly more engineering design work has to be invested to see whether such Knot Flowers can be realized as durable outdoor sculptures.

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