Regular Convex Uniform 4-Polytopes

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Regular Convex Uniform 4-Polytopes are a specific subset of 4-Polytopes in general. There are an infinite number of 4-polytopes, just as there are an infinite number of polyhedra and polygons. There are also an infinite number of uniform 4-polytopes. There are, however, only 64 convex uniform 4-polytopes and only 6 regular convex uniform 4-polytopes.

To be uniform, a 4-polytope must have equivalent vertex figures throughout, as well as uniform polyhedra for every cell. To be convex, all lines between points on the 4-polytope must themselves be totally within the 4-polytope. (There do exist non-convex regular uniform 4-polytopes, as well as non-convex regular polyhedra and non-convex regular polygons. The only non-convex regular polygon is the pentagram.) To be regular, the polyhedra making up the 4-polytope must be themselves regular.

The simplest of the regular convex uniform 4-polytopes is the simplex, or 5-cell. It is made of 5 tetrahedra. It is the 4D analog to the triangle and tetrahedron, and belongs to the $A_n$ Coxeter group. (Coxeter groups classify objects and concepts based on arrangements of reflections. Not every Coxeter group can be thought of in terms of symmetries, but the finite ones can be.) Specifically, it has a symmetry of $A_4$ in Coxeter notation. All members of the $A_n$ group have $(n+1)!$ symmetries, so the simplex has 120 symmetries.
The next in the series is the tesseract, or 8-cell. It is made of 8 cubes, and is the only of the regular convex uniform 4-polytopes to be made of cubes. It is the 4D analog to the square and cube, and belongs to the $B_n$ Coxeter group. Specifically, it has a symmetry of $B_4$ in Coxeter notation. All members of the $B_n$ group have $(2^n n!)$ symmetries, so the simplex has 384 symmetries.

The dual polytope of the tesseract is the orthoplex, or 16-cell. It is made of 16 tetrahedra. It is the 4D analog to the square or octahedron. It is considered dual to the tesseract because it is also in the $B_4$ Coxeter group and so has identical symmetries.

The octaplex, or 24-cell, is the only member of the set except for the simplex that has no dual. It is made of 24 octahedra, and is the only member which is made of octahedra. It has its own Coxeter group, which is $F_4$. These properties mean it is not considered to have an analog in 3D space, or in higher-dimensional space. It has 1152 symmetries.

The dodecaplex, or 120-cell, is the 4D analog to the pentagon and dodecahedron. It is made of 120 dodecahedra, and is the only set member which is made of dodecahedra. It has symmetry $H_4$ in Coxeter notation, and has 14400 symmetries (120 squared).

The dual of the dodecaplex is the tetraplex, or 600-cell, is the 4D analog to the icosahedron. It is made of 600 tetrahedra. It is the dual of the dodecaplex because it also belongs to the $H_4$ Coxeter group.