Nonholonomic Motion Planning:
Steering Using Sinusoids

Richard M. Murray, *Member, IEEE*, and S. Shankar Sastry, *Senior Member, IEEE*

Abstract—In this paper, we investigate methods for steering systems with nonholonomic constraints between arbitrary configurations. Early work by Brockett derives the optimal controls for a set of canonical systems in which the tangent space to the configuration manifold is spanned by the input vector fields and their first order Lie brackets. Using Brockett's result as motivation, we derive suboptimal trajectories for systems which are not in canonical form and consider systems in which it takes more than one level of bracketing to achieve controllability. These trajectories use sinusoids at integrally related frequencies to achieve motion at a given bracketing level. We define a class of systems which can be steered using sinusoids (chained systems) and give conditions under which a class of two-input systems can be converted into this form.

I. INTRODUCTION

MOTION planning for robots has a rich history. The traditional difficulty in planning robot trajectories is the avoidance of obstacles, often referred to as the piano mover's problem, in which we attempt to move an object (the piano) through a cluttered environment. This problem is solved by investigating the free configuration space of the piano—all configurations for which the piano does not intersect an obstacle. If the start and goal locations of the piano lie in the same connected component of the free configuration space, the motion planning problem is solvable.

In recent years, there has been a great deal of activity in the generation of efficient motion planning algorithms for robots. Most of this work has concentrated on the global problem of determining a path when the obstacle positions are known and dynamic constraints on the robot are not considered. This has resulted in a rather complete understanding of the complexity of the computational effort required to plan the trajectories of robots to avoid both fixed and moving obstacles [10], [22], [28]. Other approaches include the use of potential functions for navigating in cluttered environments [24], [25] and compliant motion planning for navigating in the presence of uncertainty [13], [14], [37].

Our interests in motion planning are not along the lines of the aforementioned approaches, but are complemen-

tary: they involve motion planning in the presence of nonholonomic or nonintegrable constraints. That is, we consider systems in which there are constraints on the velocities of the robots which cannot be integrated to give constraints which are exclusively a function of the configuration variables. These situations arise in a number of different ways and we describe a few of the sources of their origin:

1) *Mobile Robots Navigating in a Cluttered Environment:* The kinematics of the drive mechanisms of robot carts results in constraints on the instantaneous velocities that can be achieved. For instance, a cart with two forward drive wheels and two back wheels cannot move sideways. This was first pointed out by Laumond in the context of motion planning for the Hilare mobile robot [29], [30].

2) *Multifingered Hands Manipulating a Grasped Object:* If an object is twirled through a cyclic motion that returns the object to its initial position and orientation, and the fingers roll without slipping on the surface of the object, the fingers do not necessarily return to their initial configurations. This feature can be used to plan the regrasp of a poorly grasped object or to choose the nature of this grasp. This application of nonholonomic motion planning was first pointed out by Li [34], [35] (see also [39]).

3) *Space Robotics:* Unanchored robots in space are difficult to control with either thrusters or internal motors since they conserve total angular momentum. This is a nonintegrable constraint. The motion of astronauts on space walks is of this ilk, so that planning a strategy to reorient an astronaut is a nonholonomic motion planning problem [55]. Other examples of this effect include gymnasts and springboard divers.

Nonholonomic constraints arise either from the nature of the controls that can be physically applied to the system or from conservation laws which apply to the system. Conventional path planners implicitly assume that arbitrary motion in the configuration space is allowed as long as obstacles are avoided. If a system contains nonholonomic constraints, many of these path planners cannot be directly applied. If we attempt to ignore the constraint, the paths generated by a path planner may not be feasible (see Fig. 1). For this reason, it is important to understand how to efficiently compute paths for nonholonomic systems.

To be more specific, we are interested in mechanical systems with linear velocity constraints of the form

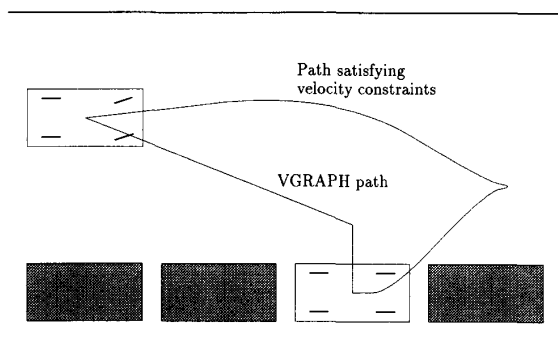$$\omega_i(x)\dot{x} = 0 \qquad i = 1, \cdots, k. \tag{1}$$

Fig. 1. Paths generated by conventional path planners may ignore nonholonomic constraints. The straight line path in the figure indicates the path that a conventional path planner might generate. The curved path is one which satisfies the nonholonomic constraints on the car's motion.

Here, $x \in \mathbb{R}^n$ is the configuration of the system being controlled and $\omega_i(x)$ is a row vector in $\mathbb{R}^n$. These are constraints on the *velocities* of the system. In some cases, the constraints may be explicitly integrable, giving constraints of the form

$$h_i(x) = c_i$$

for some constant c_i. If this is possible, motion of the system is restricted to a level surface of h_i. Such a constraint is said to be *holonomic*. By choosing coordinates for the surface, configuration space methods can be applied. In the instance that there is only one constraint on the velocity of the system, its integrability may be determined by checking the symmetry of the Jacobian matrix of $\omega_1(x)$. There is no easy extension of this characterization to the case of multiple constraints.

A constraint is said to be *nonholonomic* if it cannot be written as an algebraic constraint in the configuration space. There are many types of nonholonomic constraints, corresponding to different physical situations.

It will be convenient for us to convert problems with nonholonomic constraints into steering problems for control systems. Consider the problem of constructing a path $x(t) \in \mathbb{R}^n$ between a given x_0 and x_1 subject to the k constraints given in equation (1). We assume the ω_i's are smooth and linearly independent. Specific examples of such systems are given in Section II. Roughly speaking, we would like to convert the constraint specification from describing the directions in which the system cannot move to those in which it can. Formally, we choose a basis for the right null space of the constraints, denoted by $g_i(x) \in \mathbb{R}^n$, $i = 1, \cdots, n - k$. The path planning problem can be restated as finding an input function, $u(t) \in \mathbb{R}^{n-k}$, such that the control system

$$\dot{x} = g_1(x)u_1 + \cdots + g_{n-k}(x)u_{n-k}$$

is driven from x_0 to x_1. It can be shown that if the ω_i's are smooth and linearly independent, then the g_i's inherit these properties.

The outline of this paper is as follows: in Section II, we collect some mathematical preliminaries from the literature on controllability of nonlinear systems and on classification of free Lie algebras. These are drawn from classical references in control theory [7], [20], [21], [41], [49] and Lie algebras [18], [53]. In Section III, using some outstanding results of Brockett on optimal steering of certain classes of systems as motivation [6], we discuss the use of sinusoidal inputs for steering systems of first order, i.e., systems where controllability is achieved after just one level of Lie brackets of the input vector fields. Section IV attempts to expand the domain of applicability of these results to more complex systems, where several orders of Lie brackets are needed to obtain the full Lie algebra associated with the input distribution. The style of the paper is self-contained so as to make it accessible to both robotics and control researchers and several examples are sustained through the paper.

A target problem which we set ourselves at the start of this research was that of parking of a car with N trailers. This problem remains unsolved and indeed has generated some fascinating new ideas in the field. It is not a "toy problem" since efforts are underway to automate baggage handling by carts with multiple trailers in airports (not to mention trucks with multiple trailers). It is fair to say that the study of nonholonomic motion planning is in its infancy. There have, however, been notable contributions by Laumond *et al.* [23], [29], [31]–[33] and by Barraquand and Latombe [2] on motion planning for mobile robots in a cluttered field. While this work represents important initial progress, we feel that less computationally intensive and more insightful approaches are possible by conducting a systematic research program on motion planning of dynamical systems with nonholonomic constraints. We are joined by several complementary efforts, notably those of Li and co-workers [15], [35] and Sussmann and co-workers [27], [51]. We have also applied the techniques of this paper to steering of space robots using sinusoids in [55].

II. MATHEMATICAL PRELIMINARIES

This section describes the notation to be used throughout the paper and collects a variety of results from nonlinear control theory and Lie algebras which will be used in the sequel. For basic definitions and concepts in differential geometry, see Boothby [5] or Spivak [48]. A good introduction to nonlinear control theory which includes many of the necessary differential geometric concepts can be found in Isidori [21] or Nijmeijer and van der Schaft [41].

A. Nonlinear Control Theory

We consider the problem of steering a control system

$$\Sigma: \dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \qquad x \in U \subset \mathbb{R}^n$$

$$u \in \mathbb{R}^m \qquad (2)$$

from an initial state $x_0 \in U$ to a final state $x_1 \in U$ by appropriate choice of a control $u: [0, T] \rightarrow \mathbb{R}^m$. For simplicity, we assume U to be an open neighborhood of the origin and $\{g_i\}$ to be a collection of smooth, linearly independent vector fields defined on U. Associated with the system Σ is a distribution

$$\Delta = \text{span}\{g_1, \cdots, g_m\}$$

where we take the span over the set of smooth real-valued functions on U. $\Delta_x \subset \mathbb{R}^n$ denotes the subspace defined by evaluating Δ at a point x.

Controllability of the system Σ can be characterized in terms of the Lie algebra generated by the vector fields g_i. Define the Lie bracket between two vector fields f, g as

$$[f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$

A straightforward calculation shows that the Lie bracket has the following properties:

$$[f, g] = -[g, f] \qquad \text{(skew-symmetry)}$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]]$$

$$= 0 \qquad \text{(Jacobi identity)}.$$

Given a distribution Δ, the involutive closure of the distribution, denoted $\bar{\Delta}$, is the closure of Δ under Lie bracketing.

A system Σ is controllable if for any $x_0, x_1 \in U$ there exists a $T > 0$ and $u: [0, T] \rightarrow \mathbb{R}^m$ such that Σ satisfies $x(0) = x_0$ and $x(T) = x_1$. For a control system which is linear in the input, the time interval T is arbitrary since we can scale the inputs (and hence time) as needed. The conditions for controllability are given by Chow's theorem (see [20]).

Theorem 1 (Chow): If $\bar{\Delta}_x = \mathbb{R}^n$ for all $x \in U$ then the system Σ is controllable on U.

A useful interpretation of Chow's theorem can be obtained by using the following characterization of the Lie bracket. Let $\phi_t^f: U \rightarrow \mathbb{R}^n$ denote the flow of a vector field f for time t and consider the sequence of flows depicted in Fig. 2. The net motion consists of flowing along $g_1, g_2, -g_1, -g_2$ for time ϵ and can be shown to satisfy

$$\phi_\epsilon^{-g_2} \circ \phi_\epsilon^{-g_1} \circ \phi_\epsilon^{g_2} \circ \phi_\epsilon^{g_1}(x_0) = \epsilon^2[g_1, g_2](x_0) + o(\epsilon^3). \tag{3}$$

Thus, the Lie bracket is the infinitesimal motion that results from flowing around a square defined by two tangent vectors. If $[g_1, g_2] = 0$ then g_1 and g_2 commute and it can be shown that the right-hand side of (3) is identically zero; i.e., we return to the starting point. Roughly speaking, Chow's theorem states that if we can move in every direction using Lie bracket motions (possibly of higher order than one), then the system is controllable.
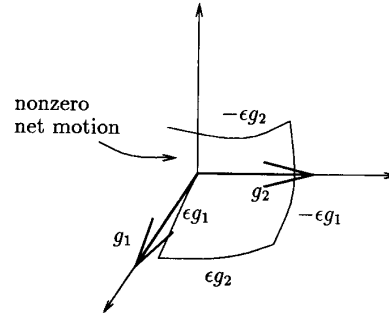

Fig. 2. A Lie bracket motion.

B. Classification of Lie Algebras

We now develop some concepts which allow us to classify nonholonomic systems. A more complete treatment can be found in the work of Vershik [16], [54]. Basic facts concerning Lie algebras are taken from Varadarajan [53]. Let $\Delta = \text{span}\{g_1, \cdots, g_m\}$ be the distribution associated with the control system (2). Define $G_1 = \Delta$ and

$$G_i = G_{i-1} + [G_1, G_{i-1}]$$

where

$$[G_1, G_{i-1}] = \text{span}\{[g, h]: g \in G_1, h \in G_{i-1}\}.$$

The set of all G_i's defines the *filtration* associated with a distribution. Each G_i is defined to be spanned by the input vector fields plus the vector fields formed by taking up to $i - 1$ Lie brackets. The Jacobi identity implies $[G_i, G_j] \subset [G_1, G_{i+j-1}] \subset G_{i+j}$.

A filtration is *regular* in a neighborhood U of x_0 if

$$\text{rank } G_i(x) = \text{rank } G_i(x_0) \qquad \forall x \in U.$$

We say a system is regular if the corresponding filtration is regular. If a filtration is regular, then at each step of its construction, G_i either gains dimension or the construction terminates. If rank $G_{i+1} = \text{rank } G_i$ then G_i is involutive and hence $G_{i+j} = G_i$ for all $j \geq 0$. Clearly, rank $G_i \leq n$ and hence if a filtration is regular, then there exists an integer $p < n$ such that $G_i = G_p$ for all $i \geq p$. We refer to p as the *degree of nonholonomy* of the distribution.

For a regular system, Chow's theorem states that a path exists between two arbitrary points in an open set $U \subset \mathbb{R}^n$ if and only if $G_p(x) = \mathbb{R}^n$ for all $x \in U$. A system (or distribution) satisfying the conditions of this theorem is said to be *maximally nonholonomic*. If a regular system is not maximally nonholonomic, then by Frobenius' theorem we can restrict ourselves to a manifold on which the system is maximally nonholonomic.

It is also useful to record the dimension of each G_i. For a regular system, we define the *growth vector* $r \in \mathbb{Z}^p$ as

$$r_i = \text{rank } G_i.$$

We define the *relative growth vector* $\sigma \in \mathbb{Z}^p$ as $\sigma_i = r_i - r_{i-1}$ and $r_0 := 0$. The growth vector for a system is a convenient way to represent information about the associ-

ated control Lie algebra. For a distribution with finite rank, the growth vector is bounded from above at each step. To properly determine this bound, we must determine the maximal rank of G_i taking into account skew-symmetry and the Jacobi identity. A careful calculation [46] gives

$$\bar{\sigma}_i = \frac{1}{i}\left((\bar{\sigma}_1)^i - \sum_{j|i,\, j<i} j\bar{\sigma}_j\right) \quad i > 1 \tag{4}$$

where $\bar{\sigma}_i$ is the maximum relative growth at the ith stage and $j|i$ means all integers j such that j divides i. If $\sigma_i = \bar{\sigma}_i$ for all i, we say Δ has *maximum growth*.

C. Examples of Nonholonomic Systems

To illustrate the classification of nonholonomic systems, we present several detailed examples. These examples are used in later sections as a basis for testing planning algorithms.

Example 1 (Hopping Robot): As our first example, we consider the dynamics of a hopping robot in flight phase [36], as shown in Fig. 3. This robot consists of a body with an actuated leg that can rotate and extend; the "constraint" on the system is conservation of angular momentum.

Let (ψ, l, θ) be the body angle, leg extension, and leg angle of the robot. For simplicity, we take the body mass to be one and concentrate the mass of the leg, m_l, at the foot. The upper leg length is also taken to be one, with l representing the extension of the leg past this point. Since we control the leg angle and extension directly, we choose their velocities as our inputs. The angular momentum of the robot is given by

$$\dot{\theta} + m_l(l+1)^2(\dot{\theta} + \dot{\psi}) = 0.$$

Thus, our equations become

$$\dot{\psi} = u_1$$
$$\dot{l} = u_2$$
$$\dot{\theta} = -\frac{m_l(l+1)^2}{1 + m_l(l+1)^2}u_1.$$

In vector field notation we have

$$g_1 = \frac{\partial}{\partial\psi} - \frac{m_l(l+1)^2}{1 + m_l(l+1)^2}\frac{\partial}{\partial\theta}$$

$$g_2 = \frac{\partial}{\partial l}$$

$$g_3 = [g_1, g_2] = \frac{2m_l(l+1)}{\left(1 + m_l(l+1)^2\right)^2}\frac{\partial}{\partial\theta}.$$

In a neighborhood of $l = 0, \{g_1, g_2, g_3\}$ is full rank and hence the hopping robot has degree of nonholonomy 2 with growth vector $(2,3)$.
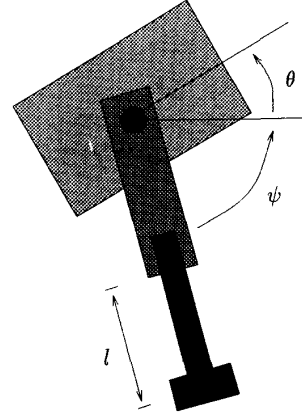

Fig. 3. A simple hopping robot. The robot consists of a leg which can both rotate and extend. The configuration of the mechanism is given by the angle of the body and the angle and length (extension) of the leg.

Example 2 (Kinematic Car): Consider a simple kinematic model for an automobile with front and rear tires [40], as shown in Fig. 4. The rear tires are aligned with the car while the front tires are allowed to spin about the vertical axes. To simplify the derivation, we model the front and rear pairs of wheels as single wheels at the midpoints of the axles. The constraints on the system arise by allowing the wheels to roll and spin, but not slip.

Let (x, y, ϕ, θ) denote the configuration of the car, parameterized by the location of the rear wheel(s), the angle of the car body with respect to the horizontal (θ), and the steering angle with respect to the car body (ϕ). The constraints for the front and rear wheels are formed by writing the sideways velocity of the wheels:

$$\frac{d}{dt}(x + l\cos\theta)\cdot\sin(\theta + \phi)$$

$$-\frac{d}{dt}(y + l\sin\theta)\cdot\cos(\theta + \phi) = 0$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$$

Written as one forms we have

$$\omega_1 = \sin(\theta + \phi)\,dx - \cos(\theta + \phi)\,dy - l\cos\phi\,d\theta$$

$$\omega_2 = \sin\theta\,dx - \cos\theta\,dy.$$

Converting this to a control system gives

$$\dot{x} = \cos\theta u_1$$
$$\dot{y} = \sin\theta u_1$$
$$\dot{\phi} = u_2$$
$$\dot{\theta} = \frac{1}{l}\tan\phi u_1.$$

For this choice of vector fields, u_1 corresponds to the forward velocity of the rear wheels of the car and u_2 corresponds to the velocity of the steering wheel.
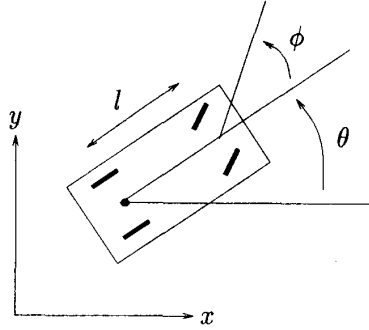
Fig. 4. Kinematic model of an automobile. The configuration of the car is determined by the Cartesian location of the back wheels, the angle the car makes with the horizontal and the steering wheel angle relative to the car body. The two inputs are the velocity of the rear wheels and the steering velocity.

To calculate the growth vector, we compute the control Lie algebra:

$$g_1 = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} + \frac{1}{l}\tan\phi \frac{\partial}{\partial \theta}$$

$$g_2 = \frac{\partial}{\partial \phi}$$

$$g_3 = [g_1, g_2] = \frac{-1}{l\cos^2\phi}\frac{\partial}{\partial \theta}$$

$$g_4 = [g_1,[g_1,g_2]] = \frac{-\sin\theta}{l\cos^2\phi}\frac{\partial}{\partial x} + \frac{\cos\theta}{l\cos^2\phi}\frac{\partial}{\partial y}$$

$$g_5 = [g_2,[g_1,g_2]] = \frac{-2\tan\phi}{l\cos^2\phi}\frac{\partial}{\partial \theta}.$$

$\{g_1, g_2, g_3, g_4\}$ are linearly independent when $\phi \neq \pm\pi/2$. Thus, the system has degree of nonholonomy 3 with growth vector $r = (2, 3, 4)$ and relative growth vector $\sigma = (2, 1, 1)$. The system is regular away from $\phi = \pm\pi/2$, where g_1 is undefined.

Example 3 (Car with N Trailers): Fig. 5 shows a car with N trailers attached. We attach the hitch of each trailer to the center of the rear axle of the previous trailer. The wheels of the individual trailers are aligned with the body of the trailer. The constraints are again based on allowing the wheels only to roll and spin, but not slip. The dimension of the state space is $4 + N$ with 2 controls.

We parameterize the configuration by the states of the automobile plus the angles of each of the trailers with respect to the horizontal. For consistency we will write θ_0 for the angle of the car. Calculation of the constraints becomes tedious since we have to write the velocity of the wheels of each trailer, which depend on all previous trailers. Instead, we choose to use the same inputs as the automobile and calculate the effect on the trailer angles.

At each trailer, we can write the hitch velocity as the sum of two components: the velocity in the direction the
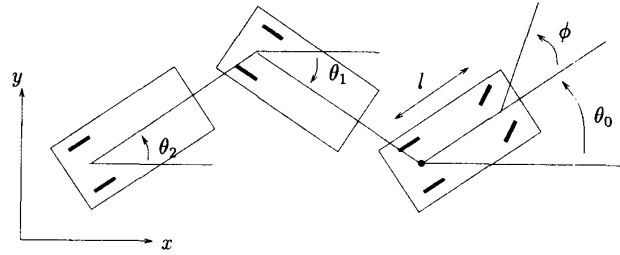

Fig. 5. Kinematic car with trailers. The trailer configuration is described the angle the trailer makes with the horizontal, θ_i. The rear wheels of the trailer are fixed and constrained to move along the line in which they point or rotate about their center. The inputs to the system are the inputs to the tow car: the driving velocity (of the front wheels) and the steering velocity.

trailer is pointing and its perpendicular. The perpendicular component causes the trailer to spin. Letting v_{i-1} be the forward velocity of the previous trailer, we have

$$\dot{\theta}_i = \frac{1}{d_i}\sin(\theta_{i-1} - \theta_i)v_{i-1}$$

$$v_i = \cos(\theta_{i-1} - \theta_i)v_{i-1}.$$

Aggregating these equations gives

$$\dot{x} = \cos\theta_0 u_1$$

$$\dot{y} = \sin\theta_0 u_1$$

$$\dot{\phi} = u_2$$

$$\dot{\theta}_0 = \frac{1}{l}\tan\phi u_1 \tag{5}$$

$$\dot{\theta}_i = \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1} - \theta_j)\right)\sin(\theta_{i-1} - \theta_i)u_1.$$

The filtration corresponding to the N trailer problem is very complex. For small values of N, controllability can be verified directly. For the general case, a very detailed and well-organized calculation by Laumond [31] shows that the system is controllable with degree of nonholonomy $N + 3$ and relative growth vector $\sigma = (2, 1, \cdots, 1)$.[1]

D. Philip Hall Bases for Lie Algebras

We will be interested in the sequel in constructing nonholonomic systems which are canonical in the sense that they allow for the maximal growth of the filtration associated with a set of vector fields $\Delta = \text{span}\{g_1, \cdots, g_m\}$.

To construct such systems with a given number of inputs and degree of nonholonomy, it is necessary to introduce some additional machinery. In constructing canonical nonholonomic systems we must observe the

[1] Laumond uses a slightly different system, obtained by ignoring ϕ and choosing u_1 and $u_1\tan\phi$ as inputs. Since setting $u_1 = 0$ allows us to steer ϕ independently, controllability for the system given here follows from Laumond's result.

fundamental restrictions imposed by the Lie bracket: skew-symmetry and the Jacobi identity. Our search for a set of vector fields which have a given degree of non-holonomy is equivalent to searching for a basis for a free, finitely generated, finite-dimensional Lie algebra. One basis set for such a distribution is a Philip Hall basis [18, 46].

Given a set of generators $\{X_1, \cdots, X_m\}$, a Lie product is any nested set of Lie brackets written in terms of the X_i's. The length of a Lie product is defined recursively as

$$l(X_i) = 1 \qquad i = 1, \cdots, m$$

$$l([A, B]) = l(A) + l(B),$$

where A and B are themselves Lie products. Alternatively, $l(A)$ is the number of generators in the expansion for A. A Lie algebra is *nilpotent* if there exists an integer k such that all Lie products of length greater than k are zero. k is called the order of nilpotency. A nilpotent Lie algebra is finite dimensional. A *P. Hall Basis* is an ordered set of Lie products $H = \{B_i\}$ satisfying
PH1) $X_i \in H$, $i = 1, \cdots, m$
PH2) If $l(B_i) < l(B_j)$ then $B_i < B_j$
PH3) $[B_i, B_j] \in H$ if and only if
    a) $B_i, B_j \in H$ and $B_i < B_j$ and
    b) either $B_j = X_k$ for some k or
      $B_j = [B_l, B_r]$ with $B_l, B_r \in H$ and $B_l \le B_i$.
The proof that a P. Hall basis is a basis for the free Lie algebra generated by $\{X_1, \cdots, X_m\}$ can be found in [18], [46]. The construction above is a clever way of keeping track of the conditions imposed by the skew-symmetry and the Jacobi identity.

A P. Hall basis with order of nilpotency k can be constructed from a set of generators using the definition. The simplest approach is to construct all possible Lie products with length less than k and use the definition to eliminate elements which fail to satisfy one of the properties. In practice, the basis can be built in such a way that only PH3) need be checked.

Example 4: A basis for the nilpotent Lie algebra of order 3 generated by $\{X, Y, Z\}$ is

$X \quad Y \quad Z$

$[X, Y] \quad [X, Z] \quad [Y, Z]$

$[X, [X, Y]] \quad [X, [X, Z]] \quad [Y, [X, Y]] \quad [Y, [X, Z]]$

$[Y, [Y, Z]] \quad [Z, [X, Y]] \quad [Z, [X, Z]] \quad [Z, [Y, Z]].$

Note that $[X, [Y, Z]]$ does not appear since

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

and two of the three terms are already present.

Example 5: A larger example, which we will use in the sequel, is a basis for a Lie algebra of order 5 with 2

generators:

$$B_1 - B_2 \colon X \quad Y$$
$$B_3 \colon [X, Y]$$
$$B_4 - B_5 \colon [X, [X, Y]] \quad [Y, [X, Y]]$$
$$B_6 - B_8 \colon [X, [X, [X, Y]]] \quad [Y, [X, [X, Y]]]$$
$$[Y, [Y, [X, Y]]]$$
$$B_9 - B_{14} \colon [X, [X, [X, [X, Y]]]]$$
$$[Y, [X, [X, [X, Y]]]]$$
$$[Y, [Y, [X, [X, Y]]]]$$
$$[Y, [Y, [Y, [X, Y]]]]$$
$$[[X, Y], [X, [X, Y]]]$$
$$[[X, Y], [Y, [X, Y]]].$$

Note that B_{13} and B_{14} have the form $[B_3, B_4]$ and $[B_3, B_5]$, requiring careful checking of the condition PH3).

III. STEERING CONTROLLABLE SYSTEMS USING SINUSOIDS

In this section, we investigate methods for steering systems with nonholonomic constraints between arbitrary configurations. Early work by Brockett derives the optimal controls for a set of canonical systems in which the tangent space to the configuration manifold is spanned by the input vector fields and their (first order) Lie brackets. Using Brockett's result as motivation, we derive suboptimal trajectories for systems which are not in canonical form and consider systems in which it takes more than one level of bracketing to achieve controllability. These trajectories use sinusoids at integrally related frequencies to achieve motion at a given bracketing level. Examples and simulation results are presented.

We consider systems of the form

$$\dot{x} = g_1(x)u_1 + \cdots + g_m(x)u_m \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (6)$$

with $\{g_i\}$ a set of smooth, linearly independent vector fields in some neighborhood of the origin. We also assume that the system is regular (as defined in Section II-B) and hence has a well-defined degree of nonholonomy and growth vector.

A. First-Order Systems

Control systems in which the first level of brackets together with the input vector fields span the tangent space at each configuration arise in many areas. In classical mechanics, systems with growth vector $r = (n - 1, n)$ are called contact structures [1]. A version of the Darboux theorem asserts that for these systems the corresponding constraint can be written as

$$dx_3 = x_2 \, dx_1$$

(using the notation of exterior differential forms). In $\mathbb{R}^3$ and using control system form, this becomes

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2 \qquad\qquad (7)$$
$$\dot{x}_3 = x_2 u_1.$$

Brockett considered a more general version of this system in [6]; we review his results here. Consider a control system as in (6) that is maximally nonholonomic with growth vector $(m, n) = (m, (m(m + 1)/2))$, We would like to find an input $u(t)$ on the interval 0 to 1 which steers the system between an arbitrary initial and final configuration and minimizes

$$\int_0^1 |u|^2 \, dt.$$

This problem is related to finding the geodesics associated with a singular Riemannian metric (Carnot–Caratheodory metric). To solve the problem, Brockett considers a class of systems which have a special canonical form. An equivalent form, which is more useful for our purposes, is

$$\dot{x}_i = u_i \qquad i = 1, \cdots, m$$
$$\dot{x}_{ij} = x_i u_j \qquad i > j. \qquad (8)$$

We see that if $m = 2$, this is exactly the contact system (7). It can be shown that the input vector fields and their pairwise brackets span $\mathbb{R}^n$ and hence the system is controllable with degree of nonholonomy equal to 2.

To find the optimal input between two points, we construct the Lagrangian

$$L(x, \dot{x}) = \sum_{i=1}^m \dot{x}_i^2 + \sum_{i,j} \lambda_{ij} (\dot{x}_{ij} - x_i \dot{x}_j). \qquad (9)$$

Here we have used the fact that $u_j = \dot{x}_j$. The λ_{ij}'s are the Lagrangian multipliers associated with the constraint imposed by the control system. Substituting (9) into the Euler–Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

it can be shown that the input must satisfy

$$u = e^{\Lambda t} u_0$$

where Λ is a constant skew-symmetric matrix. Thus, the inputs are sinusoids at various frequencies. Unfortunately, even for very simple problems, determining Λ and u_0 given an initial and final configuration is very difficult.

A great deal of simplification occurs if we consider moving between configurations where $x_i(1) = x_i(0)$. In this instance the eigenvalues of Λ must be multiples of 2π and Brockett showed that the optimal inputs are sinusoids at integrally related frequencies, namely $2\pi, 2 \cdot 2\pi, \cdots, (m/2) \cdot 2\pi$. This simplifies the problem tremendously and for many examples reduces the search to that of finding u_0. We use this result to propose the following algorithm for steering systems of this type:

Algorithm 1: (Steering first-order canonical systems):

1) Steer the x_i's to their desired values using any input and ignoring the evolution of x_{ij}'s.
2) Using sinusoids at integrally related frequencies, find u_0 such that the input steers the x_{ij}'s to their

desired values. For example, choosing

$$u_i = \sum_{k>0} (a_{ik} \sin kt + b_{ik} \cos kt)$$

yields

$$x_{ij}(2\pi) = x_{ij}(0) + \left(\sum_{k>0} \frac{a_{jk} b_{ik} - a_{ik} b_{jk}}{k} \right) \pi.$$

By the choice of input, the x_i's are unchanged.

The resulting trajectories are suboptimal but easily computable and have several nice properties which we will explore.

Example 6: We consider as an example a kinematic hopping robot, as shown in Fig. 3. This example has been studied by Li, Montgomery, and Raibert [36] using holonomy methods. We wish to reorient the body of robot while in midair and bring the leg rotation and extension to a desired final value. The kinematic equations of the robot (in center of mass coordinates) can be written as

$$\dot{\psi} = u_1$$
$$\dot{l} = u_2$$
$$\dot{\theta} = - \frac{m_l (l + 1)^2}{1 + m_l (l + 1)^2} u_1$$

where we have used units such that the mass of the body is one and the length of the leg at zero extension is also one. The last equation is a consequence of conservation of angular momentum. Expanding the equation using a Taylor series about $l = 0$:

$$\dot{\theta} = - \frac{m_l}{1 + m_l} \dot{\psi} - \frac{2m_l}{(1 + m_l)^2} l u_1 + o(l) u_1.$$

This suggests a change of coordinates, $\alpha = \theta + (m_l/1 + m_l)\psi$ to put the equations in the form

$$\dot{\psi} = u_1$$
$$\dot{l} = u_2$$
$$\dot{\alpha} = - \frac{2m_l}{(1 + m_l)^2} l u_1 + o(l) u_1 = f(l) u_1.$$

This equation has the same form locally as the canonical system in (8).

Using this as justification, we attempt to use our proposed algorithm to steer the full nonlinear system. Since we control the ψ and l states directly, we first steer them to their desired values. Then using sinusoids in the ψ and l inputs,

$$u_1 = a_1 \sin \omega t$$
$$u_2 = a_2 \cos \omega t$$

we steer θ to its desired value. By construction, this last motion does not affect the final values of ψ and l. To include the effect of nonlinearity in the first vector field, harmonic analysis can be used. Since l is periodic, we

expand $f(l)$ using its Fourier series

$$f\left(\frac{a_2}{\omega} \sin \omega t\right) = \beta_1 \sin \omega t + \beta_2 \sin 2\omega t + \cdots.$$

Integrating $\dot{\alpha}$ over one period, only the first term in the expansion contributes to the net motion

$$\alpha\left(\frac{2\pi}{\omega}\right) = \alpha(0) + \int_0^{2\pi/\omega}$$

$$\cdot \left(\frac{a_1\beta_1}{\omega} \sin^2\omega t + \frac{a_1\beta_2}{\omega} \sin \omega t \sin 2\omega t + \cdots\right) dt$$

$$= \alpha(0) + \frac{\pi a_1 \beta_1}{\omega^2}.$$

Fig. 6 shows the trajectory for the last motion segment; ψ and l return to their initial values but α (and hence θ) experiences a net change. To compute the required input amplitudes, we plot β_1 as a function of a_2 and choose a_2 such that $(\pi a_1 \beta_1(a_2)/\omega^2) = \theta_1 - \theta_0$. For example, choosing $m_l = 5$, $a_1 = a_2 = 1$, and $\omega = 1$ yields

$$\beta_1 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f\left(\frac{a_2}{\omega} \sin \omega t\right) \sin \omega t\, dt \approx -0.468.$$

The net motion in α (and hence θ) is $(\pi a_1 \beta_1/\omega^2) \approx -1.47$, which agrees with the simulated motion.

To incorporate practical considerations, such as limited leg extension and rotation, multiple cycles may be used to achieve a large body rotation.

B. Second-Order Systems

We next consider systems in which the first level of bracketing is not enough to span $\mathbb{R}^n$. We begin by trying to extend the previous canonical form to the next higher level of bracketing. Consider a system which can be expressed as

$$\dot{x}_i = u_i \qquad i = 1, \cdots, m$$
$$\dot{x}_{ij} = x_i u_j \qquad i > j$$
$$\dot{x}_{ijk} = x_{ij} u_k \qquad (\text{mod Jacobi identity}). \qquad (10)$$

Because Jacobi's identity imposes relations between certain brackets, not all x_{ijk} combinations are permissible if the system (10) is to be completely controllable. This is analogous to limiting the x_{ijk}'s according to

$$\left[g_i, [g_j, g_k]\right] + \left[g_j, [g_k, g_i]\right] + \left[g_k, [g_i, g_j]\right] = 0.$$

Using the calculation in (4) shows that a system with two levels of growth has relative growth vector $(m, (m(m-1)/2), ((m+1)m(m-1)/3))$. Constructing the Lagrangian (with the same integral cost function) and substituting into the Euler–Lagrange equations does not in general result in a constant set of Lagrange multipliers, although Brockett and Dai have shown that for $m = 2$ the optimal inputs are elliptic functions [9].

We can extend and apply our previous algorithm as follows.

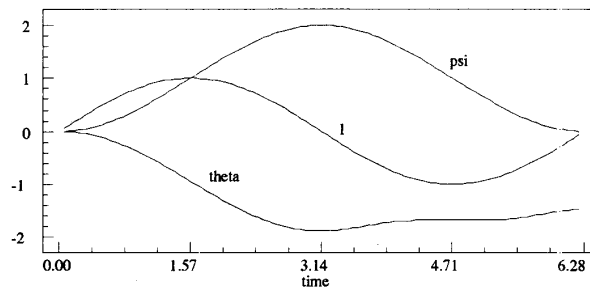*Algorithm 2:* (Steering second-order canonical systems).

Fig. 6. Nonholonomic motion for a hopping robot. Using sinusoidal inputs, the leg angle and extension return to their starting values but the body angle goes a net rotation.

1) Steer the x_i's to their desired values. This causes drift in all other states.

2) Steer the x_{ij}'s to their desired values using integrally related sinusoidal inputs. If the ith input has frequency ω_i then x_{ij} will have frequency components at $\omega_i \pm \omega_j$. By choosing inputs such that we get frequency components at zero, we can generate motion in the desired states.

3) Use sinusoidal inputs a second time to move all previously steered states in a closed loop and generate motion only in the x_{ijk} directions. This requires careful choice of the input frequencies so that $\omega_i \pm \omega_j \neq 0$ but $\omega_i \pm \omega_j \pm \omega_k$ has zero frequency components.

The required calculations for Step 2) are identical to those in Algorithm 1. A general calculation of the motion in Step 3) is quite cumbersome, although for specific systems of practical interest the calculations are straightforward. For example, if $m = 2$, (13) becomes:

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_{21} = x_2 u_1$$
$$\dot{x}_{211} = x_{21} u_1$$
$$\dot{x}_{212} = x_{21} u_2.$$

To steer x_1, x_2, and x_{21} to their desired states, we apply Algorithm 1. To steer x_{211} independently of the other states, choose $u_1 = a \sin t$, $u_2 = b \cos 2t$ to obtain

$$x_{211}(2\pi) = x_{211}(0) + \frac{a^2 b}{8} \cdot 2\pi.$$

Similarly, choosing $u_1 = b \cos 2t$ and $u_2 = a \sin t$ gives

$$x_{212}(2\pi) = x_{212}(0) + \frac{a^2 b}{8} \cdot 2\pi$$

and all other states return to their original values. Note that this approach requires separate steps to steer in each of the x_{ijk} directions. It is also possible to generate net motion in multiple coordinates simultaneously by using linear combinations of sinusoids and solving a polynomial equation for the necessary coefficients.

Example 7: To illustrate the algorithm, we consider the motion of a front wheel drive car as shown in Fig. 4. The kinematics of this mechanism were derived in the last

chapter and can be written as

$$\dot{x} = \cos \theta u_1$$

$$\dot{y} = \sin \theta u_1$$

$$\dot{\phi} = u_2 \qquad (11)$$

$$\dot{\theta} = \frac{1}{l} \tan \phi u_1.$$

In this form, u_1 does not control any state directly. We use a change of coordinates and a change of input to put the equations in the form

$$\dot{x} = v_1 \qquad\qquad v_1 = \cos \theta u_1$$

$$\dot{\phi} = v_2 \qquad\qquad v_2 = u_2$$

$$\dot{\alpha} = \frac{1}{l} \tan \phi v_1 \qquad \alpha = \sin \theta$$

$$\dot{y} = \frac{\alpha}{\sqrt{1 - \alpha^2}} v_1.$$

As before, the linear portion of the nonlinearities matches the canonical system and we can include the effects of the nonlinearities using Fourier series techniques.

An example of the algorithm applied to the car is given in Fig. 7. The first portion of the path, labeled A, drives the x and ϕ states to their desired values using a constant input. The second portion, labeled B, uses a periodic input to drive θ while bringing the other two states back to their desired values. The last step brings y to its desired value and returns the other three states to their correct values. The Lissajous figures that are obtained from the phase portraits of the different variables are quite instructive. Consider the portion of the curve labeled C. The upper left plot contains the Lissajous figure for x, ϕ (two loops); the lower left plot is the corresponding figure for x, θ (one loop) and the open curve in x, y shows the increment in the y variable. The very powerful implication here is that the Lie bracket directions correspond to rectification of harmonic periodic motions of the driving vector fields and the harmonic relations are determined by the order of the Lie bracket corresponding to the desired direction of motion. This point has also been made rather elegantly by Brockett [8] in the context of the rectification of mechanical motion.

IV. CHAINED SYSTEMS

We now study more general examples of nonholonomic systems and investigate the use of sinusoids for steering such systems. As in the previous section, we try to generate canonical classes of higher order systems, i.e., systems where more than one level of Lie brackets is needed to span the tangent space to the configuration space. We show that, in full generality, it is difficult to use sinusoids to steer such systems. This leads us to specialize to a smaller class of higher order systems, which we refer to as chained systems, that can be steered using sinusoids. We give sufficient conditions under which systems can be
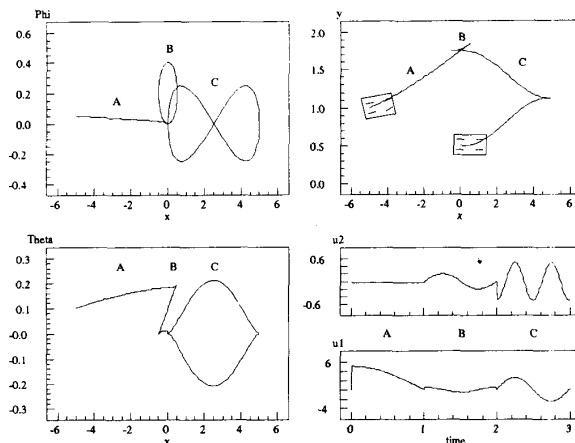

Fig. 7. Sample trajectories for a car. The trajectory shown is a three stage path which moves the unicycle from $x = -5$, $y = 1$, $\theta = 0.05$, $\phi = 1$ to $(0, 0.5, 0, 0)$. The first three figures show the states versus x; the bottom right figures show the inputs as functions of time.

transformed into a chained form and show the procedure applied to several illustrative examples.

A. Maximum Growth Canonical Systems

Using a P. Hall basis, it is possible to construct vector fields which have maximum growth; at each level of bracketing the dimension of the filtration grows by the maximum possible amount. More specifically, we wish to construct a set of vector fields $\{X_i\}$ such that when the vector fields are substituted into the expressions for the P. Hall basis elements, the resulting set of vector fields is linearly independent. The method of construction used here is due to Grayson and Grossmann [17]; similar results can be found in the work of Sussmann [50]. We present only the 2-input case.

Given a P. Hall basis element $B = [B_i, B_j]$, we convert B into a canonical form by recursively expanding B_j:

$$B = \left[B_i, \left[B_{j_1}, \cdots [B_{j_k}, X_l] \right] \right]. \qquad (12)$$

Any element of a P. Hall basis has a unique representation in this form. Thus, we can associate with each such basis element a well-defined vector $\alpha \in \mathbb{Z}^n$ which indicates the number of times each basis element occurs in the expansion (12); i.e., $\alpha_i(k)$ is the number of times B_k appears in the expansion for B_i. From the properties of a P. Hall basis, it is clear that $\alpha_i(k) = 0$ if $k \geq i$.

Given a P. Hall basis $H = \{B_1, \cdots, B_n\}$ we now construct a vector field on $\mathbb{R}^n$ using coordinates $x \in \mathbb{R}^n$. Assume $B_i = X_i$ for $i = 1, \cdots, m$. Given α_i associated with B_i, $i > m$, we define

$$x^{\alpha_i} = \prod_j x_j^{\alpha_i(j)}$$

$$\alpha_i! = \prod_j \alpha_i(j)!$$

Theorem 2 (Maximal Growth 2 Input Systems [17]): Fix $k \geq 1$ and let n be the rank of the free, nilpotent Lie algebra of order k with two generators. Then

$$X_1 = \frac{\partial}{\partial x_1} \qquad X_2 = \frac{\partial}{\partial x_2} + \sum_{i=3}^{n} \frac{x^{\alpha_i}}{\alpha_i!} \frac{\partial}{\partial x_i}$$

generate a free, nilpotent Lie algebra (of vector fields) of order k at the origin.

The vector fields generated by this theorem are extensions of the canonical forms we have seen for degree of nonholonomy 2 and 3. The degree of nonholonomy for these vector fields is always one greater than the order of nilpotency.[2] One way to interpret and gain insight into this formula is to note that a Lie product

$$B_i = \left[B_{i_1}, \left[B_{i_2}, \cdots [B_{i_k}, X_2] \right] \right]$$

corresponds to a vector field obtained by taking the derivative of the components of X_2 with respect to $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$. The coefficients of X_2 are chosen such that taking this derivative leaves 1 in the $(\partial/\partial x_i)$ term.

Example 8: Consider the two input example given previously, but with order of nilpotency 4 instead of 5. The system generated by Theorem 2 is

$$\dot{x}_1 = u_1 \qquad\qquad X$$
$$\dot{x}_2 = u_2 \qquad\qquad Y$$
$$\dot{x}_3 = x_1 u_2 \qquad\quad [X,Y]$$
$$\dot{x}_4 = \frac{1}{2}x_1^2 u_2 \qquad [X,[X,Y]]$$
$$\dot{x}_5 = x_1 x_2 u_2 \qquad [Y,[X,Y]]$$
$$\dot{x}_6 = \frac{1}{6}x_1^3 u_2 \qquad [X,[X,[X,Y]]]$$
$$\dot{x}_7 = \frac{1}{2}x_1^2 x_2 u_2 \quad [Y,[X,[X,Y]]]$$
$$\dot{x}_8 = \frac{1}{2}x_1 x_2^2 u_2 \quad [Y,[Y,[X,Y]]].$$

We can now ask ourselves if it is possible to steer these canonical systems using sinusoids. Although the form of the system is different from that we used in Section III-B, the same approach can be used to steer x_1 through x_5. That is, sinusoids at the same frequency and proper phase give motion in x_3 and sinusoids at frequency 1 and 2 give motion in x_4 and x_5 (switching the input frequency switches between x_4 and x_5). This can be verified by direct calculation.

Steering in the $x_6 - x_8$ directions is more difficult. Consider the effect of using two simple sinusoids as inputs, $u_1 = a \cos \omega_1 t$ and $u_2 = b \sin \omega_2 t$. In order to prevent motion in lower level brackets, we must have $\omega_1 \neq \pm \omega_2$, $\omega_1 \neq \pm 2\omega_2$, $\omega_2 \neq \pm 2\omega_1$. Assuming these relationships hold, we get the following frequency compo-

---

[2]This discrepancy is due to the original definition of degree of nonholonomy given in [16], to which we have adhered.

---

nents in the derivatives fo the dynamic system:

$$x_6: \omega_1 \pm \omega_2 \quad 3\omega_1 \pm \omega_2$$
$$x_7: \omega_1 \quad 2\omega_1 \quad 2\omega_2 \quad 2\omega_1 \pm 2\omega_2$$
$$x_8: \omega_2 \quad \omega_1 \pm \omega_2 \quad \omega_1 \pm 3\omega_2.$$

By choosing frequencies such that the derivative has a term at frequency 0, we get motion in that coordinate. Thus $\omega_2 = 3\omega_1$ gives motion in x_6 (only) and $\omega_1 = 3\omega_2$ gives motion in x_8 (only).

Based on these calculations, it would appear that choosing $2\omega_1 = 2\omega_2$ would give motion in x_7. This is, in fact, the case, but we *also* get motion in the x_3 direction. It is not possible to get motion *only* in the x_7 direction using simple sinusoids. A direct calculation verifies that adjusting the phasing of the inputs does not resolve this dilemma. It may still be possible to steer the system using combinations of sinusoids at different frequencies for each input or using more complicated periodic functions (such as elliptic functions, see [9]).

B. Chained Systems

Rather than explore the use of more complicated inputs for steering nonholonomic systems, we consider instead a simpler class of systems. The justification for changing the class of systems is simple—most of the systems encountered as examples do not have the complicated structure of our canonical example. Thus there may be a simpler class of systems which is both steerable using simple sinusoids and representative of systems in which we are interested.

Consider a two input system of the following form:

$$\dot{x}_0 = u_1 \qquad\qquad \dot{y}_0 = u_2$$
$$\dot{x}_1 = y_0 u_1 \qquad\quad (\dot{y}_1 := x_0 u_2)$$
$$\dot{x}_2 = x_1 u_1 \qquad\quad \dot{y}_2 = y_1 u_2$$
$$\dot{x}_3 = x_2 u_1 \qquad\quad \dot{y}_3 = y_2 u_2 \qquad (13)$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$\dot{x}_{n_x} = x_{n_x - 1} u_1 \qquad \dot{y}_{n_y} = y_{n_y - 1} u_2.$$

The equation for $\dot{y}_1$ is contained in parenthesis to indicate that y_1 is not an independent variable. In fact, choosing

$$y_1 := x_0 y_0 - x_1$$

gives the indicated value for the derivative of y_1. In vector field form, (13) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = Xu_1 + Yu_2 \qquad \begin{aligned} X &= \frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} x_{i-1} \frac{\partial}{\partial x_i} \\ Y &= \frac{\partial}{\partial y_0} + \sum_{j=2}^{n} y_{j-1} \frac{\partial}{\partial y_j}. \end{aligned}$$

We refer to the system (13) as a *two-chain system*. The first item is to check the controllability of these systems.

To this end, denote iterated Lie products as $\mathrm{ad}_X^k Y$:

$$\mathrm{ad}_X Y = [X, Y]$$
$$\mathrm{ad}_X^k Y = \left[X, \mathrm{ad}_X^{k-1} Y \right]$$
$$= [X, [X, \cdots, [X, Y] \cdots]].$$

Lemma 3 (Lie Bracket Calculations): For the vector fields in (13)

$$\mathrm{ad}_X^k Y = (-1)^k \frac{\partial}{\partial x_k}$$
$$\mathrm{ad}_Y^k X = (-1)^k \frac{\partial}{\partial y_k} \qquad k > 1.$$

Proof: By induction. Since the first level of brackets is irregular, we begin by expanding $[X, Y]$ and $[X, [X, Y]]$.

$$[X, Y] = \left(\frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum x_{i-1} \frac{\partial}{\partial x_i} \right)$$
$$\cdot \left(\frac{\partial}{\partial y_0} + \sum y_{j-1} \frac{\partial}{\partial y_j} \right)$$
$$- \left(\frac{\partial}{\partial y_0} + \sum y_{j-1} \frac{\partial}{\partial y_j} \right)$$
$$\cdot \left(\frac{\partial}{\partial x_0} + y_0 \frac{\partial}{\partial x_1} + \sum x_{i-1} \frac{\partial}{\partial x_i} \right)$$
$$= \left(y_0 \frac{\partial}{\partial y_2} - y_0 \frac{\partial}{\partial y_2} \right) - \frac{\partial}{\partial x_1} = - \frac{\partial}{\partial x_1}$$
$$[X, [X, Y]] = X \left(- \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial x_1} (X) = 0 + \frac{\partial}{\partial x_2}.$$

Now assume that $\mathrm{ad}_X^k Y = (-1)^k (\partial / \partial x_k)$. Then

$$\mathrm{ad}_X^{k+1} Y = \left[X, \mathrm{ad}_X^k Y \right] = (-1)^k \left(X \left(\frac{\partial}{\partial x_k} \right) - \frac{\partial}{\partial x_k} (X) \right)$$
$$= (-1)^{k+1} \frac{\partial}{\partial x_{k+1}}.$$

The proof for $\mathrm{ad}_Y^k X$ is identical using the facts $[Y, X] = -[X, Y]$ and $y_1 := x_0 y_0 - x_1$. □

Proposition 4 (Controllability of the Two-Chain System): The two-chain system (13) is maximally nonholonomic (controllable).

Proof: There are $n_x + n_y + 1$ coordinates in (13) and the $n_x + n_y + 1$ Lie products

$$\{X, Y, \mathrm{ad}_X^i Y, \mathrm{ad}_Y^j X\} \qquad 1 \le i \le n_x, \qquad 2 \le j \le n_y$$

are independent using Lemma 3. We require $j \ge 2$ since $\mathrm{ad}_Y X = -\mathrm{ad}_X Y$ and hence those Lie products can never be independent. □

To steer this system, we use sinusoids at integrally related frequencies. Roughly speaking, if we use $u_1 = \sin t$ and $u_2 = \cos kt$ then $\dot{x}_1$ will have components at frequency $k - 1$, $\dot{x}_2$ at frequency $k - 2$, etc. $\dot{x}_k$ will have a

component at frequency zero and when integrated we get motion in x_k while all previous variables return to their starting values. In the y variables, all frequency components will be of the form $m \cdot k \pm 1$ and hence we get no motion for $k > 1$. We make this precise with the following algorithm.

Algorithm 3:
1) Steer x_0 and y_0 to their desired values.
2) For each x_k, $k \ge 1$, steer x_k to its final value using $u_1 = a \sin t$, $u_2 = b \cos kt$, where a and b satisfy

$$x_k(2\pi) - x_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi.$$

3) For each y_k, $k \ge 2$, steer y_k to its final value using $u_1 = b \cos kt$, $u_2 = a \sin t$, where a and b satisfy

$$y_k(2\pi) - y_k(0) = \frac{(a/2)^k b}{k!} \cdot 2\pi.$$

Proposition 5: Algorithm 3 can steer (13) to an arbitrary configuration.

Proof: The proof is constructive. It suffices to consider only step 2) since step 3) can be proved by switching x and y in what follows. We must show two things:
1) moving x_k does not affect x_j, $j < k$
2) moving x_k does not affect y_j, $j = 1, \cdots, n_y$.

To verify that using $u_1 = a \sin t$, $u_2 = b \cos kt$ produces motion only in x_k, we integrate the x states. If x_{k-1} has terms at frequency ω_i, then x_k has corresponding terms at $\omega_i \pm 1$ (by expanding products of sinusoids as sums of sinusoids). Since the only way to have $x_i(2\pi) \ne x_i(0)$ is to have x_i have a component at frequency zero, it suffices to keep track only of the lowest frequency component in each variable; higher components will integrate to zero. Direct computation starting from the origin yields

$$x_0 = a(1 - \cos t)$$
$$x_1 = \int \frac{ab}{k} \sin kt \sin t = \frac{1}{2} \frac{ab}{k(k-1)} \sin (k-1) t$$
$$+ \frac{1}{2} \frac{ab}{k(k+1)} \sin (k+1) t$$
$$x_2 = \frac{1}{4} \frac{a^2 b}{k(k-1)(k-2)} \sin (k-2) t + \cdots$$
$$\vdots$$
$$x_k = \int \left(\frac{a^k b}{2^{k-1} k!} \sin^2 t + \cdots \right) dt = \frac{a^k b}{2^{k-1} k!} \frac{t}{2} + \cdots$$

$x_k(2\pi) = x_k(0) + ((a/2)^k b / k!)\pi$ and all earlier x_i's are periodic and hence $x_i(2\pi) = x_i(0)$, $i < k$. If the system does not start at the origin, the initial conditions generate extra terms of the form $x_{i-1}(0) u_2$ in the ith derivative and this integrates to zero, giving no net contribution.

To show that we get no motion in the y variables, we show that all frequency components in the y's have the form $mk \pm 1$ where m is some integer. This is true for

$y_1 := x_0 y_0 - x_1$ from the calculation above. Assume it is true for y_i:

$$\dot{y}_{i+1} = y_i u_2$$
$$= \sum_m \alpha(m) \sin(mk \pm 1)t \cdot \cos kt$$
$$= \sum_m \frac{\alpha(m)}{2}(\sin((m+1)k \pm 1)t$$
$$+ \sin((m-1)k \pm 1)t).$$

Hence, y_{i+1} only has components at nonzero frequencies $m'k \pm 1$ and therefore $y_i(2\pi) = y_i(0)$. □

To include systems with more than two inputs, we replicate the structure of (13) for each additional input. Let h_{ij}^k represent the motion corresponding to the Lie product $\mathrm{ad}_{X_i}^k X_j$. In the two input case, $x_0 = h_1^0$, $x_k = h_{21}^k$ and $y_0 = h_2^0$, $y_k = h_{12}^k$. The following system on $\mathbb{R}^n$ is an *m-chain system*:

$$\dot{h}_j^0 = u_j \qquad j = 1, \cdots, m$$
$$\dot{h}_{ij}^1 = h_i^0 u_j \qquad i > j \quad \text{and} \quad h_{ji}^1 := h_i^0 h_j^0 - h_{ij}^1 \quad (14)$$
$$\dot{h}_{ij}^k = h_{ij}^{k-1} u_j \qquad i, j = 1, \cdots, m \quad \text{and} \quad k > 1, i \neq j.$$

Proposition 6 (Multichain System Controllability): The multi-chain system of (14) is maximally nonholonomic and can be steered using sinusoids.

Proof: The system (14) can be rewritten

$$\dot{h} = X_1 u_1 + \cdots + X_m u_m$$

with

$$X_j = \frac{\partial}{\partial h_j^0} + \sum_{\substack{i=1 \\ i>j}}^m h_i^0 \frac{\partial}{\partial h_{ij}^1} + \sum_{k>1} \sum_{i \neq j} h_{ij}^{k-1} \frac{\partial}{\partial h_{ij}^k}.$$

Given any two X_i, X_j, their Lie product expansions only involve terms of the form h_{ij}^k for some k. But this is precisely the vector fields from Lemma 3 and hence

$$\mathrm{ad}_{X_i}^k X_j = (-1)^k \frac{\partial}{\partial h_{ij}^k}.$$

Taking these terms for all possible i, j, k we get a set of independent Lie products just as in the proof of Theorem 4.

To show that the system can be steered using sinusoids, pick any $i, j \in \{1, \cdots, m\}$, $i > j$. Fix $u_l = 0$ for all $l \neq i, j$. The resulting system is identical to (13) can be steered using Algorithm 3. By choosing all possible combinations of i and j, we can move to any position. □

C. Noncanonical Chained Systems

We would like to extend the class of systems which we can steer by including systems which have similar structure to (13), but with additional nonlinearities. The following example illustrates the limitations of using sinusoidal

inputs for this purpose. Consider the system

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = (x_2 + \epsilon x_2^2)u_1$$
$$\dot{x}_4 = (x_3 + \epsilon x_3^2)u_1$$
$$\dot{x}_5 = x_4 u_1.$$

This is similar to a chained system with a single chain: $\mathrm{ad}_{g_1}^k g_2$, $k = 0, 1, 2, 3$, together with g_1 forms a basis for $\mathbb{R}^5$.

If we apply inputs $u_1 = \sin t$ and $u_2 = \cos 3t$, we get the following motion, starting from $x = 0$,

$$x_1(2\pi) = 0$$
$$x_2(2\pi) = 0$$
$$x_3(2\pi) = 0$$
$$x_4(2\pi) = -\frac{7}{1440}\epsilon^2$$
$$x_5(2\pi) = \frac{\pi}{24} + 0.03\epsilon^2.$$

The reason for this perturbation in x_4 is that the (small) nonlinear terms cause zero frequency components to appear in $\dot{x}_4$. Hence, we cannot use simple sinusoids to steer this system as before.

Nonetheless, there are many special instances where sinusoids are an important tool. For example, we were able to steer the automobile with sinusoids, despite the nonlinearities. Since the automobile had degree of nonholonomy 3, the problems present in the previous example do not occur. Another example is a system which has the chained canonical form until the last coordinate. In this case, harmonic analysis is needed when finding the motion at the last step of the algorithm and zero frequency terms do not appear in any previous coordinates.

It may also be possible to use feedback transformation to convert certain systems into chained canonical form. This is similar to the technique used in nonlinear control to convert a nonlinear system into a linear one by using a change of coordinates and state feedback. Similar efforts have been used by Lafferriere and Sussmann [26] to convert systems into nilpotent form for use with their planning algorithm. It is interesting to note that in several of their examples, the converted systems were also in chained canonical form. We study this possibility in detail in the next section.

Finally, sinusoids may be useful for steering systems which are not locally in canonical form. The minimal structure necessary to attempt motion generation using sinusoids is a *triangular system* [38]. A system is triangular if we can find a set of coordinates $h = (h^0, h^1, \cdots, h^p) \in$

$\mathbb{R}^{m_0 \times m_1 \times \cdots \times m_p} = \mathbb{R}^n$ such that

$$\dot{h}^0 = v \qquad v \in \mathbb{R}^{m_0}$$

$$\dot{h}^1 = f^1(h^0)v$$

$$\dot{h}^2 = f^2(h^0, h^1)v$$

$$\vdots$$

$$\dot{h}^p = f^p(h^0, \cdots, h^{p-1})v.$$

The triangular form was necessary in our examples to insure that the differential equations driven by sinusoidal inputs could be integrated in a stepwise fashion.

D. Converting Systems to Chained Form

In this section, we introduce a set of sufficient conditions for determining if a system can be converted to chained form. This set of conditions gives a constructive method for building a feedback transformation which accomplishes the conversion. We concentrate on the two input case with a single chain.

Proposition 7 (Converting Systems to Two-Chained Form): Consider a controllable system

$$x = g_1(x)u_1 + g_2(x)u_2$$

with g_1, g_2 linearly independent and smooth. Define

$$\Delta_0 := \text{span}\left\{g_1, g_2, \text{ad}_{g_1}g_2, \cdots, \text{ad}_{g_1}^{n-2}g_2\right\}$$

$$\Delta_1 := \text{span}\left\{g_2, \text{ad}_{g_1}g_2, \cdots, \text{ad}_{g_1}^{n-2}g_2\right\}$$

$$\Delta_2 := \text{span}\left\{g_2, \text{ad}_{g_1}g_2, \cdots, \text{ad}_{g_1}^{n-3}g_2\right\}.$$

If for some open set U, $\Delta_0(x) = \mathbb{R}^n$ for all $x \in U \subset \mathbb{R}^n$ and Δ_1 and Δ_2 are involutive on U and there exists a smooth function $h_1 : U \to \mathbb{R}$ such that $dh_1 \cdot \Delta_1 = 0$ and $L_{g_1}h_1 = 1$, then there exists a local feedback transformation

$$\xi = \phi(x) \qquad u = \beta(x)v$$

such that the transformed system is in chained form:

$$\dot{\xi}_1 = v_1$$

$$\dot{\xi}_2 = v_2$$

$$\dot{\xi}_3 = \xi_2 v_1$$

$$\vdots$$

$$\dot{\xi} = \xi_{n-1}v_1.$$

Proof: By assumption, there exists a function $h_1 : U \to \mathbb{R}$ such that $dh_1 \cdot \Delta_1 = 0$ and $dh_1 \cdot g_1 = 1$. Furthermore, $dh_1 \cdot \Delta_2 = 0$ and since Δ_2 is an involutive distribution of codimension 2, we can find a second function $h_2 : U \to \mathbb{R}$ such that $dh_2 \cdot \Delta_2 = 0$ and $dh_2 \cdot \text{ad}_{g_1}^{n-2}g_2 \neq 0$.

Define the map $\phi : x \mapsto \xi$ as

$$\xi_1 = h_1$$

$$\xi_2 = L_{g_1}^{n-2}h_2$$

$$\vdots$$

$$\xi_{n-1} = L_{g_1}h_2$$

$$\xi_n = h_2.$$

To verify that ϕ is a valid change of coordinates, we use the fact that

$$L_{[f,g]}h = L_f L_g h - L_g L_f h$$

so that

$$L_{\text{ad}_{g_1}^{n-2}g_2}h_2 = L_{g_1}L_{\text{ad}_{g_1}^{n-3}g_2}h_2 - L_{\text{ad}_{g_1}^{n-3}g_2}L_{g_1}h_2$$

$$= (-1)^{n-2}L_{g_2}L_{g_1}^{n-2}h_2 \neq 0$$

and $L_{\text{ad}_{g_1}^k g_2}h_2 = 0$ for $k < n - 2$ by the same reasoning. Using this calculation,

$$\frac{\partial \phi}{\partial x} = \begin{bmatrix} dh_1 \\ dL_{g_1}^{n-2}h_2 \\ \vdots \\ dL_{g_1}h_2 \\ dh_2 \end{bmatrix}$$

$$\frac{\partial \phi}{\partial x}\Delta_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & \pm a(x) & * & \cdots & * \\ * & 0 & \pm a(x) & & \vdots \\ \vdots & \vdots & & \ddots & * \\ * & 0 & \cdots & 0 & \pm a(x) \end{bmatrix}$$

where $a(x) = L_{g_2}L_{g_1}^{n-2}h_2 \neq 0$. $(\partial \phi / \partial x)\Delta_0$ is full rank, hence ϕ is a local diffeomorphism.

Evaluating the derivatives of the coordinate transformation, we define

$$v_1 := u_1$$

$$v_2 := \left(L_{g_1}^{n-1}h_2\right)u_1 + \left(L_{g_2}L_{g_1}^{n-2}h_2\right)u_2.$$

Since $L_{g_2}L_{g_1}^{n-2}h_2 \neq 0$, this change of inputs is invertible and the resulting system is in chained form. $\qquad\square$

This proposition gives a set of sufficient conditions for converting a system with relative growth vector $\sigma = (2, 1, \cdots, 1)$ into chained form. In order to apply the results, however, we must solve two sets partial differential equations: $dh_1 \cdot \Delta_1 = 0$, $dh_1 \cdot g_1 = 1$ and $dh_2 \cdot \Delta_2 = 0$. This task can be simplified if g_1 and g_2 have the special form

$$g_1(x) = \frac{\partial}{\partial x_1} + \sum_{i=2}^{n} g_1^i(x)\frac{\partial}{\partial x_i}$$

$$g_2(x) = \sum_{i=2}^{n} g_2^i(x)\frac{\partial}{\partial x_i}.$$

In this case, choosing $h_1 = x_1$ shows that Δ_1 is always involutive and we are left with only one involutivity condi-

tion and associated partial differential equation. It is always possible to locally put g_1 and g_2 into this form using a change of input, due to the assumption that the input vector fields are linearly independent. This change of input is not unique.

One corollary to Proposition 7 is that all systems with relative growth vector $\sigma = (2, 1)$ can be converted to chained form. Using the change of basis described above, this becomes a direct consequence of the fact that all 1 dimensional distributions are involutive.

Example 9: Consider as our first example, the kinematic model of an automobile. The equations governing the motion of the system were derived in Section II-C:

$$\dot{x} = \cos \theta u_1$$
$$\dot{y} = \sin \theta u_1$$
$$\dot{\phi} = u_2 \qquad (15)$$
$$\dot{\theta} = \frac{1}{l} \tan \phi u_1.$$

To convert the system to chained form, we first scale the inputs so that u_1 enters $\dot{x}$ directly. Reusing the symbol u_1, the kinematics become:

$$\dot{x} = u_1$$
$$\dot{y} = \tan \theta u_1$$
$$\dot{\phi} = u_2$$
$$\dot{\theta} = \frac{1}{l} \sec \theta \tan \phi u_1.$$

Choose the y position of the car as the function h_2; it is easy to verify that this function annihilates Δ_2 in Proposition 7. The resulting change of coordinates is

$$\xi_1 = x$$
$$\xi_2 = \frac{1}{l} \sec^3 \theta \tan \phi$$
$$\xi_3 = \tan \theta$$
$$\xi_4 = y$$
$$u_1 = v_1$$
$$u_2 = -\frac{3}{l} \sin^2 \phi \sin \theta v_1 + \frac{1}{l} \cos^2 \theta \cos^3 \phi v_2.$$

And the transformed system has the form:

$$\dot{\xi}_1 = v_1$$
$$\dot{\xi}_2 = v_2$$
$$\dot{\xi}_3 = \xi_2 v_1$$
$$\dot{\xi}_4 = \xi_3 v_1.$$

This system can now be steered using the sinusoidal algorithm of the previous section or another method, such as Lafferriere and Sussmann's algorithm for generating motions for nilpotent systems. The motion is implemented as a feedback precompensator which converts the v inputs

into the actual system input, u. This feedback transformation agrees the that used in Lafferriere and Sussmann to nilpotentize the kinematic car example. Their formulation of the feedback transformation was not presented, although it seems clear that a similar approach must have been used.

Fig. 8 shows the results of using the chained form to steer an automobile. These trajectories are qualitatively similar to those in Fig. 7, but do not require the calculation of Fourier coefficients for determining open-loop trajectories. Instead, the system requires feedback compensation to place the system into chained form.

Example 10 (Car with N trailers): Consider first the case of a car pulling a single trailer. The equations of motion are identical to those of the car, with an additional equation specifying the motion of the attached trailer:

$$\dot{\theta}_1 = \sin(\theta_0 - \theta_1) u_1.$$

By solving the partial differential equations in the proof of Proposition 7, it can be shown that the function

$$h_2(y, \theta_1) = y - \log\left(\frac{1 + \sin \theta_1}{\cos \theta_1}\right)$$

generates a chained set of coordinates. Again we can locally steer the trailer using sinusoidal inputs or other methods.

Note that in this example, Algorithm 2 cannot be applied directly: the system is not a second order system. Further, the original system description does not have triangular structure ($\dot{\theta}_1$ depends on θ_1), so it is difficult to use sinusoids combined with Fourier analysis to generate a desired motion.

When additional trailers are added, the distribution Δ_2 is no longer involutive and hence the procedure outlined above does not apply. Since the conditions in the proposition are only sufficient conditions, this does not mean that a car with N trailers cannot be steered using sinusoids. But a more complicated change of basis would be required in order to convert the vector fields to the necessary form. This example points out the weaknesses of the theorem and provides directions for future research. We have begun this program in [52].

V. DISCUSSION AND FUTURE WORK

Most current nonholonomic motion planners rely on special system structure to generate efficient motions. In some cases the structure is very specific, as evidenced by the large number of path planners for car-like robots using the special form of the kinematics for that system. More general path planners, such as the one proposed by Lafferriere and Sussmann [26], require that either the system be nilpotent or that an iterative procedure be used. In the nonnilpotent case, the iterative algorithm generates very complex paths which can steer arbitrarily close to the goal only at the cost of additional complexity. The results of Section III are somewhat complimentary—the methods can easily be applied to
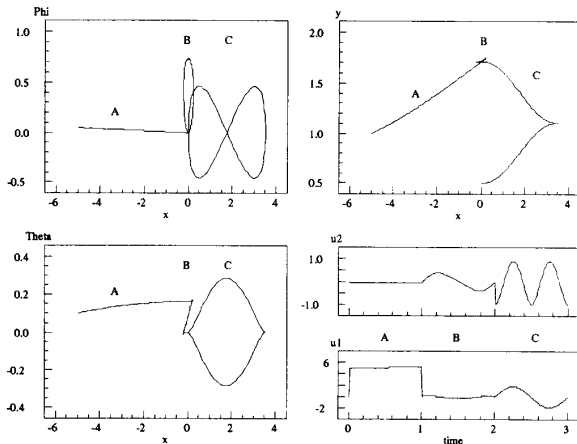
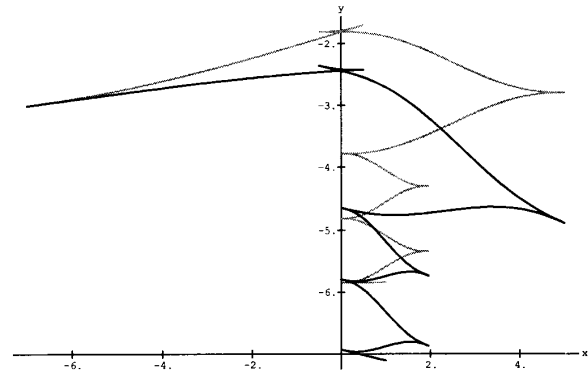Fig. 8.   Sample trajectories for a car using chained form.

Fig. 9.   Effects of initial condition errors on open-loop paths. The gray line shows a parking maneuver for an automobile. The solid path is the trajectory which is followed when the initial steering wheel angle of the car is off by 0.05 radians (approximately three degrees).

certain systems which are not nilpotent, but the general case requires a restrictive canonical form.

Research in efficient motion planning for general nonholonomic systems can proceed in many ways. More general conditions under which a distribution can be represented by a nilpotent or chained basis would clarify the extent to which particular algorithms can be applied. On the other hand, new approaches using metric or other properties of nonholonomic distributions might lead to path planners which work for more general classes of systems. Computational approaches such as those proposed by Barraquand and Latombe [2] might also be extended to handle higher dimensional systems with very few structural requirements.

The work in nonholonomic motion planning thus far has been primarily in the generation of open-loop trajectories. Closed-loop control of nonholonomic systems is very difficult, in part because of fundamental restrictions which prohibit the existence of smooth feedback controllers which asymptotically stabilize a point. Indeed, one can show using the results of Brockett [7], [47] that the class of nonholonomic systems is not stabilizable by smooth state feedback. Nonetheless, it is vital to introduce closed loop control for these systems to account for initial condition and modeling errors, noise, and other effects that are encountered in any real implementation. Fig. 9 shows an example of the effects of initial condition errors on parallel parking maneuvers for an automobile.

Several researchers have initiated research in feedback control of nonholonomic systems using either discontinuous or time-varying state feedback to circumvent the limitations imposed by Brockett's conditions. Recent work by Coron has shown that it is possible to stabilize a nonholonomic system using smooth, time-varying state feedback [11]. Constructive approaches have been presented by Samson [44], [45] and Pomet [42]. Discontinuous feedback laws for specific systems have been proposed by Canudas [12] and Pomet et al. [43]. In addition, Bloch and

McClamroch have studied problems related to stabilization to a manifold instead of a point [3], [4].

Another possible approach to the control of nonholonomic systems is the study of controllability along a reference trajectory. If we are given a desired state trajectory, we would like to construct a controller which stabilizes the system to this trajectory. The simplest example of such a controller is a control law for steering a car down the road. While the car is moving, it is quite easy to linearize the system and design linear feedback controllers which cause the car to stay aligned with a given trajectory. In fact, if the car is moving at a constant velocity, $u_1 = v_c$, then we can write

$$\dot{x} = g_1(x)v_c + g_2(x)u_2$$
$$= f(x) + g_2(x)u_2.$$

Furthermore, this system is completely controllable as a nonlinear system. Methods for extending these results to more complicated systems are currently being pursued [56].

The development of closed-loop controls may allow simplifications in planning for nonholonomic systems. Rather than attempt to find an input which steers us between the initial and desired locations, we might construct a piecewise feasible trajectory which connects the two points. We then apply a feedback controller about the piecewise feasible segments to implicitly define the input u. To illustrate this approach, we consider a parallel parking maneuver as shown in Fig. 10. This controller was constructed by using piecewise linear state feedback for each feasible segment.

Finally, we consider the problem of planning for systems with a nonzero drift vector field:

$$\dot{x} = f(x) + g(x)u.$$

The planning problem for this system is to steer between two equilibrium points of the system using u. If the equilibrium points lie on a connected manifold and the system is controllable at each point along the manifold,
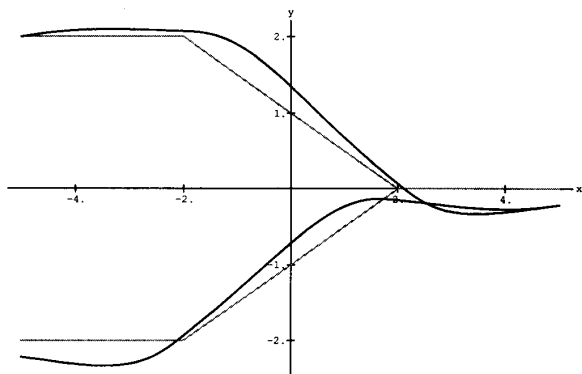
Fig. 10. Parallel parking maneuver using piecewise feasible segments (gray lines) and closed-loop control.

this problem can be solved for very general systems (see [19] for a specific example). However, if the start and goal position are not connected by an equilibrium manifold, it is not clear how to proceed. Although the existence of a trajectory is guaranteed by the appropriate controllability conditions, construction of a trajectory for systems with drift is still an open problem.
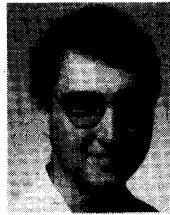
ACKNOWLEDGMENTS

The authors would like to thank Roger Brockett, Alberto Isidori, Jean-Paul Laumond, Zexiang Li, Richard Montgomery, Hector Sussmann, Dawn Tilbury, GregWalsh, and the members of the UC Berkeley Robotics Lab for many useful discussions in the area of nonholonomic motion planning.

REFERENCES

[1] V. I. Arnold, *Mathematical Methods of Classical Mechanics*. New York: Springer-Verlag, 2nd ed., 1989.

[2] J. Barraquand and J.-C. Latombe, Motion planning with many degrees of freedom and dynamic constraints, in *International Symposium on Robotics Research*, pp. 74–83, 1989.

[3] A. M. Bloch and N. H. McClamroch, "Control of mechanical systems with classical nonholonomic constraints," in *Proc. IEEE Conf. Decision Contr.*, pp. 201–205, 1989.

[4] ——, "Controllability and stabilizability properties of a nonholonomic control system," in *Proc. IEEE Conf. Decision Contr.*, 1990.

[5] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*. New York: Academic, 2nd ed., 1986.

[6] R. W. Brockett, "Control theory and singular Riemannian geometry," in *New Directions in Applied Mathematics*. New York: Springer-Verlag, pp. 11–27, 1981.

[7] ——, "Asymptotic stability and feedback stabilization," in R. W. Brockett, R. S. Millman, and H. J. Sussmann, Ed., *Differential Geometric Control Theory*, pp. 181–191. Boston: Birkhauser, 1983.

[8] R. W. Brockett, "On the rectification of vibratory motion," *Sensors and Actuators*. vol. 20, nos. 1–2, pp. 91–96, 1989.

[9] R. W. Brockett and L. Dai, Nonholonomic kinematics and the role of elliptic functions in constructive controllability, in *Progress in Nonholonomic Motion Planning*. New York: Kluwer, Feb. 1992, pp. 1–22.

[10] J. F. Canny, *The Complexity of Robot Motion Planning*. Cambridge, MA: MIT Press, 1988.

[11] J.-M. Coron, "Global asymptotic stabilization for controllable systems without drift," Tech. Rep., Université Paris-Sud, Labaratoire d'Analyse Numérique, Bâtiment 425, 91405 Orsay, FRANCE, 1991, preprint; to appear in MCSS, 1992.

[12] C. Canudas de Wit and O. J. Sordalen, "Exponential stabilization of mobile robots with nonholonomic constraints," submitted to *IEEE Trans. Autom. Contr.*, May 1991.

[13] B. R. Donald, "Planning multistep error detection and recovery strategies," *Int. J. Robotics Res.*, vol. 9, no. 1, pp. 60, 1990.

[14] M. A. Erdmann and M. Mason, "An exploration of sensorless manipulation," *IEEE J. Robotics Automation*, vol. 4, no. 4, pp. 369–379, 1988.

[15] C. Fernandes, L. Gurvits, and Z. Li, "Foundations of nonholonomic motion planning," Tech. Rep. 577, Courant Institute of Mathematical Sciences, New York University, 1991.

[16] V. Gershkovich and A. Vershik, "Nonholonomic manifolds and nilpotent analysis," *J. Geometry Phys.*, vol. 5, no. 3, pp. 407–452, 1988.

[17] M. Grayson and R. Grossman, "Models for free nilpotent Lie algebras," Tech. Memo PAM-397, Center for Pure and Applied Mathematics, University of California, Berkeley, 1987, to appear in *J. Algebra*.

[18] M. Hall, *The Theory of Groups*. New York: Macmillan, 1959.

[19] J. Hauser and R. M. Murray, "Nonlinear controllers for nonintegrable systems: the acrobot example, in *Proc. Amer. Contr. Conf.*, 1990, pp. 669–671.

[20] R. Hermann and A. J. Krener, "Nonlinear controllability and observability," *IEEE Trans. Automat. Cont.*, vol. AC-22, pp. 728–740, 1977.

[21] A. Isidori, *Nonlinear Control Systems*. New York: Springer-Verlag, 2nd Ed., 1989.

[22] J. Hopcroft, J. T. Schwartz, M. Sharir, *Planning, Geometry, and Complexity of Robot Motion*. Ablex,Norwood, New Jersey, 1987.

[23] P. Jacobs, J.-P. Laumond, M. Taix, and R. Murray, "Fast and exact trajectory planning for mobile robots and other systems with nonholonomic constraints," Tech. Rep. 90318, LAAS/CNRS, Toulouse, France, Sept. 1990; also to appear in *IEEE Trans. Robot. Automation*.

[24] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," *Int. J. Rob. Res.*, vol. 5, no. 1, pp. 90–99, 1986.

[25] D. E. Koditschek and E. Rimon, "Robot navigation functions on manifolds with boundary," *Adv. Appl. Math.*, vol. 11, no. 4, pp. 412–442, 1990.

[26] G. Lafferriere and H. J. Sussmann, "Motion planning for controllable systems without drift: A preliminary report, Tech. Rep. SYCON-90-04, Rutgers Center for Systems and Control, 1990.

[27] G. Lafferriere and H. J. Sussmann, "Motion planning for controllable systems without drift," in *IEEE Int. Conf. Robot. Automat.*, pp. 1148–1153, 1991.

[28] J.-C. Latombe, *Robot Motion Planning*. Boston: Kluwer, 1991.

[29] J.-P. Laumond, "Feasible trajectories for mobile robots with kinematic and environment constraints," in *Intelligent Autonomous Systems*. Amsterdam, The Netherlands: North-Holland, 1987.

[30] ——, "Finding collision-free smooth trajectories for a nonholonomic mobile robot," in *Int. J. Conf. Art. Intell.*, pp. 1120–1123, 1987.

[31] ——, "Nonholonomic motion planning versus controllability via the multibody car system example," Tech. Rep. STAN-CS-90-1345. Dept. Comput. Science, Stanford Univ., Oct. 1990, preprint.

[32] J.-P. Laumond and T. Siméon, "Motion planning for a two degrees of freedom mobile robot with towing," in *IEEE Int. Conf. Contr. Appl.*, 1989.

[33] J.-P. Laumond, M. Taix, and P. Jacobs, "A motion planner for car-like robots based on a mixed global/local approach," in *IEEE Int. Workshop Intell. Robots Syst.*, 1990.

[34] Z. Li, *Kinematics*, "Planning and Control of Dextrous Robot Hands," Ph.D. Dissertation, Dept. Elec. Eng. Comput. Sci., Univ. California, Berkeley, 1989.

[35] Z. Li and J. Canny, "Motion of two rigid bodies with rolling constraint," *IEEE Trans. Robot. Automation*, vol. 6, no. 1, pp. 62–71, 1990.

[36] Z. Li, R. Montgomery, and M. Raibert, "Dynamics and optimal control of a legged robot in flight phase," in *IEEE Int. Conf. Robot. Automation*, pp. 1816 –1821, 1989.

[37] T. Lozano-Pérez, M. T. Mason, and R. H. Taylor, "Automatic synthesis of fine-motion strategies for robots," *Int. J. Robot. Res.*, vol. 3, no. 1, pp. 3–24, 1984.

[38] R. Montgomery, 1990, personal communication.

[39] R. M. Murray and S. S. Sastry, "Grasping and manipulation using multifingered robot hands," in R. W. Brockett, Ed., *Robotics*: in

Proc. Symp. Appl. Math. vol. 41, pp. 91–128, American Mathematical Society, 1990.

[40] E. Nelson, *Tensor Analysis*. Princeton, NJ: Princeton University Press, 1967.

[41] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.

[42] J.-B. Pomet, "Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift," *Syst. Contr. Lett.*, vol. 18, no. 2, pp. 147–158, 1992.

[43] J. B. Pomet, B. Thuillot, G. Bastin, and G. Campion, "A hybrid strategy for the feedback stabilization of nonholonomic mobile robots," Tech. Rep. 91.74, Centre for Systems Engineering and Applied Mechanics, Université Catholique de Louvain, Belgium, 1991.

[44] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," in *Int'l. Workshop Adapt. Nonlinear Contr. Issues Robot.*, 1990.

[45] C. Samson and K. Ait-Abderrahim, "Feedback stabilization of a nonholonomic wheeled mobile robot," in *Int. Conf. Intelligent Rob. Syst.* (IROS), 1991.

[46] J.-P. Serre, *Lie Algebras and Lie Groups*. New York: W. A. Benjamin, 1965.

[47] E. D. Sontag, "Feedback stabilization of nonlinear systems," in *Mathematical Theory of Networks and Systems*. Boston: Birkhauser, 1989.

[48] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 1. Houston: Publish or Perish, 2nd Ed., 1979.

[49] H. Sussmann,"Lie brackets, real analyticity and geometric control," In R. W. Brockett, R. S. Millman and H. J. Sussmann, Ed., *Differential Geometric Control Theory*. Boston: Birkhäuser, pp. 1–116, 1983.

[50] H. J. Sussmann, "A product expansion for the Chen series," In C. Byrnes and A. Lindquist, Ed., *Theory and Applications of Nonlinear Control Systems*, pp. 323–335. Amsterdam, The Netherlands. North-Holland, 1986.

[51] H. J. Sussmann and W. Liu, "Limits of highly oscillatory controls and the approximation of general paths by admissible trajectories," Tech. Rep. SYCON-91-02, Rutgers Center for Systems and Control, 1991.

[52] D. Tilbury, J.-P. Laumond, R. Murray, S. Sastry, and G. Walsh, "Steering car-like systems with trailers using sinusoids," in *IEEE Int. Conf. Robot. Automat.*, pp. 1993–1998, 1992.

[53] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*. New York: Springer-Verlag, 1984.

[54] A. M. Vershik and V. Ya. Gershkovich, "Nonholonomic problems and the theory of distributions," *Acta Applicandae Mathematicae*, vol. 12, no. 181, pp. 181–209, 1988.

[55] G. Walsh and S. Sastry, "On reorienting rigid linked bodies using internal motions, in *IEEE Contr. Decision Conf.*, pp. 1190–1195, 1991.

[56] G. Walsh, D. Tilbury, S. Sastry, R. Murray, and J.-P. Laumond, "Stabilization of trajectories for systems with nonholonomic constraints, in *IEEE Int. Conf. Robot. Automation*, pp. 1999–2004, 1992.

Richard M. Murray (S'84–M'91) received the B.S. degree in electrical engineering from California Institute of Technology, Pasadena, in 1981 and the M.S. and Ph.D. degrees in electrical engineering and computer sciences from the University of California, Berkeley, in 1988 and 1991, respectively.

He held a postdoctoral appointment at University of California, Berkeley, in 1991, and is currently an Assistant Professor of Mechanical Engineering at the California Institute of Technology, Pasadena. His research interests include nonlinear control of mechanical systems and robotics.

S. Shankar Sastry (S'79–M'80–SM'90) received the B. Tech. degree from the Indian Institute of Technology India, in 1977, the M.S. and Ph.D. degrees in electrical engineering, in 1979 and 1981, respectively, and the M.A. degree in mathematics in 1980, all from the University of California, Berkeley.

After having been on the faculty of the Massachusetts Institute of Technology, Cambridge, from 1980 to 1982, he is currently a Professor in the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley. He has held visiting appointments at the Australian National University, Canberra, the University of Rome, Italy, the laboratory LAAS of the French Centre National Rechereche Scientifique in Toulouse, and the Center for Intelligent Control Systems at MIT. His areas of research interests are nonlinear control systems, adaptive control, robotic motion planning, kinematics, and control of multifingered robot hands. He is a coauthor (with M. Bodson) of *Adaptive Control: Stability, Converence and Robustness*, Prentice-Hall, 1989.

Dr. Sastry was an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL and the IEEE Transactions on Circuits and Systems and is an Associate Editor of the *IMA Journal of Control and Information*, the *International Journal of Adaptive Control and Signal Processing* and the *Journal of Mathematical Systems, Estimation and Control*. He received the President of India Gold Medal in 1977, the IBM Faculty Development award for 1983–1985, the NSF Presidential Young Investigator Award in 1985 and the Eckman Award of the American Automatic Control Council in 1990.