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Aircraft Conflict Resolution: Lie-Poisson Reduction for Game on $SE(2)$[1]

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Abstract

In this paper the problem of conflict resolution for two noncooperative aircraft on a fixed altitude is considered. We formulate the problem into a differential game, and then apply the Lie-Poisson reduction on the dual of the Lie algebra of the special Euclidean group. The reduced Hamiltonian dynamics is thus derived and can be integrated explicitly backwards in time. We use hybrid automata to describe the solution to the reduced dynamics as well as the aircraft maneuvers in the game. The safe portion of the target set frontier is calculated, and the solution to the boundary of the safe portion is also derived.

1 Introduction

With the rapid growth of air traffic over the past decade, air traffic management is currently an area of great research interest [1, 2, 3]. An important notion in the next generation air traffic management is *free flight*, in which each aircraft would be able to optimize its own trajectory according to certain factors such as safety, fuel economy and passenger comfortableness. Consequently, the aircraft would have much more flexibility to choose airway than in the conventional fixed route system. In order to maintain the standard safety requirement, free flight renders a tremendous complexity on conflict detection and resolution. In [4], a differential game approach was proposed for noncooperative conflict resolution in which each aircraft develops a resolution strategy for the worst possible action of the other aircraft. For the case in which many different agents have conflicting objectives, noncooperative dynamic game theory provides a natural framework.

In this paper, following the work of [5], we develop a solution for two noncooperative aircraft conflicts on a fixed altitude. The main contribution of this paper is to derive the explicit solution to the reduced Hamiltonian dynamics of the differential game. This solution can be nicely described by the executions of hybrid automata. With this solution, we are able to find the possible aircraft maneuvers.

The outline of the paper is as follows. Section 2 presents some background material. In Section 3, a differential game framework is set up to solve the conflicts between two noncooperative aircraft on a fixed altitude, and the reduced Hamiltonian dynamics is derived by Lie-Poisson reduction. In Section 4, the solution to the reduced dynamics is obtained and the hybrid automata to describe this solution is given. With this solution, all the possible aircraft maneuvers for evader and pursuer under optimal control and worst disturbance are derived. Section 5 presents the computation of the safe portion of the target set frontier and the solution to the boundary of the safe portion. A conclusion is given in Section 6.

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2 Lie-Poisson reduction on Poisson manifold

In this section, we present some background material. Readers can find more details in [6, 7, 8].

Let G be a finite dimensional Lie group with identity e. Denote the Lie algebra of the Lie group G as $\mathfrak{g}$. Let X_e be a controlled curve in $\mathfrak{g}$, that is,

$$X_e(u) = \sum_{i=1}^{m} u_i X_i, \tag{1}$$

where u_i are controls and the X_i span an m-dimensional subalgebra of $\mathfrak{g}$, $m \leq n = \dim(\mathfrak{g})$.

Let $\{X_1, \ldots, X_n\}$ be a basis for $\mathfrak{g}$, $\{X^1, \ldots, X^n\}$ be the corresponding dual basis for $\mathfrak{g}^*$. The structure constants C_{ab}^d are defined by

$$[X_a, X_b] = \sum_{d=1}^{n} C_{ab}^d X_d \tag{2}$$

where a, b run from 1 to n. Identify the set of functions on $\mathfrak{g}^*$ with the set of left invariant functions on T^*G

endows $\mathfrak{g}^*$ with Poisson structures given by

$$\{F,G\}_{\pm}(\mu) = \pm \sum_{a,b,d=1}^{n} C_{ab}^{d}\mu_d \frac{\partial F}{\partial \mu_a}\frac{\partial G}{\partial \mu_b}. \quad (3)$$

where $\mu = \sum_{i=1}^{n} \mu_i X^i$. We present the standing theorem of this paper as follows.

Theorem 1 (Theorem 13.6.2, [7]) *Let G be a Lie group and let $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian. Let $h : \mathfrak{g}^* \to \mathbb{R}$ be the restriction of H to T_e^*G. For a curve $p(t) \in T_{g(t)}^*G$, let $\mu(t) = (T_e^* L_{g(t)}) \cdot p(t)$ be the induced curve in $\mathfrak{g}^*$. Assuming that $g(t)$ satisfies the differential equation*

$$\dot{g} = T_e L_g \frac{\delta h}{\delta \mu}, \quad (4)$$

where $\mu = p(0)$, the following are equivalent:

(a) $p(t)$ is an integral curve of X_H; i.e., Hamilton's equations on T^*G hold;

(b) for any $F \in C^\infty(T^*G)$, $\dot{F} = \{F,H\}$, where $\{,\}$ is the canonical bracket on T^*G;

(c) for any $f \in C^\infty(\mathfrak{g}^*)$, we have $\dot{f} = \{f,h\}_-$, where $\{,\}_-$ is the minus Lie-Poisson bracket;

(d) $\mu(t)$ satisfies the Lie-Poisson equations

$$\frac{d\mu}{dt} = ad^*_{\delta h/\delta\mu}\mu \quad (5)$$

where $ad_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $ad_\xi \eta = [\xi,\eta]$ and ad_ξ^* is its dual, i.e.,

$$\dot{\mu}_i = \{\mu_i, h\}_- = -\sum_{b,d=1}^{n} C_{ib}^{d}\mu_d \frac{\partial h}{\partial \mu_b}. \quad (6)$$

3 Differential game approach to aircraft conflict resolution

In this section, we introduce the kinematics of an individual aircraft on a fixed altitude, and then present a differential game approach to aircraft conflict resolution. For the early work on this topic, see [4, 5].

Suppose that there are two aircraft on a fixed altitude, the dynamics of aircraft i can be described by the following state equations

$$\begin{aligned} \dot{x}_i &= v_i \cos\phi_i \\ \dot{y}_i &= v_i \sin\phi_i \\ \dot{\phi}_i &= w_i, \end{aligned} \quad (7)$$

where (x_i, y_i) is the position and ϕ_i is the heading of aircraft i, and v_i, w_i are the airspeed and angular velocity respectively. Let

$$g_i = \begin{bmatrix} \cos\phi_i & -\sin\phi_i & x_i \\ \sin\phi_i & \cos\phi_i & y_i \\ 0 & 0 & 1 \end{bmatrix}, \quad (8)$$

thus $g_i \in SE(2)$, the special Euclidean group on the plane. Take the standard coordinate map (ϕ, x, y) of $SE(2)$, and let

$$X_e = w\frac{\partial}{\partial\phi}\Big|_e + v\frac{\partial}{\partial x}\Big|_e, \quad (9)$$

we have

$$\begin{aligned} X_g &= T_e L_g(X_e) \\ &= w\frac{\partial}{\partial\phi}\Big|_g + v\cos(\phi(g))\frac{\partial}{\partial x}\Big|_g + v\sin(\phi(g))\frac{\partial}{\partial y}\Big|_g. \end{aligned} \quad (10)$$

Hence system (7) is left invariant in $SE(2)$. The Lie algebra of $SE(2)$ is $se(2)$, which is spanned by

$$\begin{aligned} X_1\big|_g &= \frac{\partial}{\partial\phi}\Big|_g, \\ X_2\big|_g &= \cos(\phi(g))\frac{\partial}{\partial x}\Big|_g + \sin(\phi(g))\frac{\partial}{\partial y}\Big|_g, \\ X_3\big|_g &= -\sin(\phi(g))\frac{\partial}{\partial x}\Big|_g + \cos(\phi(g))\frac{\partial}{\partial y}\Big|_g. \end{aligned} \quad (11)$$

Since $[X_1, X_2] = X_3$, $[X_1, X_3] = -X_2$, and $[X_2, X_3] = 0$, where $[\,,\,]$ is the Lie bracket on smooth vector fields, the structure constants are given by

$$\begin{array}{lll} C_{12}^1 = 0, & C_{12}^2 = 0, & C_{12}^3 = 1; \\ C_{13}^1 = 0, & C_{13}^2 = -1, & C_{13}^3 = 0; \\ C_{23}^1 = 0, & C_{23}^2 = 0, & C_{23}^3 = 0. \end{array} \quad (12)$$

According to the current standards of Federal Aviation Administration, the aircraft must be separated by 5 nautical miles laterally and 2,000 ft vertically [5]. Hence for two aircraft on the same altitude, the goal of each aircraft is to remain outside a target set

$$T = \{(g_1, g_2) \in SE(2) \times SE(2) | l(g_1, g_2) < 0\}, \quad (13)$$

where $l(g_1, g_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2 - 5^2$. When aircraft 1 is uncertain about the action of aircraft 2, its safest strategy is to fly beyond the minimum separation with aircraft 2, regardless of the action of aircraft 2. Therefore, we formulate this problem as a zero-sum noncooperative differential game with two players: control vs. disturbance. The control is the input of aircraft 1 $u := (v_1, w_1) \in U$ and the disturbance is the input of aircraft 2 $d := (v_2, w_2) \in D$. Here the control and disturbance sets U and D are compact sets in $\mathbb{R}^2$:

$$\begin{aligned} U &= ([v_1^m, v_1^M], [w_1^m, w_1^M]), \\ D &= ([v_2^m, v_2^M], [w_2^m, w_2^M]), \end{aligned} \quad (14)$$

and the corresponding control and disturbance spaces are defined as: $\mathcal{U} = \{ u(\cdot) \in PC^0(\mathbb{R}) \times PC^0(\mathbb{R}) \,|\, u(\tau) \in U, \forall \tau \in \mathbb{R}\}$, $\mathcal{D} = \{ d(\cdot) \in PC^0(\mathbb{R}) \times PC^0(\mathbb{R}) \,|\, d(\tau) \in D, \forall \tau \in \mathbb{R}\}$, where $PC^0(\mathbb{R})$ is the space of piecewise continuous functions on $\mathbb{R}$.

Consider the two-player, zero-sum differential game over the time interval $[t, 0]$, where $t < 0$, with cost functions defined by

$$J(g_1(\cdot), g_2(\cdot), u(\cdot), d(\cdot), t) = l(g_1(0), g_2(0)). \tag{15}$$

Because we are mainly concerned with the safety issue, the cost function depends only on the final state and the Lagrangian is identically zero. When the Lagrangian is chosen as a function of the control and disturbance inputs, the Lie-Poisson reduction still works.

Since aircraft 1 attempts to maximize the cost index, regardless of the action aircraft 2 might take, thus the optimal control and worst disturbance action are obtained as

$$\begin{aligned} u^* &= \arg\max_{u\in\mathcal{U}}\min_{d\in\mathcal{D}} J(g_1(\cdot), g_2(\cdot), u(\cdot), d(\cdot), t) \\ d^* &= \arg\min_{d\in\mathcal{D}}\max_{u\in\mathcal{U}} J(g_1(\cdot), g_2(\cdot), u(\cdot), d(\cdot), t) \end{aligned} \tag{16}$$

The game is said to have a *saddle solution* (u^*, d^*) if the optimal cost $J^*(g_1, g_2, t)$ does not depend on the order of the play [9]:

$$\begin{aligned} J^*(g_1, g_2, t) &= \max_{u\in\mathcal{U}}\min_{d\in\mathcal{D}} J(g_1(\cdot), g_2(\cdot), u(\cdot), d(\cdot), t) \\ &= \min_{d\in\mathcal{D}}\max_{u\in\mathcal{U}} J(g_1(\cdot), g_2(\cdot), u(\cdot), d(\cdot), t) \end{aligned} \tag{17}$$

To solve this differential game, let $p_i = p_i^1 d\phi + p_i^2 dx + p_i^3 dy \in T^*_{g_i} SE(2)$, $i = 1, 2$, then

$$T^*_e L_{g_i}(p_i) = \mu_i^1 d\phi + \mu_i^2 dx + \mu_i^3 dy, \tag{18}$$

where

$$\begin{aligned} \mu_i^1 &= p_i^1 \\ \mu_i^2 &= p_i^2 \cos(\phi(g)) + p_i^3 \sin(\phi(g)) \\ \mu_i^3 &= -p_i^2 \sin(\phi(g)) + p_i^3 \cos(\phi(g)). \end{aligned} \tag{19}$$

The Hamiltonian is given by

$$\begin{aligned} &H(g_1, g_2, p_1, p_2, u, d) \\ &= \langle p_1, T_e L_{g_1}(X_{e,1})\rangle + \langle p_2, T_e L_{g_2}(X_{e,2})\rangle \\ &= \langle T_e^* L_{g_1} p_1, X_{e,1}\rangle + \langle T_e^* L_{g_2} p_2, X_{e,2}\rangle \\ &= \mu_1^1 w_1 + \mu_1^2 v_1 + \mu_2^1 w_2 + \mu_2^2 v_2, \end{aligned} \tag{20}$$

where $X_{e,i} = w_i \frac{\partial}{\partial\phi}\big|_e + v_i \frac{\partial}{\partial x}\big|_e$, $i = 1, 2$. It follows that the optimal input u^* and worst disturbance d^* are given by

$$\begin{aligned} w_1^* &= \frac{w_1^M + w_1^m}{2} + \mathrm{sgn}(\mu_1^1)\frac{w_1^M - w_1^m}{2}, \\ v_1^* &= \frac{v_1^M + v_1^m}{2} + \mathrm{sgn}(\mu_1^2)\frac{v_1^M - v_1^m}{2}, \\ w_2^* &= \frac{w_2^M + w_2^m}{2} - \mathrm{sgn}(\mu_2^1)\frac{w_2^M - w_2^m}{2}, \\ v_2^* &= \frac{v_2^M + v_2^m}{2} - \mathrm{sgn}(\mu_2^2)\frac{v_2^M - v_2^m}{2}, \end{aligned} \tag{21}$$

and the optimal Hamiltonian H^* is

$$H^*(g_1, g_2, p_1, p_2) = \mu_1^1 w_1^* + \mu_1^2 v_1^* + \mu_2^1 w_2^* + \mu_2^2 v_2^*. \tag{22}$$

Since H^* is G-invariant, by Theorem 1, regular extremals are given by integral curves of the reduced Hamiltonian H^* on $se^*(2)_- \times se^*(2)_-$. The Poisson bracket of two functions ϕ and ψ on $se^*(2)_-$ can be calculated through (3):

$$\{\phi, \psi\}_-(\mu) = -\sum_{a,b,d=1}^{3} C^d_{ab}\mu_d \frac{\partial\phi}{\partial\mu_a}\frac{\partial\psi}{\partial\mu_b}. \tag{23}$$

From (6), the reduced Hamilton's equations are

$$\begin{array}{ll} \dot\mu_1^1 = -\mu_1^3 v_1^* & \dot\mu_2^1 = -\mu_2^3 v_2^* \\ \dot\mu_1^2 = \mu_1^3 w_1^* & \dot\mu_2^2 = \mu_2^3 w_2^* \\ \dot\mu_1^3 = -\mu_1^2 w_1^*, & \dot\mu_2^3 = -\mu_2^2 w_2^*. \end{array} \tag{24}$$

The terminal conditions for p_1, p_2 are given by

$$\begin{aligned} p_1(0) &= (0, 2(x_1(0) - x_2(0)), 2(y_1(0) - y_2(0)))^T, \\ p_2(0) &= (0, 2(x_2(0) - x_1(0)), 2(y_2(0) - y_1(0)))^T. \end{aligned} \tag{25}$$

Therefore from (19), the terminal conditions for μ_1^j, μ_2^j are

$$\begin{aligned} \mu_1^1(0) &= 0 \\ \mu_1^2(0) &= 2(x_1(0) - x_2(0))\cos\phi_1(0) \\ &\quad + 2(y_1(0) - y_2(0))\sin\phi_1(0) \\ \mu_1^3(0) &= -2(x_1(0) - x_2(0))\sin\phi_1(0) \\ &\quad + 2(y_1(0) - y_2(0))\cos\phi_1(0), \\ \mu_2^1(0) &= 0 \\ \mu_2^2(0) &= 2(x_2(0) - x_1(0))\cos\phi_2(0) \\ &\quad + 2(y_2(0) - y_1(0))\sin\phi_2(0) \\ \mu_2^3(0) &= -2(x_2(0) - x_1(0))\sin\phi_2(0) \\ &\quad + 2(y_2(0) - y_1(0))\cos\phi_2(0). \end{aligned} \tag{26}$$

Thus the optimal trajectory can be obtained by solving the differential equation (7), (24) with the boundary condition (26) and initial condition of g_1 and g_2 at time t.

4 Solutions to the differential game

From (21), it is clear that the optimal control is determined by the sign of μ_1^1 and μ_1^2, and the worst disturbance by the sign of μ_2^1 and μ_2^2. The closed form solution to the differential game can be derived provided that the dynamics of the reduced Hamiltonian H^* on $se^*(2)_- \times se^*(2)_-$ can be solved together with the aircraft dynamics (7). In this section, we assume the final state for the optimal cost index is known, and solve the differential equation backwards.

Assume that $(x_1(0) - x_2(0))^2 + (y_1(0) - y_2(0))^2 = 5^2$, $\cos\alpha = (x_2(0) - x_1(0))/5$, and $\sin\alpha = (y_2(0) - y_1(0))/5$, we have that

$$\begin{aligned}
\mu_1^1(0) &= 0 \\
\mu_1^2(0) &= -10\cos(\phi_1(0) - \alpha) \\
\mu_1^3(0) &= 10\sin(\phi_1(0) - \alpha), \\
\mu_2^1(0) &= 0 \\
\mu_2^2(0) &= 10\cos(\phi_2(0) - \alpha) \\
\mu_2^3(0) &= -10\sin(\phi_2(0) - \alpha).
\end{aligned} \tag{27}$$

Without loss of generality, let $-\pi \le \phi_1(0) - \alpha \le \pi$ and $-\pi \le \phi_2(0) - \alpha \le \pi$. Further, we assume the constraints on the dynamics of the aircraft:

$$v_i^m \ge 0, \quad v_i^M > 0, \quad w_i^m < 0, \quad w_i^M > 0, \tag{28}$$

which implies that the aircraft can turn either clockwise or counterclockwise, and fly only forward.

Solving the differential equation (24) with the boundary condition (27) explicitly, we find that the solution to μ_1 can be described by the non-zeno executions of the hybrid automata shown in Figure 1. For the details on the hybrid automaton, see [10]. Here all the resets of the hybrid automata in Figure 1 are identity maps, and all the discrete transitions are triggered by the sign change of μ_1^1. Note that the hybrid automata are endowed with the final states instead of the initial states and the time goes from negative infinity to final time instant $t = 0$. If $-\frac{\pi}{2} \le \phi_1(0) - \alpha < 0$ or $0 < \phi_1(0) - \alpha \le \frac{\pi}{2}$, the solution to μ_1 is accepted by the hybrid automaton on the top, which have only two discrete states; if $\frac{\pi}{2} < \phi_1(0) - \alpha \le \pi$ or $-\pi \le \phi_1(0) - \alpha < -\frac{\pi}{2}$, the solution is accepted by the hybrid automaton on the bottom, which have four discrete states. Further, it turns out that the executions of the hybrid automata for μ_1 are periodic: the continuous part has a minimum positive period $T_1 = 2|\phi_1(0) - \alpha|(\frac{1}{w_1^M} - \frac{1}{w_1^m})$, and the discrete part has a minimum positive period either 2 or 6, depending on which quadrant $\phi_1(0) - \alpha$ lies in. The solution to μ_2 can be obtained similarly.

Since the signs of μ_i determine (v_1^*, w_1^*) and (v_2^*, w_2^*) through (21), the aircraft dynamics (7) switches the

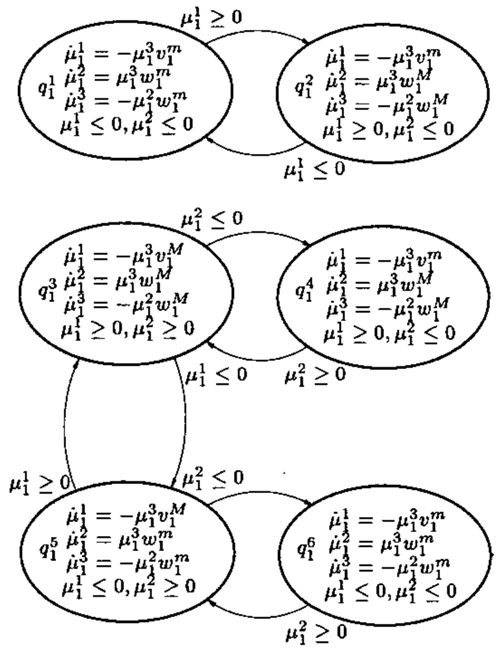

Figure 1: Hybrid automata for μ_1

optimal control and worst disturbance between their boundaries at the same time instant and in the same transition sequence as μ_i. From the trajectory of reduced dynamics μ_i, the aircraft state g_i can be derived explicitly by solving the system dynamics (7). Depending on which quadrant $\phi_i(0) - \alpha$ lies in, there are four different cases for the solution to $g_i(t)$. Each case yields an aircraft maneuver as shown in Figure 2. For $t < -2|\phi_i(0) - \alpha|/|w_i^*(0)|$, the trajectory of g_i switches the maneuvers in the following sequences:

(A)	←	(D)	←	(A)	←	(D)	←	...
(B)	←	(C)	←	(B)	←	(C)	←	...
(C)	←	(B)	←	(C)	←	(B)	←	...
(D)	←	(A)	←	(D)	←	(A)	←	...

For example, the maneuver ahead of (A) is (D), and what precedes even further is (A) again. The complete trajectory of g_i can be obtained by piecing together all the maneuvers according to the above sequences.

5 Safe portion computation

Our goal is to maintain the two aircraft beyond the safe separation, i.e., keep the trajectory of system (7) from entering the prespecified target set T defined in (13). The Lie derivative of $l(g_1, g_2)$ with respect to X_g is

$$\begin{aligned}
L_{X_g} l(g_1, g_2) = {} & \frac{\partial l}{\partial g_1}(w_1, v_1\cos\phi_1, v_1\sin\phi_1)^T \\
& + \frac{\partial l}{\partial g_2}(w_2, v_2\cos\phi_2, v_2\sin\phi_2)^T.
\end{aligned} \tag{29}$$

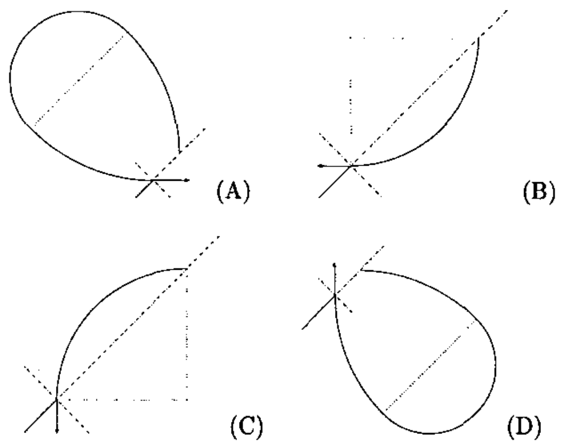

Figure 2: Aircraft maneuvers

For those (g_1, g_2) on ∂T,

$$L_{X_g} l(g_1, g_2) = 10(-v_1(0)\cos(\phi_1(0) - \alpha) + v_2(0)\cos(\phi_2(0) - \alpha)). \quad (30)$$

Let

$$L_1(g_1) = \frac{v_1^M + v_1^m}{2}\cos(\phi_1(0) - \alpha) - |\cos(\phi_1(0) - \alpha)|\frac{v_1^M - v_1^m}{2}, \quad (31)$$

$$L_2(g_2) = \frac{v_2^M + v_2^m}{2}\cos(\phi_2(0) - \alpha) - |\cos(\phi_2(0) - \alpha)|\frac{v_2^M - v_2^m}{2}, \quad (32)$$

then

$$L_{X_g} l(g_1, g_2) = 10(L_2(g_2) - L_1(g_1)). \quad (33)$$

We define the safe and unsafe portions of the target set frontier ∂T as follows [4]:

$$\text{Safe portion of } \partial T = \{\, (g_1, g_2) \in \partial T \,|\, \exists u, \forall d, L_{X_g} l(g_1, g_2) \geq 0\}, \quad (34)$$

$$\text{Unsafe portion of } \partial T = \{\, (g_1, g_2) \in \partial T \,|\, \forall d, \exists u, L_{X_g} l(g_1, g_2) < 0\}. \quad (35)$$

Since $L_{X_g} l(g_1, g_2)$ is separable in u and d, it is clear that those (g_1, g_2) on ∂T for which $L_2(g_2) \geq L_1(g_1)$ constitute the safe portion of ∂T, and those (g_1, g_2) on ∂T for which $L_2(g_2) < L_1(g_1)$ constitute the unsafe portion. Now we consider only the case $v_1^m = v_2^m = v^m$, $v_1^M = v_2^M = v^M$, and $w_i^m = -w$, $w_i^M = w$, where $w \in \mathbb{R}^+$. Then we have

$$\text{Safe portion of } \partial T = \{(g_1, g_2) \in \partial T \,|\, -|\phi_1(0) - \alpha| \leq \phi_2(0) - \alpha \leq |\phi_1(0) - \alpha|\}, \quad (36)$$

$$\text{Unsafe portion of } \partial T = \{(g_1, g_2) \in \partial T \,|\, -\pi \leq \phi_2(0) - \alpha < -|\phi_1(0) - \alpha|, \text{ or } |\phi_1(0) - \alpha| < \phi_2(0) - \alpha \leq \pi\}. \quad (37)$$

We present the relative kinematic model for two aircraft, which describes the motion of aircraft 2 with respect to aircraft 1:

$$\begin{aligned} x_r(t) &= \cos\phi_1(t)(x_2(t) - x_1(t)) \\ &\quad + \sin\phi_1(t)(y_2(t) - y_1(t)) \\ y_r(t) &= -\sin\phi_1(t)(x_2(t) - x_1(t)) \\ &\quad + \cos\phi_1(t)(y_2(t) - y_1(t)) \\ \phi_r(t) &= \phi_2(t) - \phi_1(t). \end{aligned} \quad (38)$$

We illustrate the safe portion of ∂T in the relative model in Figure 3.

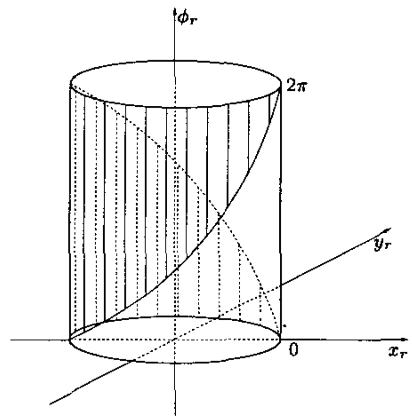

Figure 3: The safe portion of ∂T in relative model.

With the solution for μ_i, it can be derived that the Hamiltonian H^* keeps constant along the optimal trajectory, i.e., $H^*(\tau) = H^*(0)$, for all $\tau \in [t, 0]$. Furthermore,

$$\begin{aligned} &H^*(0) \\ &= v_1^*(0)\mu_1^2(0) + v_2^*(0)\mu_2^2(0) \\ &= \frac{1}{2}(v_1^M + v_1^m + \mathrm{sgn}(\mu_1^2(0))(v_1^M - v_1^m))\mu_1^2(0) \\ &\quad + \frac{1}{2}(v_2^M + v_2^m - \mathrm{sgn}(\mu_2^2(0))(v_2^M - v_2^m))\mu_2^2(0) \\ &= 10(L_2(g_2) - L_1(g_1)). \end{aligned} \quad (39)$$

Therefore $H^*(0) = 0$ implies $\phi_1(0) - \alpha = \pm(\phi_2(0) - \alpha)$. Note that it is nothing but the boundary of the safe portion of ∂T. With the solutions of the differential game, we can compute the solution to the boundary of the safe portion of ∂T by integrating the system equation backwards in time.

Case I $\phi_1(0) = \phi_2(0)$

We have the solution as follows:

$$\begin{aligned} x_r(t) &= K\cos(\phi_1(t)-\alpha) \\ y_r(t) &= -K\sin(\phi_1(t)-\alpha) \\ \phi_r(t) &= 0, \end{aligned} \tag{40}$$

where $\phi_1(t) = w_1^* t + \phi_1(0)$. Furthermore,

$$\frac{dy_r}{dx_r} = \tan(\frac{\pi}{2} - (\phi_1(t) - \alpha)). \tag{41}$$

Therefore for any $t \leq 0$, the trajectory of (y_r, x_r) is tangential to ∂T, and $\phi_1(t) = \phi_2(t)$. The trajectory for this case is depicted in Figure 4.

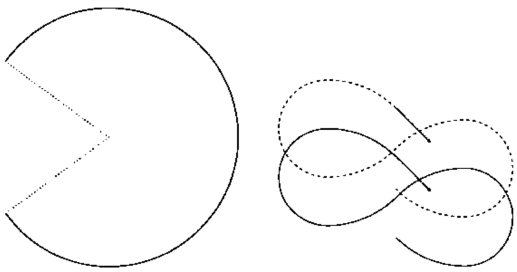

Figure 4: $\phi_1(0) = \phi_2(0)$: Left: Top view of the trajectory in the relative model; Right: trajectories for aircraft 1 (solid) and aircraft 2 (dashed) respectively.

Case II $\phi_1(0) - \alpha = -(\phi_2(0) - \alpha)$

When $t = 0$, we have

$$\left.\frac{dy_r}{dx_r}\right|_{t=0} = \tan(\frac{\pi}{2} - (\phi_1(0) - \alpha)), \tag{42}$$

thus the trajectory of (y_r, x_r) is tangential to ∂T at $t = 0$. The solution for this case is omitted due to its length, and the trajectories are shown as in Figure 5. It turns out that for all $t \leq 0$, $v_1^*(t) = v_2^*(t)$, $w_1^*(t) = w_2^*(t)$, and $\phi_1(t) - \alpha = -(\phi_2(t) - \alpha)$.

Note that in both cases, the trajectories are periodic and the minimum positive period is $4|\phi_1(0) - \alpha|/w$.

6 Conclusion

By applying Lie-Poisson reduction to a differential game on the special Euclidean group, we study the two aircraft conflict resolution problem in a noncooperative circumstance. The hybrid automata is used to describe the solution to the reduced Hamiltonian dynamics as well as the trajectory switching among different aircraft maneuvers for evader and pursuer. The safe portion of the target set frontier is calculated, and the solution to its boundary is also obtained. It may then be possible to perform efficient computation of solutions to hybrid systems.

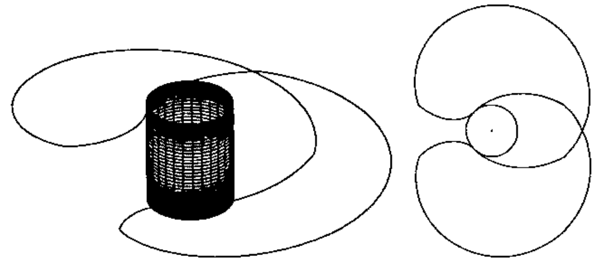

Figure 5: Left: Trajectory in the relative model; Right: Top view of the left figure

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