Optimal Sojourn Time Control within an Interval[1]

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Abstract

In this paper we study the problem of find optimal feedback control that can keep an agent within an interval for at least a certain amount of expected time with the least energy. The dynamics of the agent are subject to the perturbation of random noises, and thus are given by a stochastic differential equation. By formulating the problem as an optimal control problem, we propose a solution by using the Maximum Principle and finding a first integral to the resulting differential equations. Some numerical simulations are presented to illustrate the results.

1 Introduction

In many practical applications where safety is the principal concern, for example, in traffic systems such as air traffic management system [4] and automated highway system [5], the evolution of the state of the agents to be controlled can often be modeled as a proper dynamical system subject to the perturbation of random noises, or more precisely, by a stochastic differential equation, with the control to the system of the form of state feedback control. The system is defined to be safe as long as its state evolves inside a certain subset of the state space called the safe region, and whenever this is violated, some emergency procedures have to be evoked in order to reset the state within the safe region. Due to the usually much higher cost of the emergency procedures, it is preferable to design a feedback control law of a reasonable cost that can keep the state of the system within the safe region for as long as possible. Or equivalently, one wishes to find a feedback control with the least cost that can keep the system safe long enough. In this paper we study a simple instance of such a problem.

As a concrete example, consider the following scenario. Suppose that there are three consecutive cars driving in the same direction on a road, numbered 1, 2, and 3 from front to end. The body length of each car is d. Denote by D the distance between car 1 and car 3 and by x the distance between car 2 and car 3, both excluding body length. Hence the distance between car 1 and car 2 is $D - d - x$ (see Fig. 1). Suppose that in a period of time, D remains constant, i.e., car 1 and car 3 have identical velocities, while x is given by the equation:

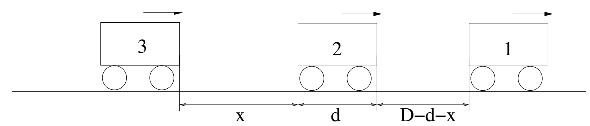

Figure 1: Three cars on a highway.

$$\frac{dx(t)}{dt} = f[x(t)] + w(t).$$

Here f is the feedback control law for car 2 that depends on the state x, and $w(t)$ is the white noise modeling random perturbations such as air resistance and road frictions. Car 2 is safe if its distance from either car 1 or car 3 is at least $\epsilon > 0$, i.e., if $x \in [\epsilon, D - d - \epsilon]$. Suppose that at time $t = 0$ car 2 is at its ideal position $x(0) = (D - d)/2$. Due to the presence of noises, $x(t)$ will eventually venture outside of the safe region. Denote by T the first time $x(t)$ escapes from the safe region, which is a positive random variable. A natural question arises: among all the feedback control laws f satisfying the constraint $E[T] > t_0$ for some threshold t_0, find the one with the minimal cost

$$\int_{\epsilon}^{D-d-\epsilon} f^2(x)\, dx.$$

The solution to a version of this problem will be discussed in this paper.

This paper is organized as following. First of all, the problem under study is defined in its most general form in Section 2. In Section 3, some properties of its optimal solutions are established, which inspire us to propose a simplified version of the problem. By formulating the simplified problem as an optimal control problem in Section 4, we use the Maximal Principle to find the equations for the optimal solutions. Due to presence of the two-point boundary conditions, these equations are hard to solve directly. However, by finding

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a first integral, we can significantly simplify their solution. These results are illustrated through numerical simulations. Finally, conclusions and possible extensions are discussed in Section 5.

2 Problem Formulation

Given an interval $[-a, a]$ on the real line, consider the solution X_t to the following stochastic differential equation (SDE):

$$dX_t = f(X_t)dt + dB_t,$$

where B_t is the standard Brownian motion, and f is a piecewise Lipschitz continuous function. Denote by $T_a = \inf\{t \geq 0 : X_t = a\}$ (respectively, $T_{-a} = \inf\{t \geq 0 : X_t = -a\}$) the first time X_t hits a (respectively, $-a$). Then $T_a \wedge T_{-a} \triangleq \min\{T_a, T_{-a}\}$ is the first time X_t escapes from the interval $[-a, a]$. For each $x \in [-a, a]$, define

$$g(x) = E_x[T_a \wedge T_{-a}],$$

where E_x means that the expectation is taken under the initial condition that $X_0 = x$. Thus $g(x)$ is the expected escaping time from $[-a, a]$ given that X_t starts from x.

It is a standard result in stochastic calculus (see, e.g., [2]) that g satisfies the following ordinary differential equation (ODE):

$$g''(x) + 2f(x)g'(x) + 2 = 0, \quad \forall x \in [-a, a], \qquad (1)$$

with the boundary condition $g(-a) = g(a) = 0$. Note that g belongs to $\mathcal{C}$, the family of functions on $[-a, a]$ with piecewise Lipschitz continuous second order derivative.

Remark 1 *For certain special cases, equation (1) can be solved explicitly [3]. For example, if $f \equiv u$ on $[-a, a]$, then $X_t = B_t + ut$ is the BM with drift u, and*

$$g(x) = \frac{(a-x)e^{2ua} + (x+a)e^{-2ua} - 2ae^{-2ux}}{u(e^{2ua} - e^{-2ua})}.$$

In particular, if $f \equiv 0$ on $[-a, a]$, then $X_t = B_t$ is the standard BM. Let $u \to 0$ in the above equation, we have $g(x) = a^2 - x^2$. Figure 2 plots $g(x)$ for different values of u.

We now formulate our problem.

Problem 1 *Among all the control functions f for which the corresponding g satisfies*

$$g_{\max} \triangleq \max\{g(x) : x \in [-a, a]\} \geq t_0$$

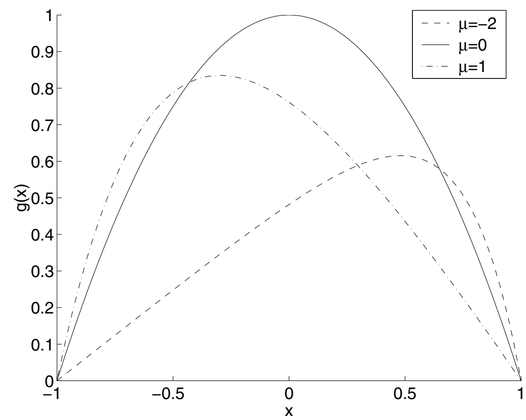

Figure 2: Plots of $g(x)$ for $\mu = -2, 0, 1$.

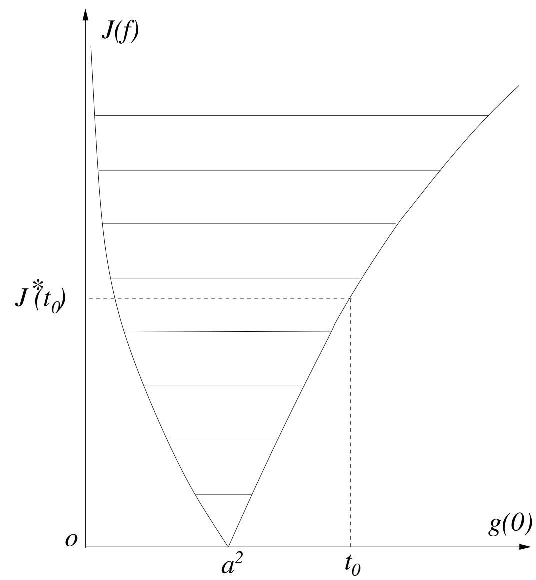

Figure 3: Feasible region of control function f in $(g(0), J(f))$ coordinates.

for some fixed threshold $t_0 > 0$, find the one that minimizes the energy

$$J(f) = \int_{-a}^{a} f^2(x)\, dx.$$

In other words, we try to find the minimal energy control f subject to the constraint that the expected escaping time from at least one point in the safe region $[-a, a]$ is no less than t_0.

By Remark 1, if f has the minimum energy $J(f) = 0$, i.e., if $f \equiv 0$, then $g(x) = a^2 - x^2$, hence $g_{\max} = a^2$. As a result, unless $t_0 > a^2$, the solution to Problem 1 is always $f \equiv 0$. Therefore, we shall in the following assume that $t_0 \geq a^2$.

For each threshold t_0, denote by $J^*(t_0)$ the energy of the solution to Problem 1, i.e.,

$$J^*(t_0) = \inf\{J(f) : f \text{ such that } g_{\max} \geq t_0\}.$$

Then it is clear that $J^*(t_0)$ is an increasing function of t_0. $J^*(t_0)$ has the following interpretation. Represent each control function f as a point on a plane such that its x coordinate is given by $g_{\max}$ and its y coordinate is given by $J(f)$. In this coordinate system, the feasible region of all possible f is plotted as the shaded region in Fig. 3. The region is bounded from the right by the graph of the function $J^*(t_0)$ for $t_0 \geq a^2$, and from the left by another curve of no practical interest. The vertex of the region corresponds to the control function $f \equiv 0$.

As a by-product of the above analysis, we have an equivalent (dual) formulation of the problem as follows.

Problem 2 *Among all the control functions f whose energy is at most $J_0 > 0$, find the one for which the corresponding g_{max} is maximized.*

The value of $g_{\max}$ for the solution to Problem 2 as a function of the threshold J_0 is exactly the inverse of the function J^*. We shall focus on Problem 1 only in the rest of the paper.

3 Some Properties of Optimal Solutions

We first show some useful properties of the optimal solutions to Problem 1.

Proposition 1 *For any control function f, the solution g to equation* (1) *is strictly concave. Therefore, $g'' \leq 0$, and there is a unique $x^* \in [-a, a]$ such that $g(x^*) = g_{max}$.*

Proof: We adopt a probabilistic proof using the definition of g. For any $-a \leq y < x < z \leq a$,

$$g(x) = E_x[T_{-a} \wedge T_a] = P_x(T_y < T_z)g(y) + P_x(T_z < T_y)g(z) + E_x[T_y \wedge T_z],$$

where T_y and T_z denote the first time X_t hits y and z respectively. Note that $P_x(T_y < T_z) + P_x(T_z < T_y) = 1$ and $E_x[T_y \wedge T_z] > 0$. Therefore, g is strictly concave. ∎

Proposition 2 *Suppose that f is an optimal solution to Problem 1, and the corresponding g assumes its maximum at x^*. Then $f \geq 0$ on $[-a, x^*]$ and $f \leq 0$ on $[x^*, a]$.*

In other words, the optimal control function f will direct toward x^* from both side.

To prove this proposition, we need two technical lemmas on the monotonicity of solutions to the ODE (1).

Lemma 1 *For a fixed control function f and $x_0 \in \mathbb{R}$, denote by $g(x; g_0, u_0)$ the solution to the ODE* (1) *with initial condition $g(x_0) = g_0$ and $g'(x_0) = u_0$. Then for any fixed x and g_0, $g(x; g_0, u_0)$ is an increasing function of u_0 if $x > x_0$, and a decreasing function of u_0 if $x < x_0$.*

Proof: Suppose that for some $u_0 < v_0$, $x > x_0$, we have $g_1(x) \triangleq g(x; g_0, u_0) > g_2(x) \triangleq g(x; g_0, v_0)$. Then since $g_1(x) < g_2(x)$ for $x > x_0$ in a neighborhood of x_0, there exists an $y \in (x_0, x]$ where $g_2 - g_1$ achieve its maximum over $[x_0, x]$. At y we have $g_1(y) < g_2(y)$ and $g_1'(y) = g_2'(y)$. Now $g_1(x)$ and $g_2(x) - g_2(y) + g_1(y)$ are two solutions to the ODE (1) with identical values of $g(y)$ and $g'(y)$, but nonetheless differ at x_0 and x, a contradiction to the the uniqueness of solutions to the ODE (1). ∎

Lemma 2 *Suppose that g_1 and g_2 are the solutions to the ODE* (1) *with initial condition $g(x_0) = g_0$, $g'(x_0) = 0$, that correspond to the control functions f_1 and f_2, respectively, and that $f_1 \geq f_2$ on $[x_0, \infty)$. Then $g_1 \geq g_2$ on $[x_0, \infty)$.*

The proof of Lemma 2 follows the same line as that of Lemma 1, hence is omitted.

We now prove Proposition 2. Suppose otherwise that the optimal solution f is positive on some nontrivial subinterval $I = [x_1, x_2]$ of $[x^*, a]$. Define a new control function $\hat{f}$ such that $\hat{f} = -f$ on I and $\hat{f} = f$ at everywhere else. Note that $\hat{f}$ and f have the same energy. We claim that by adopting $\hat{f}$ as the control function, the expected escaping time, $\hat{g}$, is higher at x^*, a contradiction to our assumption that f is optimal. To prove this claim, choose $x_0 = x^*$ and $g_0 = g(x^*)$ in Lemma 2 to conclude that the solution to equation (1) with initial condition $g(x^*)$ and $g'(x^*) = 0$ corresponding to the new control function $\hat{f}$ is no larger than g everywhere on $[x^*, a]$. Piece such a solution on $[x^*, a]$ with g on $[-a, x^*]$ together to obtain a solution to (1) on $[-a, a]$ with initial condition $g(-a) = 0$ and $g(a) < 0$. Now Lemma 1 shows that a solution to equation (1) with initial condition $g(-a) = 0$ and $g(a) = 0$ is smaller than the pieced together solution at any $x \in (-a, a]$. In other words, the expected escaping time corresponding to the control function $\hat{f}$ starting from x^* is larger than $g(x^*)$, a contradiction. The other half of the proposition can be proved similarly.

We conjecture that for optimal solution f to Problem 1, the corresponding g assumes its maximum at $x^* = 0$. However, a rigorous proof is yet to be found. If this is indeed the case, then it is clear that there is a version of f that is odd on $[-a, a]$ which generates an escaping time of the same value at 0. In fact, starting any optimal solution f, if $\int_{-a}^{0} f^2(x)\,dx \le \int_0^a f^2(x)\,dx$, then by choosing $\hat{f}(x) = f(x)$ on $[-a, -0]$ and $\hat{f}(x) = -f(-x)$ on $[0, a]$, we have a control function $\hat{f}$ for which the escaping time $\hat{g}$ satisfies $\hat{g}(x) = g(x)$ on $[-a, 0]$ and $\hat{g}(x) = g(-x)$ on $[0, a]$.

Inspired by the above discussions, we propose a restricted version of Problem 1 as follows.

Problem 3 *Among all the control functions f that satisfy $g_{max} \ge t_0$ and that are odd on $[-a, a]$, find the one with minimal energy.*

Because of the restriction, f is odd and g is even on $[-a, a]$. Hence we need only to design the functions f and g on $[-a, 0]$, and then extend them to the whole interval $[-a, a]$ by their respective symmetries. The equation that g satisfies is now given by

$$g''(x) + 2f(x)g'(x) + 2 = 0, \quad x \in [-a, 0], \tag{2}$$

with initial condition $g(-a) = 0$, $g(0) = t_0$, $g'(0) = 0$. Here we use the obvious fact that the optimal solution f satisfies $g(0) = t_0$.

4 Solution Using Optimal Control Theory

Define $y_1(x) = g(x)$, $y_2(x) = g'(x)$. Then equation (2) implies

$$y_1' = y_2, \tag{3}$$
$$y_2' = -2fy_2 - 2, \tag{4}$$

for $x \in [-a, 0]$. The boundary conditions are $y_1(-a) = 0$, $y_1(0) = t_0$, and $y_2(0) = 0$. The above equations specify the dynamics of a control system whose states are (y_1, y_2) and whose input is f. The cost of the system is $\int_{-a}^{0} f^2(x)\,dx = \frac{1}{2}J(f)$. So Problem 1 is equivalent to

Problem 4 *Solve the following optimal control problem:*

$$\begin{aligned} &\textit{Minimize} \quad \int_{-a}^{0} f^2(x)\,dx \\ &\textit{subject to} \quad \begin{cases} y_1' = y_2, \\ y_2' = -2fy_2 - 2, \end{cases} \quad \forall x \in [-a, 0], \\ &\qquad y_1(-a) = 0,\ y_1(0) = t_0,\ y_2(0) = 0. \end{aligned}$$

Define the Hamiltonian

$$\begin{aligned} H(y_1, y_2, \lambda_1, \lambda_2, u) &= \lambda_1 y_2 + \lambda_2(-2fy_2 - 2) + f^2 \\ &= \lambda_1 y_2 - 2\lambda_2(fy_2 + 1) + f^2. \end{aligned}$$

Then the optimal control f satisfies [1]:

$$\lambda_1' = -\frac{\partial H}{\partial y_1} = 0, \tag{5}$$
$$\lambda_2' = -\frac{\partial H}{\partial y_2} = 2f\lambda_2 - \lambda_1, \tag{6}$$

and

$$\frac{\partial H}{\partial f} = 2f - 2\lambda_2 y_2 = 0, \tag{7}$$

with the boundary conditions

$$y_1(-a) = 0,\ y_1(0) = t_0,\ y_2(0) = 0,\ \lambda_2(-a) = 0.$$

By (5), λ_1 is constant on $[-a, 0]$. From (7) we have

$$f = \lambda_2 y_2. \tag{8}$$

Substituting this into (4) and (6), we obtain

$$y_2' = -2\lambda_2 y_2^2 - 2, \tag{9}$$
$$\lambda_2' = 2\lambda_2^2 y_2 - \lambda_1, \tag{10}$$

whose boundary conditions are $y_2(0) = 0$ and $\lambda_2(-a) = 0$.

We now try to solve the above dynamic equations. Using equations (9) and (10), we have

$$\begin{aligned} (\lambda_2 y_2)' &= \lambda_2' y_2 + \lambda_2 y_2' \\ &= (2\lambda_2^2 y_2 - \lambda_1) y_2 - \lambda_2(2\lambda_2 y_2^2 + 2) \\ &= -(\lambda_1 y_2 + 2\lambda_2), \end{aligned}$$

and

$$\begin{aligned} (2\lambda_2 - \lambda_1 y_2)' &= 2\lambda_2' - \lambda_1 y_2' \\ &= 2(2\lambda_2^2 y_2 - \lambda_1) + \lambda_1(2\lambda_2 y_2^2 + 2) \\ &= 2\lambda_2 y_2(\lambda_1 y_2 + 2\lambda_2). \end{aligned}$$

Therefore,

$$(2\lambda_2 - \lambda_1 y_2)' = -2(\lambda_2 y_2)(\lambda_2 y_2)' = -(\lambda_2^2 y_2^2)'.$$

Integrating, we have a first integral

$$2\lambda_2 - \lambda_1 y_2 = -\lambda_2^2 y_2^2 + C, \quad \forall x \in [-a, 0], \tag{11}$$

for some constant C. The initial conditions $y_2(0) = 0$ and $\lambda_2(-a) = 0$ imply that

$$C = 2\lambda_2(0) = -\lambda_1 y_2(-a).$$

Now λ_2 and y_2 can be solved from (11) as

$$\lambda_2 = \frac{-2 \pm 2\sqrt{1 + y_2^2(\lambda_1 y_2 + C)}}{2y_2^2},$$
$$y_2 = \frac{\lambda_1 \pm \sqrt{\lambda_1^2 - 4\lambda_2^2(2\lambda_2 - C)}}{2\lambda_2^2},$$

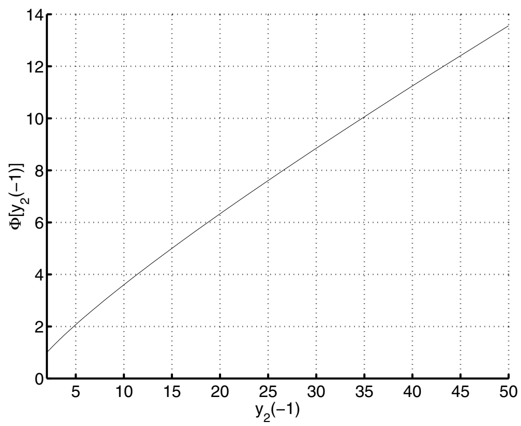

Figure 4: Plot of $\Phi[y_2(-a)]$ for $a = 1$.

which, after substitution into (9) and (10), yield

$$y_2' = \pm 2\sqrt{1 + y_2^2(\lambda_1 y_2 + C)},$$
$$\lambda_2' = \pm\sqrt{\lambda_1^2 - 4\lambda_2^2(2\lambda_2 - C)},$$

By Proposition 1, the sign in the first equation can be easily determined as:

$$\begin{aligned} y_2' &= -2\sqrt{1 + y_2^2(\lambda_1 y_2 + C)} \\ &= -2\sqrt{1 + \lambda_1 y_2^2[y_2 - y_2(-a)]}, \end{aligned} \quad (12)$$

with boundary conditions $y_2(0) = 0$. Write equation (12) as

$$dy_2/\sqrt{1 + \lambda_1 y_2^2[y_2 - y_2(-a)]} = -2dx,$$

and integrating from $x = -a$ to $x = 0$, we have

$$\begin{aligned} & \Psi(\lambda_1, y_2(-a)) \\ \triangleq \quad & \int_0^{y_2(-a)} \frac{dy_2}{\sqrt{1 + \lambda_1 y_2^2[y_2 - y_2(-a)]}} \\ = \quad & 2a, \end{aligned} \quad (13)$$

which defines λ_1 as an implicit function of $y_2(-a) \in [2a, \infty)$. In fact, for each fixed $y_2(-a) \geq 2a$, since $y_2^2[y_2 - y_2(-a)] < 0$ for $0 < y_2 < y_2(-a)$, $\Psi(\lambda_1, y_2(-a))$ is strictly increasing with λ_1. Note that $\Psi(0, y_2(-a)) = y_2(-a) \geq 2a$, and $\lim_{\lambda_1 \to -\infty} \Psi(\lambda_1, y_2(-a)) = 0 \leq 2a$. So there exists a unique $\lambda_1 \leq 0$, denoted as $\varphi[y_2(-a)]$, such that

$$\Psi(\varphi[y_2(-a)], y_2(-a)) = 2a.$$

From the above discussion, if $y_2(-a) \geq 2a$, then the solution y_2 to equation (12) satisfies the boundary condition $y_2(0) = 0$ if and only if we choose $\lambda_1 = \varphi[y_2(-a)]$.

To determine $y_2(0)$, note that y_1 satisfies $y_1' = y_2$, $y_1(-a) = 0$, $y_1(0) = t_0$. Therefore, if we define

$$\Phi[y_2(-a)] = \int_{-a}^{0} y_2(x)\, dx,$$

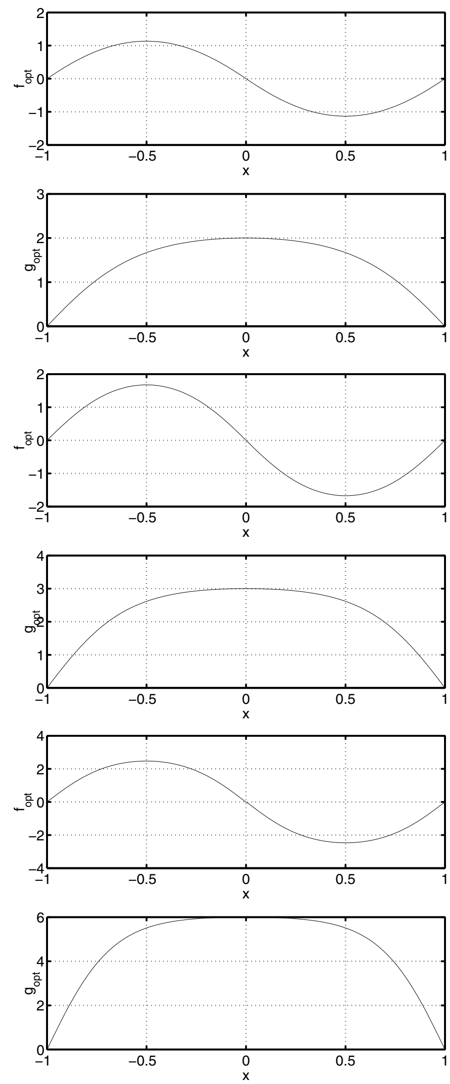

Figure 5: Plots of optimal f and g when $a = 1$, with $t_0 = 2$ (first two rows), $t_0 = 3$ (middle two rows), and $t_0 = 6$ (last two rows).

where y_2 is the solution to equation (12) with initial condition $y_2(-a) \geq 2a$ and with $\lambda_1 = \varphi[y_2(-a)]$, then

$$\Phi[y_2(-a)] = y_1(0) - y_1(-a) = t_0. \quad (14)$$

From experiments, $\Phi[y_2(-a)]$ is strictly increasing for $y_2(-a) \geq 2a$. See Figure 4 for a plot of $\Phi[y_2(-a)]$ when $a = 1$. Therefore we can find a $y_2(-a)$ such that $\Phi[y_2(-a)] = t_0$ if $t_0 \geq a^2$. Using this $y_2(-a)$ and $\lambda_1 = \varphi[y_2(-a)]$, the solution to Problem 4 can be found by integrating the following differential equations:

$$\begin{aligned} y_1' &= y_2, \qquad y_1(-a) = 0, \\ y_2' &= -2\sqrt{1 + \lambda_1 y_2^2[y_2 - y_2(-a)]}. \end{aligned}$$

Note that other boundary conditions $y_1(0) = t_0$ and $y_2(0) = 0$ are automatically satisfied by our choice of $y_2(0)$ and λ_1. Hence the difficulty of solving the two-point boundary problem specified by (9) and (10) is avoided.

Figure 5 plots the optimal f and g computed as above when $a = 1$ for $t_0 = 1$, 3, 6 respectively. The first two rows are the optimal f and g for $t_0 = 1$, the middle two rows for $t_0 = 3$, and the last two rows for $t = 6$. As expected, as t_0 increases, the optimal feedback control law f becomes more aggressive. An interesting fact is that the optimal f is nearly zero around the center of the interval, while much of the effort is spent near 1/4 and 3/4 positions of the interval.

5 Conclusion

In this paper the problem of optimal sojourn time control for an agent moving in an interval is studied. Using tools from optimal control theory, and by finding a first integral to the resulting equations, we propose a solution to the problem under the symmetric assumption.

As future directions, we are currently trying to prove the conjecture that solutions to the symmetric version of the problem are also solutions to the general problem. It is also interesting to see if our approach can be extended to the higher dimensional case, for example, when the safe region is a disk in $\mathbb{R}^n$.

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