Global Controllability of Hybrid Systems with Controlled and Autonomous Switchings*

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Abstract. In this paper we investigate the question of the global controllability posed for control hybrid systems with autounomous and controlled swithchings. The main tool for our analysis is the notion of the *controlled hybrifold*. New sufficient conditions for the global controllability are obtained in terms of the so-called *hybrid fountains*.

1 Introduction

In this paper we consider systems which have a hybrid nature, in the sense that the dynamics of the system combines continuous and discrete components. We model *control hybrid systems* as a tuple consisting of a state space, a set of admissible continuous and discrete controls, a family of controlled vector fields assigned to each discrete state, a collection of autonomous and controlled switching surfaces, and a collection of the correspondint reset maps.

The main question investigated in the paper is the controllability of control hybrid systems. This issue has been addressed in [1,5,12,13]. In particular, in [12], the notion of controllability for hybrid systems is formalized by continuity of system functions. In [1], the authors derive a necessary and sufficient algebraic condition for a certain subclass of piecewise affine hybrid systems. In [13], a sufficient condition for controllability of hybrid systems is formulated in terms of the so-called *arrival sets*.

Because of the complexity of the problem of the global controllability, its unlikely to find uniform sufficient conditions for general hybrid systems. Thus, we restrict our study to a special subclass of control hybrid systems, namely, the systems that can be represented as *hybrifolds*. The notion of the hybrifold

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was originally introduced in [14] and extended to control hybrid systems with autonomous switchings in [9] (see also [6], where the hybrifold notion is used in problems of optimal control for hybrid systems). In this paper we generalize the results formulated in [9] to systems that admit both autonomous and controlled switchings. New sufficient conditions for the global controllability are obtained in terms of the so called *hybrid fountains.* The advantage of the approach proposed in this paper is in the fact, that the fountain property can be verified at each particular state and, hence, there is no need to invoke a dynamic programming-like procedure to determine arrival sets of the system.

The paper is organized as follows. In Section 2, we formally define the class of control hybrid systems H under our consideration and specify the standard assumptions on the continuous and discrete parts of the dynamics of H. In Section 3, we generalize the notion of the *hybrifold* to control hybrid systems with controlled and autonomous switchings and define a controlled flow on the hybrifold. Section 4 relates the global controllability of H to the global controllability of the associated controlled hybrifold. In Section 5, we introduce the notion of a *hybrid fountain* and provide new sufficient conditions for the global controllability of control hybrid systems.

2 Regular Control Hybrid Systems: Standing Assumptions

We consider *control hybrid systems* which in this paper are taken to be of the following form.

Definition 1. An *n-dimensional control hybrid systems* H is a 6-tuple

$$H = \{Q, \mathcal{D}, \mathcal{S}, \mathcal{R}, \Sigma, \mathcal{F}\}, \tag{1}$$

where

$Q = \{1, \cdots, k\}$, $1 \le k < \infty$, is a set of discrete states (which are called *control locations*);
$\mathcal{D} = \{D_i;\ i \in Q, D_i \subset \mathbb{R}^n\}$ is a collection of *domains* of H;
$\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_c$ is a collection of autonomous and controlled *switching surfaces*;
$\mathcal{R} = \mathcal{R}_a \cup \mathcal{R}_c$ is a collection of autonomous and control *resets.*
$\Sigma = \Sigma_c \cup \Sigma_d$ is the set of admissible continuous and discrete *controls*;
$\mathcal{F} = \{f_i;\ i \in Q, f_i : D_i \times \mathbb{R}^{n_u} \to \mathbb{R}^n\}$ is a collection of control vector fields assigned to each location;

□

Each of these components shall be further specified in the next part of the section.

The collections of autonomous swithching surfaces (called *guards*) and autonomous resets

$$\mathcal{S}_a = \{S_a^{ij};\ \ (i,j) \in E_a\} \qquad \mathcal{R}_a = \{R_a^{ij};\ \ (i,j) \in E_a\},$$

where $E_a \subset Q \times Q$, are such that each guard S_a^{ij} is a subset of D_i and each autonomous reset R_a^{ij} is a continuous injective map acting from S_a^{ij} to D_j.

Similarly, for controlled switching surfaces and resets we have:

$$\mathcal{S}_c = \{S_c^{ij};\ \ (i,j) \,\epsilon\, E_c\} \qquad\qquad \mathcal{R}_c = \{R_c^{ij};\ \ (i,j) \,\epsilon\, E_c\},$$

where $E_c \subset Q \times Q$, each controlled switching surface S_c^{ij} is a subset of D_i, and each controlled reset R_c^{ij} is a continuous injective map acting from S_c^{ij} to D_j.

The set of discrete controls Σ_d is taken to be $\{\sigma_{ij};\ \ (i,j) \,\epsilon\, E_c\}$, where each σ_{ij} is a discrete control that can be applied at (and only at) states $x \,\epsilon\, S_c^{ij}$.

Take an arbitrary initial state (i, x_0) which does not lie on any of the switching surfaces. Then, for any control $u \,\epsilon\, \Sigma_c$, the systems evolves according to the ODE

$$\dot{x} = f_i(x,u), \qquad\qquad x(0) = x_0$$

until it hits (at some point $\bar{x}$) either (i) a guard S_a^{ij} or (ii) a controlled switching surface S_c^{ik}.

In the former case (i), the system necessarily switches to the discrete location j and the continuous component of the states resets to $R_a^{ij}(\bar{x})$. Next, the system evolves according to the dynamics f_j in the domain D_j.

In the latter case (ii), we distinguish two possibilities.

(ii.a) The discrete control σ_{ik} is applied at $\bar{x}$; then the system switches to the location k and the continuous component of the state resets to $R_c^{ik}(\bar{x})$. Next, the system evolves according to f_k in D_k.

(ii.b) The discrete control σ_{ik} is not applied; the system continues evolving according to f_i in D_i.

The following definition of a hybrid time trajectory is based on [10,11].

Definition 2 (Forward Hybrid Time Trajectory).
A *(forward) hybrid time trajectory* is a sequence of semi-closed intervals

$$\tau = \{[\tau_i, \tau_{i+1});\ 1 \le i \le N \le \infty, \tau_i < \tau_{i+1}\}.$$

We shall use the symbol $N(\tau)$ to denote the size of the time trajectory (i.e. the number of semi-intervals in the sequence τ), the symbol $\langle \tau \rangle$ to denote the set $\{1, 2, \cdots, N(\tau)\}$, and the symbol τ_∞ to denote the *execution time*, which, for a finite $N(\tau)$, is defined to be $\tau_\infty \,\underline{\Delta}\, \tau_{N(\tau)+1} - \tau_1$. □

Based on the above description of the evolution of H, for any control pair (u, σ), where u is a continuous control in Σ_c and σ is a sequence of discrete controls $\{v_1, v_2, \cdots, v_k;\ \ v_i \,\epsilon\, \Sigma_d\}$, we can define the notion of the control execution $\chi = \{\tau, q, \phi\}$ of H starting at the initial state $p \,\epsilon\, D$, where

(i) τ is a hybrid time trajectory that contains the sequence of the switching times;

(ii) $q : \langle \tau \rangle \to Q$ is a map that contains the sequence of discrete locations visited by the hybrid trajectory;

(iii) $\phi = \{\phi_j; j \in \langle\tau\rangle\}$ is the collection of continuously differentiable maps of t that satisfies the corresponding ODEs and the switching conditions as described above.

As in [14], we shall restrict ourselves to the study of hybrid systems that are subject to the following assumptions.

A1 The control hybrid system H_w is deterministic and non-blocking, for any control pair $w = (u, \sigma)$.

A2 For each $i \in Q$, D_i is assumed to be a non-empty, closed, contractible n-dimensional sub-manifold of $\mathbb{R}^n$, with a piecewise smooth boundary.

A3 For each $e \in E_a$ and $\tilde{e} \in E_c$, the guard S_a^e and the controlled switching surface $S_c^{\tilde{e}}$ are closed $(n-1)$-dimensional submanifolds with a piecewise smooth boundary. These sets have finite number of connected components.

A4 All resets maps are continuous and injective.

A5 None of the autonomous transition sets (i.e. $\{S_a^e, R_a^e(S_a^e);\ \ e \in E_a\}$; denoted $ATrans$) have intersections with the controlled transition sets (i.e. $\{S_c^{\tilde{e}}, R_c^{\tilde{e}}(S_c^{\tilde{e}});\ \ \tilde{e} \in E_c\}$; denoted $CTrans$). Further, for any two (autonomous or controlled) transition sets B_1, B_2 (denoted $Trans$), we have

$$B_1 \cap B_2 \neq \emptyset \Rightarrow B_1 = S_c^{ij_1} = B_2 = S_c^{ij_2},$$

for some $i, j_1, j_2 \in Q$.

Remark 1. We note that the restriction $S_a^e \cap S_c^{\tilde{e}} = \emptyset$ comes from the fact that H is assumed to be deterministic. The rest of the restrictions of $A5$ can be somewhat relaxed. We impose $A5$ to avoid cumbersome technical details, while illustrating the point that certain hybrid systems can be represented as manifolds (termed *hybrifolds*), and thus, results on the global controllability formulated for manifolds can be transformed to hybrid systems. □

Next we list the assumptions on the continuous part of the dynamics of H.

B1 For each $i \in Q$, $X_i \in C^r(D_i \times U; \mathbb{R}^n)$, $r \in \{1, 2, \cdots, \infty, \omega\}$, where C^ω denotes the class of analytic functions.

B2 The set of *admissible control functions*

$$\Sigma_c = \Sigma_c{}^s(\mathbb{R}; \mathbb{R}^{n_u}), s \in \{1, 2, \cdots, \infty\},$$

is the set of all $\mathbb{R}^{n_u}$-valued bounded piecewise $C^s(\mathbb{R}; \mathbb{R}^{n_u})$ functions of time with limits from the right. Hence any $u \in \Sigma_c$, defined on some $[T_1, T_2)$, $T_2 < \infty$, is C^s on $[T_1, T_2)$ with the exception of a finite number of points.

For the results formulated in this paper we shall need $r = 1$, $s = 1$.

Definition 3. A control hybrid system satisfying assumptions A1-A5 and B1-B2 is called a *regular* control hybrid system with controlled and autonomous switchings. □

Finally, it shall be assumed that the system H is *non-Zeno* in the sense that in finite time only a finite number of discrete transitions may be generated.

Lemma 1. Let H be a regular control hybrid system. For any control pair (u, σ) and any $p \in D$, there exists a unique control execution of H starting at p. $\square$

3 Controlled Hybrifold

In [14], a set M_H (called the *hybrifold*) is constructed from a hybrid system with autonomous switchings H. In this section we generalize this procedure to hybrid systems with autonomous and controlled switchings, prove that the resulting set M_H is a manifold and, finally, define the *controlled hybrid flow* on M_H.

The basic idea in the construction of the hybrifold is to *glue* together each switching surface to the image of the corresponding reset map by identifying any state $p \in S_s^e$, where $e \in E_s$, $s = a, c$, with the corresponding image $R_s^e(p)$. So an equivalence relation $\sim$ on $D \underline{\Delta} \bigcup_{i=1}^{\|Q\|} D_i$ is generated by

$$p \sim R_s^e(p),$$

for all $e \in E_s$ and $p \in S_s^e$. This relation gives rise to the quotient space

$$M_H = D/\sim,$$

where each equivalence class is collapsed to a point.

Let π be the natural projection map

$$\pi : D \to M_H$$

which assigns to each p its equivalence class. We put the quotient topology on M_H, i.e. the smallest topology in which $V \subset M_H$ is open if and only if $\pi^{-1}(V) \subset D$ is open (in the relative topology of D).

Definition 4. The set M_H with the quotient topology defined on it is called the *controlled hybrifold* associated with H. $\square$

The following result is based on [14].

Theorem 1. M_H is a topological n-manifold with boundary. $\square$

Henceforth we shall deal not with the original domains D_i but rather with the hybrifold M_H. We shall assume, without loss of generality, that M_H is embedded in $\mathbb{R}^m$, for some $n \leq m < \infty$.

Definition 5 (Hybrid Control Flow). Take an arbitrary continuous control $u \in \Sigma_c$ defined on some $[T_1, T_2)$, $T_2 < \infty$, a sequence of discrete controls σ, and a state $x \in M_H$. Let $p \in \pi^{-1}(x)$.

As follows from Lemma 1, there exists a unique control execution $\chi = \{\tau, q, \phi\}$ of H starting at p which corresponds to the control pair (u, σ).

We shall use the symbol $\Psi^H(t, x, u, \sigma)$, $t \in [T_1, T_2)$, to denote the *controlled hybrid flow* on M_H. $\Psi^H(t, x, u, \sigma)$ is defined as follows:

$$\Psi^H(t, x, u, \sigma) \,\underline{\Delta}\, \pi(\phi_i(t)), \text{ for any } i \in \langle\tau\rangle \text{ and } t \in [\tau_i, \tau_{i+1}).$$

In particular, we have $\Psi^H(\tau_1, x, u, \sigma) = \pi(\phi_1(\tau_1)) = \pi(p) = x$. □

Remark 2. We note that, as follows from the Assumption A5, the definition of the control flow on M_H does not depend on the choice of the representative p in the equivalence class x. □

Lemma 2. For any control u, the controlled hybrid flow $\Psi^H(\cdot, x, u, \sigma)$ is continuous on M_H with respect to the argument t.

Proof: This follows from the fact that all points of discontinuity of the control hybrid execution are removed by identifying them with their images under the corresponding reset maps. □

4 The Global Controllability of Hybrid Systems

Let H be an arbitrary regular control hybrid system and M_H its controlled hybrifold. In this section we relate the global controllability of the total domain D of H with the global controllability of M_H.

Definition 6 (Accessible sets of the control hybrid system H).
Let $p \in D$. We shall say that a state $p' \in D$ is *accessible from p (with respect to $V \subset D$)* if there exists a continuous control $u \in \Sigma_c$, defined on some $[T_1, T_2)$, $T_2 < \infty$, and a sequence of discrete controls $\sigma = \{v_1, \cdots, v_k\}$ such that the corresponding control execution $\chi = (\tau, q, \phi)$ of H starting at p satisfies

(i) $\phi_{N(\tau)}(T) = p'$, for some $T \in [\tau_{N(\tau)}; \tau_{N(\tau)+1})$; and
(ii for any $j \in \langle\tau\rangle$ and $t \in [\tau_j; \tau_{j+1})$, $\phi_j(t) \in V$.

The set of all states in D accessible from p (with respect to V) shall be denoted by $A_D^V(p)$. In the case $V = D$, we shall write $A_D(p)$. □

Thus we assumed that an accessible state p' can be reached from p in finite time using a finite number of switching (or *jumps*) between control locations.

Remark 3. We observe that, as follows from the definition of the control execution of H, $R_s^e(p) \in A_D(p)$, for any state $p \in S_s^e$, $e \in E_s$, $s = a, c$. □

Similarly, we can define the accessible states using the dynamics of the controlled hybrifold M_H.

Definition 7 (Accessible sets of the controlled hybrifold M_H). Let $x \epsilon M_H \subset \mathbb{R}^m$. We shall say that a state $x' \epsilon M_H$ is *accessible from* x *(with respect to* $V \subset M_H$*)* if there exists a continuous control $u \epsilon \Sigma_c$ defined on some $[T_1, T_2)$, $T_2 < \infty$, and a sequence of discrete controls $\sigma = \{v_1, \cdots, v_k\}$ such that

(i) $x' = \Psi^H(T, x, u, \sigma)$, for some $T \epsilon [T_1, T_2)$; and
(ii) for any $T_1 \leq t \leq T$, $\Psi^H(t, x, u, \sigma) \epsilon V$.

The set of all states in M_H accessible from x (with respect to V) shall be denoted by $A^V(x)$. In the case $V = M_H$, we shall write $A(x)$. □

The set of all states *co-accessible* to p (to x), with respect to $V \subset D$ (with respect to $V \subset M_H$), in H (in M_H) is defined dually and shall be denoted as $CA^V_D(p)$ (as $CA^W(x)$).

Remark 4. We observe that for any $p \epsilon D$ and any neighborhood V of p in D, we have

$$\pi(A^V_D(p)) \subset A^{\pi(V)}(\pi(p)), \tag{2}$$

where $\pi : D \longrightarrow M_H$ is the natural projection map. This is because any orbit in D is projected by π onto an orbit in M_H.

On the other hand, let $p, p' \epsilon D$ and let $\pi(p') \epsilon A^V(\pi(p))$. Then there exist some $y, y' \epsilon D$ such that (i) $p \sim y$, $p' \sim y'$ and (ii) $y' \epsilon A_D^{\pi^{-1}(V)}(y)$. In other words, the existence of a trajectory from $\pi(p)$ to $\pi(p')$ in M_H does not necessarily imply the existence of a control execution connecting p to p'; it only implies the existence of a control execution from some $y \epsilon D$ to some $y' \epsilon D$, where $y \sim p$ and $y' \sim p'$.

This is particularly easy to see in the situation, where at some controlled switching surface S^{ij}_c at least two discrete controlled σ_{ij_1}, σ_{ij_2} can be applied. Take $x \epsilon S^{ij}_c$ and consider $y_1 = R^{ij_1}_c(x)$ and $y_2 = R^{ij_2}_c(x)$. Then x, y_1, y_2 lie in the same equivalence class (they are glued together in M_H) and, hence, $\pi(y_1)$ and $\pi(y_2)$ are mutually accessible in M_H. At the same time y_1 and y_2 are not necessarily mutually accessible in D.

Hence in general, we do not have the reverse to (2) inclusion and we can only guarantee that for any $x \epsilon M_H$ and $V \subset M_H$,

$$A^V(x) \subset \pi \left\{ \bigcup_{p \epsilon \pi^{-1}(x)} A_D^{\pi^{-1}(V)}(p) \right\}. \tag{3}$$

□

Definition 8. We say that a set $D_1 \subset D$ is *controllable with respect to* $D_2 \subset D$ for the control hybrid system H if $A^{D_2}_D(p) = D_1$, for all $p \epsilon D_1$.

In the particular case when $D_1 = D$, $D_2 = D$, and $A_D(p) = D$, for all $p \epsilon D$, we shall say that the total domain D is *globally controllable* for H.

Similarly, we shall say that a set $C_1 \subset M_H$ is *controllable with respect to* $C_2 \subset M_H$ if $A^{C_2}(x) = C_1$, for all $x \epsilon C_1$. M_H is *globally controllable* if $A(x) = M_H$, for all $x \epsilon M_H$. □

Theorem 2. Let H be a regular control hybrid system. Then the total domain D is globally controllable if and only if the associated hybrifold M_H is globally controllable.

Proof:
$\Longrightarrow$ Let D be globally controllable. Then, using Remark 4 (2), we obtain for any $x \epsilon M_H$,

$$M_H = \pi(D) = \pi(A_D(p)) \subset A^{\pi(D)}(\pi(p)) = A(x) \subset M_H,$$

where p is an arbitrary point in the set $\pi^{-1}(x) \subset D$. Hence $A(x) = M_H$, for any $x \epsilon M_H$, and M_H is globally controllable.

$\Longleftarrow$ Conversely, let M_H be globally controllable. Take any $p, p' \epsilon D$. Each of them could lie in any of the sets

$$CTrans, ATrans, \tilde{D} \,\underline{\Delta}\, D - Trans,$$

i.e. there are 9 possible cases.

Consider, for instance, the case when $p \epsilon R_c^e(S_c^e)$ and $p' \epsilon R_c^{e'}(S_c^{e'})$, for some $e = (i,j), e' = (i,j') \epsilon E_c$. Take the inverse image $y' = \{R_c^{e'}\}^{-1}(p')$. As follows from the description of the hybrid executions given in Section 2, there exist states $z \epsilon D_j \cap \tilde{D}$ and $z' \epsilon D_i \cap \tilde{D}$ such that z is accessible from y and z' is co-accessible to y'. Next note, that since $z, z' \epsilon D - Trans$ and π is 1 to 1 on $\tilde{D}$, from the existence of an orbit connecting $\pi(z)$ to $\pi(z')$ in M_H follows the existence of a control execution that drives z to z'. Finally, combining all the accessibility relations for p, z, z', y', p' we conclude that $p' \epsilon A_D(p)$.

The rest of the cases can be considered in an analogous manner. Thus $A_D(p) = D$, for any $p \epsilon D$, and D is globally controllable. □

The above result allows us to use the hybrifold and the continuous controlled hybrid flow defined on it in order to study the global controllability of the original control hybrid system. The advantage of this approach is in the fact that the controllability results formulated for differential control systems acting on subsets or sub-manifolds of $\mathbb{R}^n$ can be transformed to control hybrid systems. This shall be demonstrated in the next section.

5 Hybrid Fountains

In this section we introduce the notion of a *hybrid fountain* which we shall use as the main hypothesis in our controllability result. Henceforth the symbol $B_\delta(x)$, where $x \epsilon M_H$, $0 < \delta \epsilon \mathbb{R}^1$, shall denote the m-dimensional ball with the center x and the radius δ. The sets $A^{B_\delta(p)}(p)$ and $CA^{B_\delta(p)}(p)$ shall be denoted as $A^\delta(p)$ and $CA^\delta(p)$, respectively.

Definition 9. A state $x \epsilon M_H$ is called a *hybrid fountain* if

$$\exists\, \mu > 0\ \forall\, \delta,\ 0 < \delta < \mu,\ A^\delta(x) - \{x\} \text{ and } CA^\delta(x) - \{x\} \text{ are non-empty, open sets.} \tag{4}$$

If the function $\rho \underline{\Delta} \sup\{\mu;$ such that the condition (4) holds$\}$ is continuous at x, we shall say that x is a *continuous* hybrid fountain. If ρ is unbounded at x we consider it to be continuous at x. □

The reader is referred to [2,3,7] for applications of the fountain condition to the study of ordinary differential systems acting on subsets of $\mathbb{R}^n$. See also [8], where a set of algebraic conditions for verification of the fountain property is presented, and [4] where applications to *hierarchical hybrid control theory* are outlined.

Henceforth we shall use the term *controlled closed orbit* in the sense of *controlled loop.*

Theorem 3. Let each $x \in M_H$ be a continuous hybrid fountain and let for each $x \in M_H$ there exist a control $u \in \Sigma_c$ such that x lies on a nontrivial (controlled under u) closed orbit in M_H. Then each connected component of $[M_H]^\circ$ is controllable with respect to M_H.

Proof: Let C denote one of (the finite number of) the connected components of $[M_H]^\circ$. For any two states x, x' in C we define a relation $\sim_o$ in such a way that $x \sim_o x'$ if and only if there exists a (controlled) nontrivial closed orbit in M_H passing through both x and x', i.e. there exists a control pair u, σ defined on some $[T_1, T_2)$, $T_2 < \infty$, such that

(i) $\exists\, T,\ T_1 < T < T_2, \quad \Psi(T_1, x, u, \sigma) = \Psi(T, x, u, \sigma)$; and
(ii) $\exists\, \bar{t},\ T_1 < \bar{t} \leq T, \quad \Psi(\bar{t}, p, u, \sigma) = p'$.

Clearly, the relation $\sim_o$ is reflexive (since each state in M_H lies on a nontrivial orbit), symmetric and transitive. Hence there exists a partition of C on the equivalence classes of $\sim_o$. Let $[x]$, for an arbitrary $x \in C$, denote the equivalence class containing x. We **claim** that $[x]$ is an open subset in C.

Indeed, take any $z \in [x]$. Let u and $0 \leq t < \infty$ be such that $z = \Psi(t, x, u, \sigma)$. Define $a = \Psi(t - \Delta, x, u, \sigma)$ and $b = \Psi(t + \Delta, x, u, \sigma)$, $\Delta > 0$. Then, since a and b are hybrid fountains, the sets $A^\delta(a) - \{a\}$ and $CA^\delta(b) - \{b\}$ are open, for sufficiently small $\delta > 0$. Choose Δ so small that $z \in A^\delta(a)$ and $z \in CA^\delta(b)$ (this is possible since a, b are *continuous* hybrid fountains). Then there exists an open neighborhood $N(z)$ of z which lie in the intersection $(A^\delta(a) - \{a\}) \cap (CA^\delta(b) - \{b\})$. Each state $z' \in N(z)$ is accessible from a and co-accessible to b. Moreover, since $a, b \in [x]$, we conclude that z' lies on a non-trivial orbit passing through x. This is true for all $z' \in N(z)$, hence $N(z) \subset [x]$ and $[x]$ is open, as **claimed**.

For any $x, x' \in C$ we have $[x] \cap [x'] \neq \emptyset \Longrightarrow [x] = [x']$, so any two equivalence classes are either disjoint or coincide. Thus the set C can be represented as the disjoint union $C = A \cup B$, where $A \underline{\Delta} [x]$, for some $x \in C$, and $B \underline{\Delta} \bigcup_{\substack{x' \in C \\ x' \notin [x]}} [x']$.

A and B are open and disjoint. Since C is connected, we conclude that B is empty, i.e. any $x' \in C$ is such that $x \sim_o x'$. In other words, any two states in C lie on a nontrivial controlled orbit in M_H and hence, C is controllable with respect to M_H. □

Remark 5. We note at this point that weaker recurrence conditions can be used instead of the existence of closed orbits. Also, for the proof of the above result, the continuous hybrid fountain condition (4) can be relaxed to

$\rho(x) \mathrel{\underline{\Delta}} \sup\{\mu > 0; A^{\mu}(x) - \{x\}, CA^{\mu}(x) - \{x\}$ are non-empty, open sets$\}$
is continuous, for all $x \in M_H$. □

Theorem 4. Assume that the hybrifold M_H is connected and the conditions of Theorem 3 are satisfied. Then M_H is globally controllable.

Proof: As has been shown in [14], M_H is n-dimensional manifold (possibly with boundary). This implies, by definition, that for any boundary state in ∂M_H there exists a neighborhood which is homeomorphic to $\mathbb{R}^n_+$. Hence $[M_H]^\circ$ and M_H have the same number of connected components; in particular, $[M_H]^\circ$ is connected if and only if M_H is connected.

Take any boundary state $x \in \partial M_H$. Then, since x is a hybrid fountain, the sets $A^\delta(x)-\{x\}$ and $CA^\delta(x)-\{x\}$ are non-empty and open, for sufficiently small $\delta > 0$. Hence there exist $a \in (A^\delta(x)-\{x\}) \cap [M_H]^\circ$ and $b \in (CA^\delta(x)-\{x\}) \cap [M_H]^\circ$.

For any state $p' \in [M_H]^\circ$ we can find a control $u \in \Sigma_c$ which would drive a p' and a control $u' \in \Sigma_c$ which would drive p' to b. This is because a, b, p' lie in $[M_H]^\circ$ and, as follows from Theorem 3, $[M_H]^\circ$ is controllable. We conclude that arbitrary $p \in \partial M_H$ and $p' \in [M_H]^\circ$, and thus arbitrary $p, p' \in M_H$, are mutually accessible. Hence M_H is globally controllable. □

Consider the *directed graph* Γ of H which has vertices Q and edges E. We can treat it as a finite state machine, by defining the transition function $\Phi : Q \to Q$ in such a way that for any $i, j \in Q$, $\Phi(i) = j$ if and only if $(i, j) \in E$ or $i = j$.

Theorem 5. Assume that the conditions of Theorem 3 are satisfied. Then M_H is globally controllable if and only if the graph $\Gamma = \{Q, E\}$ is controllable as a finite state machine.

Proof:

$\Longrightarrow$ Assume that M_H is globally controllable. Then for any $i, j \in Q$, $i \neq j$, take some states $p \in D_i$ and $p' \in D_j$. There exists a trajectory ψ from p to p' in M_H. Let the sequence $i = r_1, r_2, \cdots, r_\ell = j$, $\ell > 1$, be such that ψ switches consecutively from the domain D_{r_s} to the domain $D_{r_{s+1}}$, where $s = 1, 2, \cdots, \ell-1$, using the corresponding guards and the images of the reset maps. Hence each consecutive pair (r_s, r_{s+1}) belongs to E and hence, there exists a trajectory from the state i to the state j in the graph Γ. Since this holds for an arbitrary pair $(i, j) \in Q$, we conclude that Γ is controllable as a finite state machine.

$\Longleftarrow$ Conversely, assume that Γ is controllable as a finite state machine. Then for any two states $p, p' \in D$ take i and j such that $p \in D_i$ and $p' \in D_j$. If $i \neq j$, find a trajectory $i = r_1, r_2, \cdots, r_\ell = j$, $\ell > 1$, in the graph Γ. Since each consecutive pair (r_s, r_{s+1}) belongs to E, there exists a guard $G_{(r_s, r_{s+1})}$ in the domain D_{r_s} which is identified with the image of the reset map $R_{(r_s, r_{s+1})}$ in the domain $D_{r_{s+1}}$. Hence the domains D_{r_s} and $D_{r_{s+1}}$, and thus D_i and D_j, lie in one connected component of M_H. This can be shown for all $i, j \in Q$. Hence M_H is connected and, as follows from Theorem 4, M_H is globally controllable. □

An application of the obtained results can be illustrated on a two water tank system example, which, for the lack of space, shall be described briefly. The water can be added to the system at some rate $w > 0$ (where we treat the parameter w as control) in two different modes:

1: the water is added (exclusively) via tank 1;
2: the water is added (exclusively) via tank 2.

In addition to that, the water is removed from tank i, $i = 1, 2$, at some constant rate $v_i > 0$. The two tank system can be modeled as a control hybrid system in the following way. We shall distinguish two control locations - each corresponds to one of the modes, i.e. $Q = \{1, 2\}$. The continuous dynamics at the locations are as:

$$q = 1 : \quad \begin{cases} \dot{x} = w - v_1 \\ \dot{y} = -v_2 \end{cases} \quad (x, y) \in D_1 \underline{\Delta} \{[l_1, \infty) \times [l_2, \infty)\},$$

$$q = 2 : \quad \begin{cases} \dot{x} = -v_1 \\ \dot{y} = w - v_2 \end{cases} \quad (x, y) \in D_2 \underline{\Delta} \{[l_1, \infty) \times [l_2, \infty)\},$$

where x, y denote the levels of water in the tanks 1 and 2, respectively.
The class of control functions is taken to be the set of all functions taking values in $\Re$ and satisfying B_2.
The guards are defined as

$$G_{(1,2)} = 1 \times \{(x, y) \in D_1;\ y = l_2\}, \quad G_{(2,1)} = 2 \times \{(x, y) \in D_2;\ x = l_1\}.$$

The resets are defined in such a way that when hitting a guard in one domain the system switches to the other control location, without changing the continuous part of the state, i.e.

$$R_{(1,2)}(1; x, l_2) = (2; x, l_2), \quad R_{(2,1)}(2; l_1, y) = (1; l_1, y).$$

Furthermore, assume that for some level $y = l$, $l \geq l_2$, in the first tank, a discrete switching to the second tank is allowed.
To construct the corresponding controlled hybrifold we identify (via the identity reset maps) the $x = l_1$, $y = l_2$, $y = l$ axes of D_1 with the $x = l_1$, $y = l_2$, $y = l$ axes of D_2, respectively.

Using the obtained results, it can be verified that each state of the hybrifold is a hybrid fountain lying on a closed orbit. Hence, the two water tank system can be shown to be globally controllable.

Remark 6. In conclusion we note that algebraic conditions for verification of the fountain property at each state $x \in M_H$ shall be presented in a future version of the paper.

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