Nonlinear Systems: Analysis, Stability, and Control

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This book is gratefully dedicated to

Charles Desoer, Jerrold Marsden, and Roger Brockett

Visionaries of a new and beautiful world of nonlinear science

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