EE 147/247A Prof. Pister
Fall 2020

**Homework Assignment #4**

Due on bcourses Wednesday 9/30/2020 (zero credit after 9 AM Thursday)

1. [14 total] In an SOI process with a 40 um device layer you etch a cantilevered beam that is 2 um wide and has length L. Assume that the Young's modulus is 170 GPa and the fracture strain is 1%.
   a. What is the ratio of the spring constants in the transverse (y) and out-of-plane (z) directions? In general, for a beam with rectangular cross section dimensions (a,b), what is the ratio of the spring constants in the a and b directions?
   [4 points. 1 point for any formula, 1 point for the right answer]

$$k_y = \frac{Etw^3}{4L^3} \quad k_z = \frac{Ewt^3}{4L^3}$$
$$\frac{k_y}{k_z} = \frac{w^2}{t^2} = \frac{1}{400}$$

$$k_{a\ direction} = \frac{Eab^3}{4L^3} \quad k_{b\ direction} = \frac{Eba^3}{4L^3}$$
$$\frac{k_{a\ direction}}{k_{b\ direction}} = \frac{b^2}{a^2}$$

   b. What is the minimum value for L that would allow the beam to be bent into the shape of a circle? What is the end load that would bend this beam into a circle?
   [4 points. Formula/answer for the first part; and for the second part, 1 point if you say *something* related to a pure moment, 2 points if you say a pure moment]

$$if\ the\ beam\ is\ bent\ into\ a\ circle, the\ circumfrence(L) = 2\pi r\ (where\ r\ is\ \rho)\ so,$$
$$\rho = \frac{L}{2\pi}$$
$$\epsilon = \frac{t}{2p} = \frac{t\pi}{L}$$
$$\epsilon_{max} = 0.01 = \frac{t\pi}{L} \rightarrow L = 12.5\ mm$$

$$\epsilon = \frac{t}{2\rho} = \frac{t}{2}\frac{M}{EI} = \frac{t}{2}\frac{12M}{Ewt^3} = \frac{6M}{Ewt^2} = \frac{t2\pi}{L}$$
$$M = \frac{E\pi wt^3}{3L} = \frac{170Gpa * \pi * 2um * (40um)^3}{3 * 12.5mm} = 1.8u\ Nm$$

A pure moment of 1.8u Nm would bend this beam into a circle. Why could this be useful? What basic circuit device that rhymes with "conductor" is difficult to make without being lossy and why could this structure help? Come to Office Hours to discuss.

   c. If you apply a transverse end load, and need to be able to deflect at least 10 um, what is the minimum value for L?
   [2 pts]

$$\epsilon = \frac{t}{2p} = \frac{tFL}{2EI} \text{ and } y = \frac{FL^3}{3EI}$$

$$\epsilon = \frac{3ty}{2L^2} \rightarrow L = \sqrt{\frac{3ty}{2\epsilon}} = \sqrt{\frac{3(40um)(10um)}{2(0.01)}} = 2.5 * 10^{-4}m = 250um$$

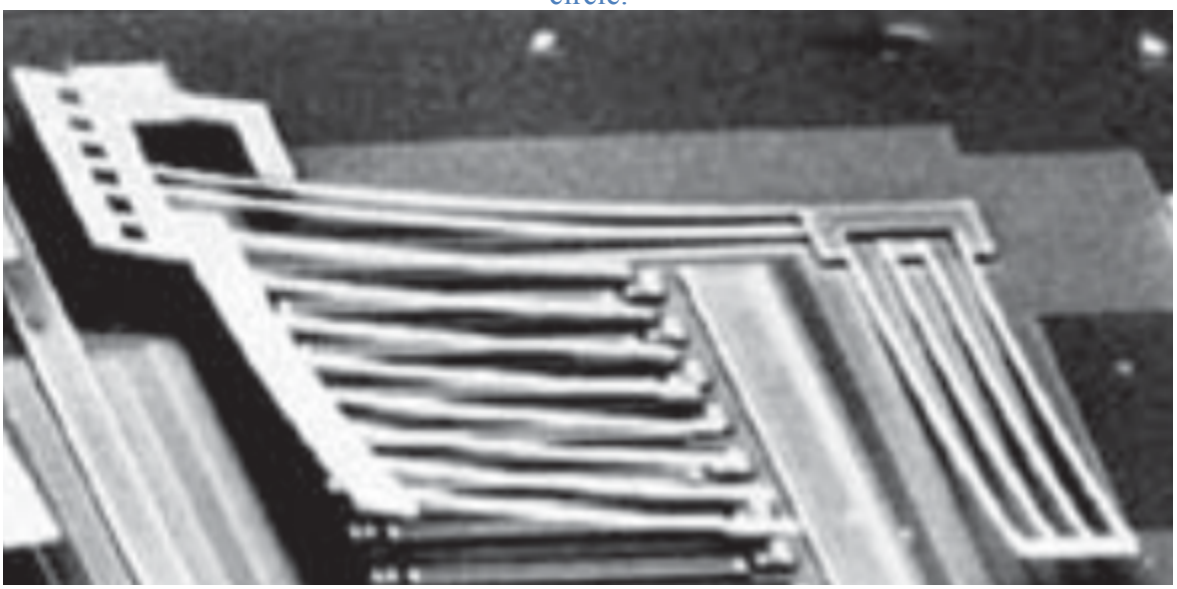
d. If you use two such beams in parallel attached to a rigid plate to make a suspension that is stiff in rotation, what is the minimum value for L if a 10 um deflection is needed? What is the spring constant of the suspension, relative to the spring constant of the single cantilever of the same length?
[4 pts]

Two parallel beams→ $F\ halved\ \rightarrow y\ halved \rightarrow L\frac{1}{\sqrt{2}} = 173um$

Beams in parallel add so stiffness is 2x

2. [3 pts total] In figure 1.6 from Kubby, "A guide to hands-on MEMS design and prototyping" (free pdf online, linked from class homepage),
   a. [1 pt] The structures on the left and right are each anchored at only one point. Do they look like they are curving up in the shape of a hemisphere?

Yes they do. Look at the top right near the anchor and the top left. Looks like the arc length of a circle.



   b. [2 pts] Assuming that the structures are in POLY1, estimate the residual stress gradient in the film.

Guess of $\rho$: 1mm Any reasonable guess is fine. 100um is way too small and 100m is way too big.

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{\sigma_1 \frac{a^2 b}{6}}{E\frac{a^3 b}{12}} = \frac{\sigma_1}{E}\frac{2}{a}$$

$$\rho = \frac{E}{\sigma_1}\frac{a}{2} = \frac{1}{\epsilon_{residual}}\frac{a}{2}$$

$$\sigma_1 = \frac{150GPa(2um)}{2(1mm)} = 150MPa$$

Beyond the scope of the problem:

Different approach using a method we will talk more about later in the class to relate residual stress to the amount of bending. The large springs at the top and bottom of the device appear to be curving up off the substrate ~25um, L~250um, t=2um. This is what I guesstimate from the scalebar at the bottom, but it is fine to have anything in the same ballpark.

$$y = \frac{\sigma_1 L^2}{Et} \text{ (a good equation to know but also be able to derive)}$$

$$\sigma_1 = \frac{Ety}{L^2} = \frac{(150GPa)(2um)(25um)}{(250um)^2} = 120MPa$$

Getting answers in the same ballpark with two different methods is a good sign and something to consider on exams.

3. [10 pts total] For the structure in Figure 2.20 in Kubby,
   a. Answer questions 2a and 2b from Kubby
      [4 pts]

$$k_{sys} = k_0$$

$$k_0 = k_y = \frac{Etw^3}{4L_{half\ beam}{}^3} = \frac{2Etw^3}{L_{full\ beam}{}^3} = \frac{2(160GPa)(2um)(2um)^3}{(150um)^3} = 1.5\frac{N}{m}$$

$$Use\ k_{cantilever}\ because\ model\ each\ beam\ as\ two\ cantilevers\ in\ series$$

$$M_p = ptA \approx 2330\frac{kg}{m^3} * 2um(2 * 170um * 10um + 2 * 85um * 15um + 55um * 20um) = 1.4*10^{-5}kg$$

$$M = ptA = 2330\frac{kg}{m^3} * 2um * 80um * 15um = 5.5 * 10^{-12}kg$$

$$f_r = \frac{1}{2\pi}\sqrt{\frac{k_{sys}}{M_p + 0.3714M}} = \frac{1}{2\pi}\sqrt{\frac{1.5\frac{N}{m}}{M_p + 0.3714M}} = 8600\frac{1}{s}$$

   b. Estimate the stiffness in the x and z directions, and in rotation about an axis perpendicular to the wafer.
      [6 pts]

For $k_z$ and $k_x$ the overall spring constant is $k_0$ but they have different equations because they're in differnt directions

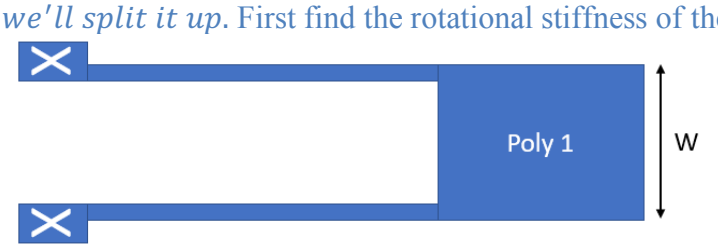
$$k_x = k_0 = \frac{EA}{L_{half\ beam}} = \frac{160GPa * (2um * 2um)}{75um} = 8500N/m$$

$$k_z = k_0 = \frac{Ewt^3}{4L_{half\ beam}{}^3} = \frac{2Ewt^3}{L_{full\ beam}{}^3} = \frac{2(160GPa)(2um)(2um)^3}{(150um)^3} = 1.5\frac{N}{m}$$

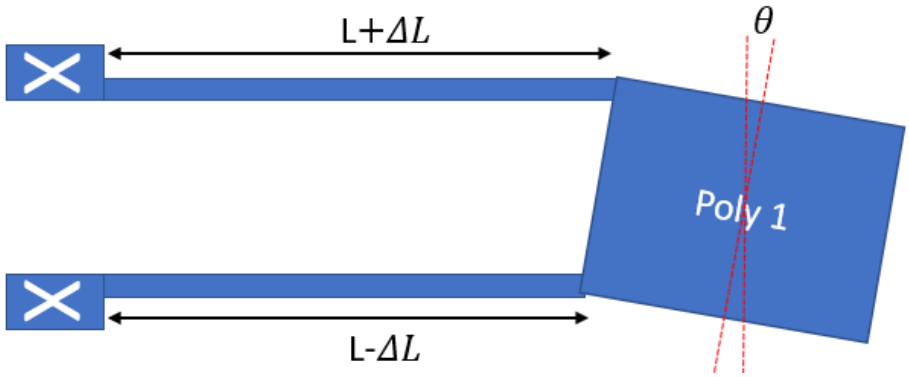
$$k_z = k_y\ because\ the\ beams\ are\ square.$$

$$Note, k_x\ is\ VERY\ stiff\ and\ many\ order\ of\ magnitude\ larger\ than\ k_z$$

$k_\theta$ is tricker so we'll split it up. First find the rotational stiffness of the structure below.

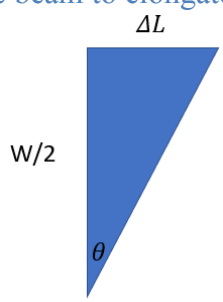



As shown in class, $\Delta L\ of\ each\ beam = \theta\frac{w}{2}$




Applying a torque $\tau$ causes a rotation

This rotation causes one beam to elongate and the other to shrink




From this triangle you can infer

$$\tan(\theta) = \frac{\Delta L}{\frac{w}{2}} \text{ and at small angles } \tan(\theta) \approx \theta \; so \; \Delta L = \theta \frac{w}{2}$$

This elongation/compression causes a force F that the beams exert on the Poly1 plate to counteract the torque applied. Each beam is $\frac{w}{2}$ from the axis of rotation so

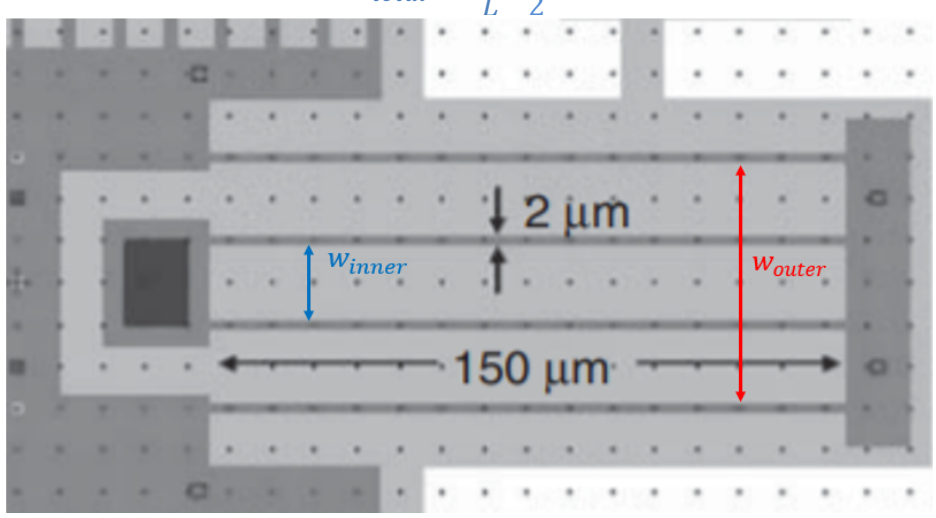
$$\tau_{one\ beam} = F\frac{w}{2} \text{ and } \tau_{total} = Fw$$

We also know from the axial stiffness of a beam F= $k_{axial}\Delta L$

$$F = \frac{EA}{L}\Delta L = \frac{EA}{L}\frac{w}{2}\theta$$

$$Putting\ the\ \tau_{total}\ and\ axial\ stiffness\ equations\ together$$

$$\tau_{total} = \frac{EA}{L}\frac{w^2}{2}\theta$$




Looking at the figure (half suspension) above we can see that this is just 2 of the structures we already solved for in series, but the lever arm is different for the inner and outer structures.

$$\frac{1}{k_{half\ suspension}} = \frac{1}{k_{inner}} + \frac{1}{k_{outer}} = \frac{2L}{EA(w_{inner})^2} + \frac{2L}{EA(w_{outer})^2} = \frac{2L}{EA}\left(\frac{1}{(w_{inner})^2} + \frac{1}{(w_{outer})^2}\right)$$

$$\frac{1}{k_{half\ suspension}} = \frac{2(150um)}{150GPa(2um)(2um)}\left(\frac{1}{(20um)^2} + \frac{1}{(60um)^2}\right) = 1.4 * 10^6$$

$$k_{half\ suspension} = 7.5 * 10^{-7}\frac{Nm}{rad}$$

*Since there are 2 of these hals suspensions in parallel,*

$$k_{theta\ total} = 1.5 * 10^{-6}\ \frac{Nm}{rad}$$

4. [9 pts] For a comb drive resonator with K=1 N/m, w=10krad/s, Q=100, and N=50 comb fingers, calculate the amplitude, phase, and frequency of all components of the deflection due to a 1.5 V 10 krad/s AC signal applied to one comb and a 15V DC signal applied to the body of the resonator.
[1 pt each for amplitude, phase, frequency of each of three frequencies, DC, omega, 2 omega]

$$F(t) = \frac{1}{2}\epsilon_0(2N_f)\frac{t+g}{g}\left[V_{DC}^2 + \frac{1}{2}V_0^2 - \frac{1}{2}V_0^2\cos 2\omega t - 2V_0V_{DC}\sin\omega t\right]$$

ff

$\omega$

$$\frac{2um + 2um}{2um}$$

At resonance → phase= -90$^o$ x(t) from Force and $X_{resonance} = QX_{DC}$

$$F_{comb} = \frac{N_{fingers}\epsilon_0 V^2(t+g)}{g} = \frac{50(8.85 * 10^{-12}\frac{F}{m})V^2(2um + 2um)}{2um} = 8.85 * 10^{-10}\frac{F}{m} * V^2$$

$$X_{DC} = \frac{F_{comb}}{k}$$

$$X_{resonance} = QX_{DC} = Q\frac{F_{comb}}{k} = \frac{100}{1N/m} * 8.85 * 10^{-10}\frac{F}{m} * V^2 = 8.85 * 10^{-8}\frac{F}{N} * V^2$$

$$x_{res}(t) = 8.85 * 10^{-8}\left(227.25 - 1.125\cos\left(20{,}000\frac{rad}{sec} * t - 90^o\right) + 45\sin\left(10{,}000\frac{rad}{sec} * t - 90^o\right)\right)m$$

What the difference between sin(x) and -cos(x-90$^o$)?

5. [7 pts total] In figure 2.21 from Kubby, assume that the torsion beams are 300um long, 25 um thick, and 2 um wide.
   a. [3 pts]Calculate the torsional spring constant of the suspension (2 beams) using the formula from class. How does your estimate differ from the more accurate formula given in Kubby Eqn. 2.26? (be careful – he defines a and b differently).

$$k_{torsion} = \frac{KG}{L}\ (I\ know\ it'sconfusing\ with\ multiple\ Ks\ but\ pay\ attention\ to\ subscripts)$$

G = shear modulus = $\frac{E}{2(1+\nu)} = \frac{150GPa}{2(1+0.22)} = 60GPa$

$K = \frac{ab^3}{3} = \frac{25um(2um)^3}{3} = \frac{2}{3}10^{-22}m^4$ where the cubed term is the shorter dimension

$$k_{torsion} = 2 * \frac{\left(\frac{2}{3}10^{-22}m^4\right)60GPa}{300um} = 2.5 * 10^{-8}\frac{N\ m}{rad}$$

*2 because there are two beams in parallel

   b. [2 pts] Estimate the torque necessary to rotate the plates 90 degrees.

$$90\ degrees = 1.57\ rad$$

$$\tau = \frac{\theta KG}{L} = \theta k_{torsion} = 40 * 10^{-9}\ N\ m$$

c. [2 pts] Estimate the rotation angle at which the beams will fracture.

$$Assume\ 1\%\ fracture\ strain$$

*2 because 2 beams in parallel

$$\theta = 2\frac{L\gamma}{w} = \frac{300um * 0.01}{2um} = 3\ radians$$

6. [5 pts total] The structure on the left below consists of a rigid body attached to the end of a beam of length L, width a, and thickness b. The goal is that a vertical force $F_y$ generates only deflection in the y direction, and no rotation θ at the tip of the beam. The force acts at a distance r from the end of the beam.

a. [2 pts] Write an expression for the rotation of the tip of the beam as a function of the moment arm r.

See W4L2 for in-depth explanation

$$\theta = y'(L) = \frac{1}{EI}\left(M_0 L + \frac{FL^2}{2}\right) = \frac{1}{EI}\left(FrL + \frac{FL^2}{2}\right)$$

b. [1 pts] Solve for the value of r that sets the tip rotation to 0.

$$\theta = y'(L) = \frac{1}{EI}\left(M_0 L + \frac{FL^2}{2}\right) = \frac{1}{EI}\left(FrL + \frac{FL^2}{2}\right) = 0$$

$$r = -\frac{L}{2}$$

c. [2 pts] Compare the stiffness of the mechanism in part b to the simple beam (i.e. $F_y$ applied at r=0).

$$k_{y-simple} = \frac{Eba^3}{4L^3}$$

$$k_{y-no\ rotation} = \frac{Eba^3}{L^3}$$ Since no rotation →guided end condition

$$\frac{k_{y-no\ rotation}}{k_{y-simple}} = 4$$

7. [5 pts] Repeat the previous problem, but with the goal of getting zero tip deflection.

a. [2 pts] Expression for deflection of the tip

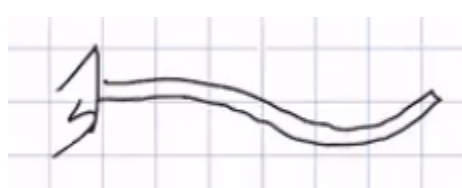
See W4L2 for in-depth explanation

$$y(L) = \frac{1}{EI}\left(\frac{M_0 L^2}{2} + \frac{FL^3}{3}\right) = \frac{1}{EI}\left(\frac{FrL^2}{2} + \frac{FL^3}{3}\right) = 0$$

b. [1 pts] Find r that sets tip deflection to 0

$$y(L) = 0 \rightarrow\ r = -\frac{2L}{3}$$

c. [2 pts. 1 pt each: tip at 0 deflection, rest of beam below] Sketch the shape of the beam under load.



8. [6 pts] In the structure on the right below, the two beams both have a width a and thickness b.

a. [2 pts] Choose $L_2$ such that the spring constants in the x and y directions are equal.

See W4L3 for in-depth explanation

Same spring constant in x and y directions → $\frac{F_x}{\Delta x} = \frac{F_y}{\Delta y}$

$$k_y = \frac{Eba^3}{4{L_1}^3} \text{ (negligible axial bending of } L_2\text{)}$$

$k_x =$ (normal transverse spring constant for beam 2) in series with (the theta spring constant from beam 1)

Apply $F_x \rightarrow M_0 = F_x L_2$ (Pure Force so no $additional\ M_0\ applied$) → $\theta = {y_{beam1}}'(L) = \frac{12F_xL_2}{Eab^3}L_1$

$$k_{theta} = \frac{Eba^3}{12L_2L_1}$$

$$k_{transverse\ beam2} = \frac{Eba^3}{4{L_2}^3}$$

$$k_x = \frac{1}{k_{theta} + k_{transverse\ beam2}} = \frac{1}{\frac{1}{\frac{Eba^3}{12L_2L_1}} + \frac{1}{\frac{Eba^3}{4{L_2}^3}}}$$

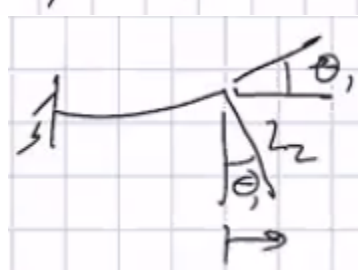
Since it's not explicitly said, I assumed top view so diagram which makes a the cubed term because the direction of deflection for both beam is along the axis that the width is defined. If you assumed side view and appropriately made b the cubed term that is fine. Make sure to state that though.

b. [2 pts] For your choice in part a, calculate the compliance $C_{xy}$, which relates the force in the y direction to the displacement in the x direction. $x=C_{xy}$ Fy

$F_{2x} = F_x$, $M_2 = M_0$, $F_{2y} = F_y$

$F_{1x} = F_x$ $M_1 = M_0 + F_x L_2$

$F_{1y} = F_y$




$$M_1 = F_yL_2\ (Pure\ force\ so\ no\ M_0) \rightarrow \theta\_1 = y\_1'(L) = \frac{1}{EI}(M_1L_1) = \frac{1}{EI}(F_yL_2\ L_1)$$

$$\Delta x = L_2 \tan(\theta) \approx L_2\theta = \frac{1}{EI}(F_y{L_2}^2\ L_1)$$

$$C_{xy} = \frac{F_y}{\Delta x} = \frac{EI}{{L_2}^2\ L_1}$$

c. [2 pts] For your choice in part a, calculate the compliance $C_{yx}$ and explain what it means.

$$F_x \rightarrow \Delta y$$

$$M_{beam1}(L_1) = F_xL_2$$

$$\frac{M}{EI} = \frac{d^2y_{beam1}}{dx^2} = \frac{F_xL_2}{EI}$$

$$\frac{dy_{beam1}}{dx} = \frac{F_x L_2}{EI} x + C_1$$

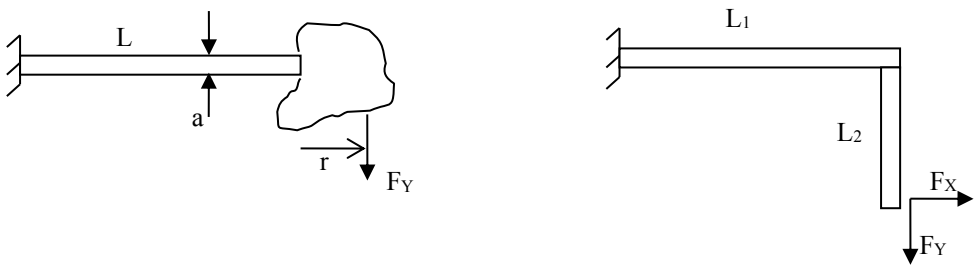
$$y_{beam1}(x) = \frac{F_x L_2}{2EI} x^2 + C_1 x + C_2$$

$$Boundary\ Conditions:$$

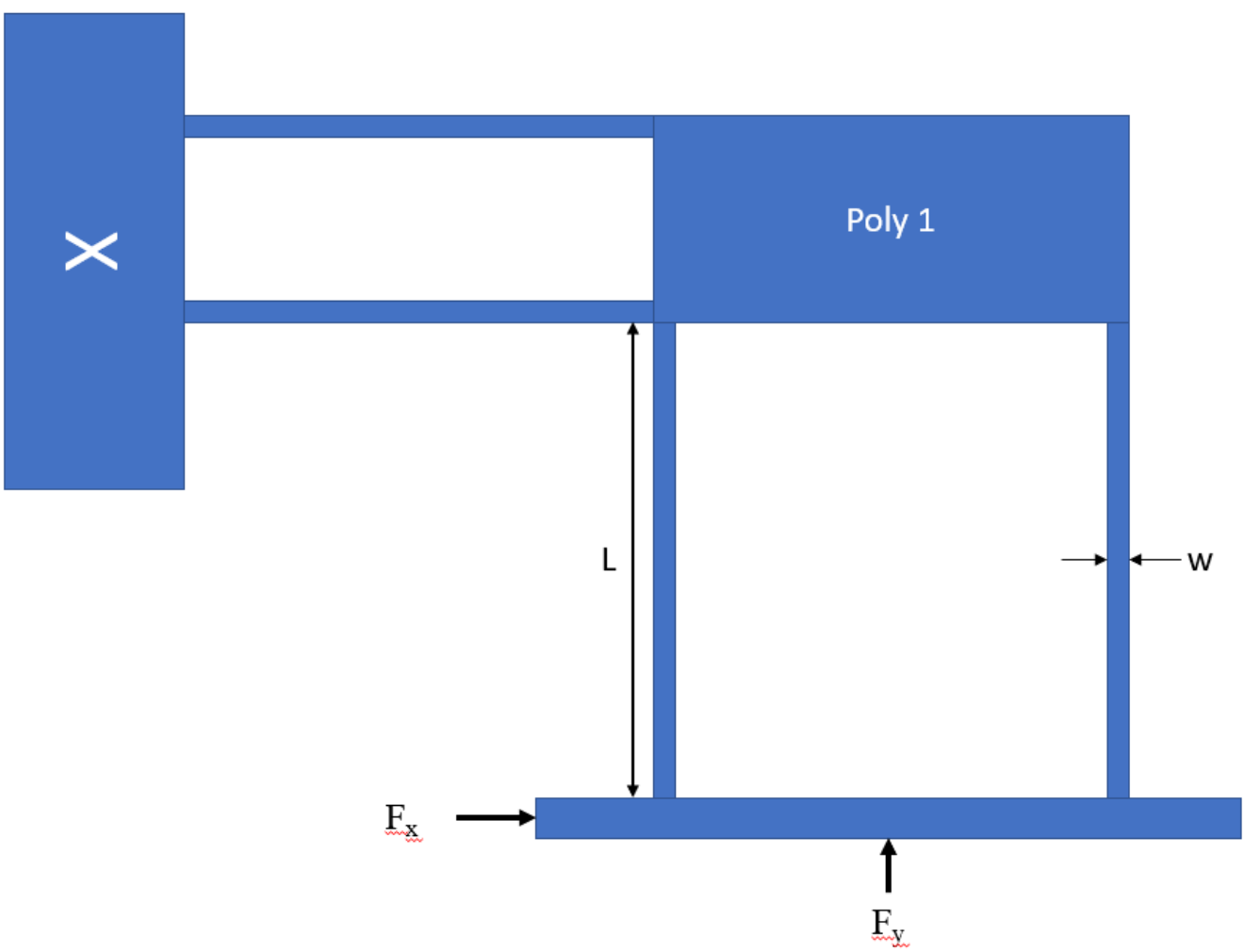
$$Fixed\ end - y(0)=0 \text{ and } y'(0)=0$$

$$Thus\ y_{beam1}(x) = \frac{F_x L_2}{2EI} x^2 = \frac{6F_x L_2}{Eab^3} x^2$$

$$y_{beam1}(L_1) = \frac{6F_x L_2}{Eab^3} (L_1)^2$$




9. [4 pts] Design a suspension in POLYMUMPS to have a stiffness of 1 N/m in x and y with no cross-coupling.



$The\ X\ denotes\ an\ anchors, while\ the\ rest\ of\ the\ structure\ is\ free. Design\ spring\ constants\ to\ be\ 1\frac{N}{m}$

Poly 1 thickness is fixed at 2um and the minimum width is also 2um. Approximating E=150GPa

$k_x = k_y$ if the 4 beams are identical

Because the axial stiffness of a beam is much larger than its stiffness due to a transverse force. Thus, the vertical beams are essentially rigid due to $F_y$ so the stiffness comes from the horizontal beams in the y direction. The same is true for the beams in the x direction so

$$k_x = k_y$$

$Since\ there\ are\ 2\ beams\ in\ parallel\ for\ each\ direction, each\ beam\ needs\ to\ \ have\ a\ stuffness\ of\ 0.5\frac{N}{m}$

$$k_{x,one\ beam} = k_{y,one\ beam} = \frac{Etw^3}{4L^3} = 0.5\frac{N}{m} = \frac{150G\ Pa(2um)(2um)^3}{4L^3} \rightarrow L = 100um$$

10. [247] [8 pts] In the structure on the right above, is it possible to attach a rigid body to the end of L2 and choose the point of action of the two forces such that $C_{xy}=C_{yx}=0$? If so, sketch your design.