Planar depth and planar subalgebras

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We consider the notion of *planar depth* of a planar algebra, *viz.*, the smallest n for which the planar algebra is generated by its 'n-boxes'. We establish a simple result which yields a sufficient condition, in terms of the principal graph of the planar algebra, for the planar depth to be bounded by k. This suffices to determine the planar depth of the E_6, E_8 and the $\frac{5+\sqrt{13}}{2}$ subfactors.

We then consider a planar subalgebra of the 'group planar algebra' which is naturally associated with a group Θ of automorphisms of the given group G. We show that this planar algebra corresponds to the 'subgroup-subfactor' associated with the inclusion $\Theta \subset (G \rtimes \Theta)$ (given by the semi-direct product extension). We conclude with a discussion of the planar depth of this planar algebra P^Θ in some examples.

Key Words: operator algebras, subfactor, planar algebra, standard invariant.

1. INTRODUCTION

Jones defined the notion of a *planar algebra* in [6], where he showed, using Popa's characterisation in [10] of the so-called λ-*lattices*, that (spherical, connected C^*-) planar algebras are in bijection with standard invariants of extremal subfactors. In particular, each planar algebra is the planar algebra associated with at least one subfactor.

Given a planar algebra $P = \{P_n\}$, let us define P^n to be the planar subalgebra (of P) generated by P_n; then we have a tower

$$P^0 \subseteq P^1 \subseteq P^2 \subseteq \cdots \subseteq P^n \subseteq P^{n+1} \subseteq \cdots \cup_{n=0}^{\infty} P^n = P \tag{1.1}$$

of planar subalgebras of P. (Note that P^0 is nothing but the Temperley-Lieb planar algebra TL - with the same δ as the given planar algebra.) It is fairly easy to see that this tower will stabilise at a finite stage *provided* the initial planar algebra P 'has finite depth'. Of course, this finite depth condition is far from necessary for the tower to stabilise; for instance, the Temperley-Lieb planar algebra TL - cf. [6] - satisfies $TL = (TL)^k, \forall k \geq 0$ (since it is generated by its zero boxes) although TL has infinite depth in the 'generic case'. We shall call the smallest integer for which the tower (1.1) stabilises the *planar depth*; i.e. P has planar depth k, $0 \leq k \leq \infty$ if $P = P^l \Leftrightarrow l \geq k$. Since 'planar depth $\leq$ depth', it is natural to ask what the planar depth of a finite-depth planar algebra can be.

In section 2, we prove a fairly simple fact (cf. Corollary 2.2.2 and Proposition 2.2.1) that suffices to determine the planar depth of some planar algebras of finite depth.

The first author(in [9]) gave a presentation of the planar algebra $P(G)$ of the group subfactor corresponding to the fixed-points of an outer action of a finite group G on a II_1 factor. He then considered the scenario of a finite group Θ acting on the group G as group-automorphisms. He showed (Theorem 9 of [9]) that the planar algebra that is generated by the elements in $P(G)_2 \equiv \mathbb{C}G$ which are fixed by (the linear extension of) Θ, is strictly smaller than $P(G)$. He discussed this subplanar algebra for several examples.

In section 3, we begin by observing that if Θ acts on G as above, then there is a natural associated action of Θ on the planar algebra $P(G)$. The invariants of this action, call it P^Θ (the group G and the action of Θ on it will be fixed once and for all), yield a planar subalgebra of $P(G)$. We see that the 'subplanar algebra' of the last sentence of the previous paragraph is just what we call $(P^\Theta)^2$.

We show that P^Θ can in fact be identified with the planar algebra associated with the subgroup-subfactor corresponding to the subgroup Θ of the semi-direct product $G \rtimes \Theta$ and in particular *has finite depth.* This finite depth statement implies, as observed earlier, that P^Θ has finite planar depth. We conclude by discussing some examples that illustrate several possible features of the tower (1.1).

2. A CRITERION FOR ESTIMATING PLANAR DEPTH

We shall use the terminology of [6] for planar algebras. Thus a planar algebra P is a tower $\{P_n : n = 0, 1, 2, \cdots\}$ of finite-dimensional C^*-algebras equipped with 'an action of the coloured operad $\mathcal{P}$ of labelled tangles'. We shall, as in [6], use the term 'k-boxes' to denote elements of P_k.

As stated in the introduction, if $P = \{P_n\}$ is a planar algebra, we shall denote by P^n the planar subalgebra of P generated by P_n. We shall denote by P^n_k the vector space of k-boxes in P^n.

PROPOSITION 2.2.1. *Suppose the principal graph Γ of P satisfies the following condition for some $n \in \mathbb{N}$:*

($\dagger$) each vertex at distance $(n+1)$ from $*$ is adjacent to a unique vertex at distance n from $*$, and these latter vertices ($=$ neighbours) are all distinct.

Then $P^n = P^{n+1}$.

Proof: The proof relies on an inspection of the tower

$$P^n_n \subset P^n_{n+1} \subseteq P^{n+1}_{n+1} \ , \tag{2}$$

showing that the condition ($\dagger$) implies that the second inclusion is actually an equality, and thus $P^n = P^{n+1}$.

We introduce some notation for the Bratteli diagrams for the inclusions in (2):

(i) Denote by X, Y, and Z the vertex set for P^n_n, P^n_{n+1}, and P^{n+1}_{n+1} respectively.

(ii) Define the set $B =$

$$\{(x,y) : x \in X \text{ is connected to } y \in Y \text{ in the Bratteli diagram.}\}$$
$$\coprod$$
$$\{(y,z) : y \in Y \text{ is connected to } z \in Z \text{ in the Bratteli diagram.}\},$$

where $\coprod$ denotes the disjoint union.

(iii) For sets S and T we shall write $(S,T) = Id$ if there is a bijection $f : S \to T$ such that for all $s \in S$, $t \in T$

$$(s,t) \in B \Leftrightarrow t = f(s).$$

(iv) Since the inclusion $P_n = P^n_n \subset P^{n+1}_{n+1} = P_{n+1}$ is part of a Jones tower, there is a natural partition of X and Z into 'old' and 'new' vertices (see [4] for this terminology) which we write as $X = X^O \coprod X^N$ and $Z = Z^O \coprod Z^N$.

(v) The inclusion $P^n_n \subset P^n_{n+1}$ is also part of a Jones tower and thus X and Y have a natural partition into 'old' and 'new' vertices as well. Furthermore, this partition for X is the same as the partition in (v) above since the two Jones towers are identical up to $P^n_n = P_n$. We denote the partition of Y by $Y = Y^O \coprod Y^N$.

With this notation, showing equality between P^n_{n+1} and P^{n+1}_{n+1} is equivalent to showing $(Y,Z) = Id$. We shall accomplish this by showing the four statements:

$$\begin{array}{ll} (Y^N, Z^N) = Id, & (Y^O, Z^O) = Id, \\ (Y^N \times Z^O) \cap B = \emptyset, & (Y^O \times Z^N) \cap B = \emptyset. \end{array} \tag{3}$$

We have the following facts about the Bratteli diagram:

(a) $(\{y\} \times Z) \cap B \neq \emptyset$ for all $y \in Y$.

(b) The nature of the partition of X, Y and Z into old and new vertices implies that

(i) $(X^O \times Y^N) \cap B = \emptyset$,

(ii) for all $y \in Y^O$, $(X^O \times \{y\}) \cap B \neq \emptyset$,

(iii) there is no $y \in Y$ such that

$$(X^O \times \{y\}) \cap B \neq \emptyset \textit{ and } (\{y\} \times Z^N) \cap B \neq \emptyset.$$

(c) It follows from (bii) and (biii) that $(Y^O \times Z^N) \cap B = \emptyset$.

(d) Since both Y^O and Z^O correspond to the set of minimal central projections in the 'basic construction ideal' $P_n e_n P_n$ (with $e_n \in P_{n+1}$ denoting the n-th Jones projection), we have $(Y^O, Z^O) = Id$ and $(Y^N \times Z^O) \cap B = \emptyset$.

Thus we have shown the last three statements in (3); it remains to show that $(Y^N, Z^N) = Id$. But given $(Y^N \times Z^O) \cap B = \emptyset$ and (a), this is just what condition (†) ensures. □

Corollary 2.2.2. *Let P be a planar algebra with associated principal graph Γ. Suppose Γ has no double bonds, no vertices of degree greater than 3, and a unique vertex of degree 3. If this degree 3 vertex is of distance $(k-1)$ from $*$, then P has planar depth k and is generated (as a planar algebra) by one k-box.*

Proof: It follows directly from Proposition 2.2.1 that $P^l = P^{l+1}$ for all $l \geq k$; hence the planar depth of P is finite and at most k. On the other hand, a simple dimension argument shows that $P^{k-1} = P^0$ is nothing but the Temperley-Lieb planar algebra TL and in particular $P^{k-1} \neq P^k$, so the planar depth is exactly k. Finally, the hypothesis shows that dim $P_k =$ dim $TL_k + 1$, and this shows that any element of P_k which is not in P^{k-1}_k will generate P as a planar algebra. □

Remark 2.2.3. *Two remarks are in order here.*

(a) Special cases to which this Corollary applies are the cases when Γ *is* D_{2n}, E_6, E_8 *and the principal graph of the* $\frac{5+\sqrt{13}}{2}$ *subfactor of [1].*

(b) Of the pair of principal graphs associated to the $\frac{5+\sqrt{13}}{2}$ *subfactor of [1], one graph has a unique triple point, while the other has two triple points at different distances from *. (This feature also holds in each case of the heirarchy of pairs of graphs listed (see [5], case (2)) by Haagerup as other possible finite principal graphs of subfactors of index less than* $3+\sqrt{3}$*.)*

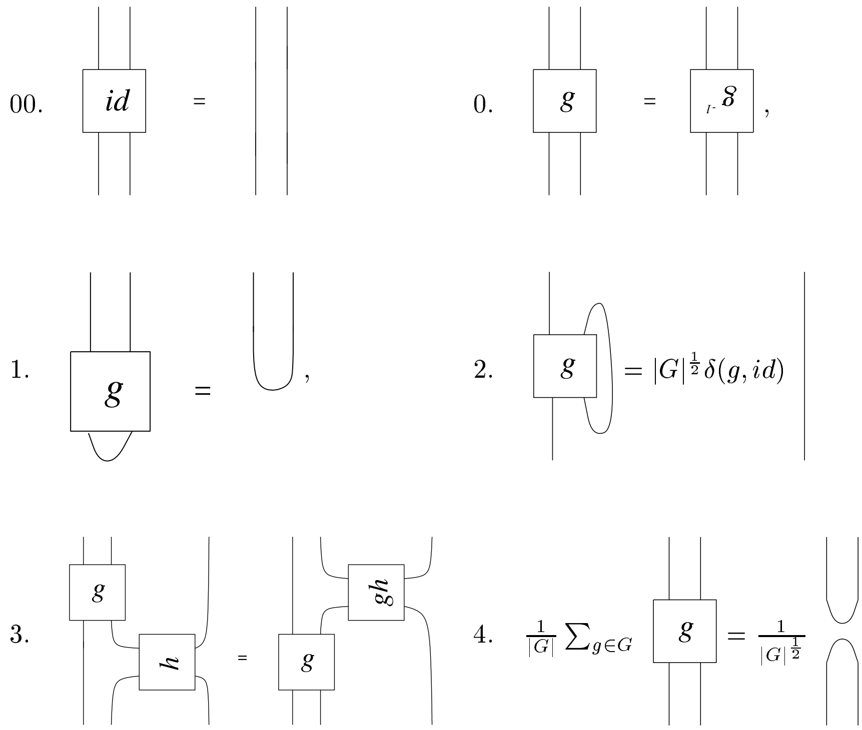
Since the planar depth of a subfactor is the same as that of its dual, we see (as illustrated by the graph (in [5]), referred to above, with two triple points) that it is possible for a principal graph to have a triple point at a distance $(k-1)$ *from* $*$ *and still have planar depth strictly smaller than k.*

(c) As remarked above (and as can be seen from Jones' description of $P^{M\subset M_1}$*), a planar algebra and its dual planar algebra have the same planar depth. This is not the case for the usual depth of a subfactor; the* $\frac{5+\sqrt{13}}{2}$ *subfactor provides an example where the depths of the subfactor and its dual differ. This may be cited as one reason why 'planar depth' is a more natural notion than the usual depth.*

3. SOME PLANAR SUBALGEBRAS OF THE GROUP PLANAR ALGEBRA

We shall only be concerned with planar algebras P which come equipped with a 'presentation (whose symbol Φ we shall suppress) by a collection $L = \coprod_{n=0}^{\infty} L_k$ of labels'. (Again see [6] for notation.)

In fact, following [9], we shall be primarily concerned with the planar algebra $P(G)$, which has a presentation as above with generators given by $L_2 = G$ and $L_k = \emptyset$ for $k \neq 2$, the relation that a simple closed loop (of either orientation) be the scalar $|G|^{\frac{1}{2}}$ and the additional six relations labelled 00,0,1,2,3,4 below (and in Theorem 5 of [9]). We shall assume that we are given a group Θ and an action $\alpha : \Theta \to Aut(G)$. Nothing is changed if we replace Θ by $\Theta/ker\ \alpha$, so we assume that Θ acts faithfully, i.e., that α is 1-1.

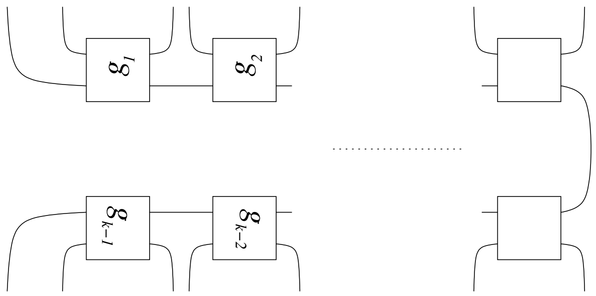


We consider the map on tangles that replaces the label of each 2-box with the label's image under $\theta \in \Theta$. Since θ is a group automorphism, it is seen that the set of relations defining $P(G)$ is unchanged by this map and thus this map defines an automorphism of $P(G)$ which we shall continue to denote by θ. It follows that the set P^{Θ} of invariants for this action of Θ on $P(G)$ is a sub-planar algebra of P, and that the set of Θ-invariant k-boxes of $P(G)$ constitutes precisely the set of k-boxes of P^{Θ}.

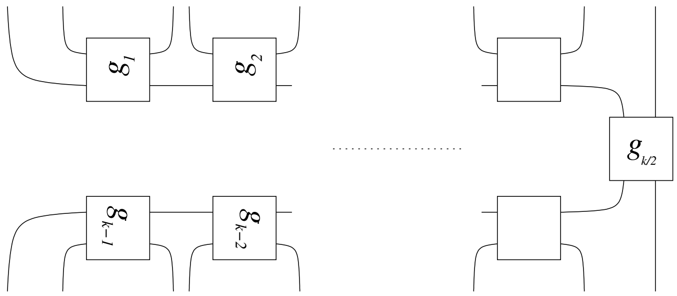
For each $k = 1, 2, \cdots$ and $\theta \in \Theta$, let $\alpha_\theta^{(k)} \in Aut(G^k)$ be defined by $\alpha_\theta^{(k)}(g_1, g_2, \cdots, g_k) = (\alpha_\theta(g_1), \alpha_\theta(g_2), \cdots, \alpha_\theta(g_k))$. When the context is clear, we shall simply write $\theta(g_1, \cdots, g_k)$ for what we have defined above as $\alpha_\theta^{(k)}(g_1, \cdots, g_k)$.

For convenience of reference, we shall gather various simple facts about bases for the spaces $P(G)_k$ in the form of the following remark.

REMARK 3.3.1. (a) It was shown in Theorem 6 of [9] that if we let $T(g_1, \cdots, g_{k-1})$ denote the labelled k-tangle given by

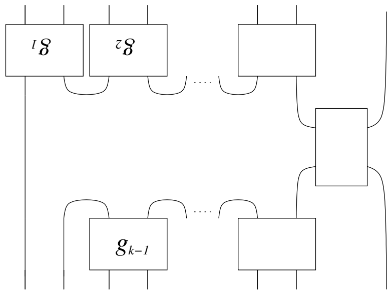


for k odd, and

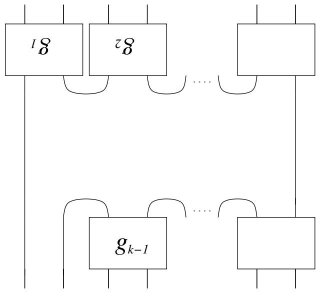


for k even, then $\{T(\overline{g}) : \overline{g} \in G^{k-1}\}$ is an orthonormal basis of $P(G)_k$ (with respect to the inner product given by the natural trace).

(b) We shall find it convenient to use a slightly different basis (which, as we shall see, is actually just a rearrangement of the basis in (a)): define $S(\overline{g})$, $\overline{g} \in G^{k-1}$ to be the labelled k-tangle given by



for k odd, and



for k even.

(c) On the one hand, the map $G^{k-1} \ni \overline{g} \mapsto \overline{h} \in G^{k-1}$ defined by

$$h_i = \begin{cases} g_i^{-1} g_{i+1} & \text{if } \ i < k-1 \\ g_{k-1}^{-1} & \text{if } \ i = k-1 \end{cases} \tag{4}$$

is clearly a bijection with inverse given by $\overline{h} \mapsto \overline{g}$, where $g_i = (h_i h_{i+1} \dots h_{k-1})^{-1}$.

On the other hand, it is an easy exercise to use relation 3. above to show that - with $\overline{g}$ and $\overline{h}$ related as above - we have $S(\overline{g}) = T(\overline{h})$, and hence that $\{S(\overline{g}) : \overline{g} \in G^{k-1}\}$ is an orthonormal basis of $P(G)_k$ as well.

(d) It follows from the definitions that $\{S(1, g_2, g_3, \dots g_{k-1}) : (g_2, g_3 \dots g_{k-1}) \in G^{k-2}\}$ is an orthonormal basis for $P(G)_{1,k}$. (*Reason:* Clearly $S(\overline{g}) \in P(G)_{1,k}$ if $g_1 = 1$, while relation 2. above shows that $S(\overline{g}) \perp P(G)_{1,k}$ if $g_1 \neq 1$.)

(e) We will also need to decompose an $S(\overline{g}), \overline{g} \in G^{k-2}$, when regarded as an element of $P(G)_k$ under the natural inclusion of $P(G)_{k-1}$ into $P(G)_k$, in terms of the basis $\{S(\overline{h}) : \overline{h} \in G^{k-1}\}$. The desired decomposition is:

$$S(g_1, \dots g_{k-2}) \ = \ \begin{cases} \frac{1}{\sqrt{|G|}} \sum_{h \in G} S(g_1, \dots g_{\frac{k-1}{2}}, h, g_{\frac{k+1}{2}}, \dots g_{k-2}) & \text{if } \ k \text{ is odd} \\ S(g_1, \dots g_{\frac{k-2}{2}}, g_{\frac{k}{2}}, g_{\frac{k}{2}}, \dots g_{k-2}) & \text{if } \ k \text{ is even.} \end{cases} \tag{5}$$

(f) In what follows, we shall find it convenient to use the notation

$$\lceil x \rceil \ = \ \min\{n \in \mathbb{Z} : x \leq n\} \ .$$

The relations in $P(G)$ are seen to imply that for arbitrary $\overline{g}, \overline{h} \in G^{k-1}$, we have:

$$\begin{aligned} & S(g_1, g_2, \dots g_{k-1}) S(h_1, h_2, \dots h_{k-1}) \\ = \; & |G|^{\lceil \frac{k}{2} \rceil - 1} \Big(\prod_{i=2}^{\lceil \frac{k}{2} \rceil} \delta(h_1 g_{k+1-i}, h_i) \Big) \\ & \quad \times S(h_1 g_1, h_1 g_2, \dots h_1 g_{\lceil \frac{k}{2} \rceil}, h_{\lceil \frac{k}{2} \rceil + 1}, h_{\lceil \frac{k}{2} \rceil + 2} \dots h_{k-1}) \end{aligned} \tag{6}$$

where $\delta(a, b)$ is zero unless $a = b$ in which case it is one.

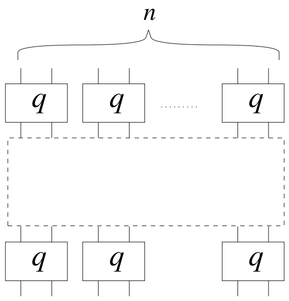
(g) The action of Θ on $P(G)_k$ maps an orthonormal basis onto itself and consequently yields a unitary representation of Θ; in particular, the orthogonal projection of $P(G)_k$ onto P_k^Θ is given by the usual averaging operator. So, if we define $\Theta S(\overline{g}) = \sum_{\theta \in \Theta} S(\theta(\overline{g}))$, it is seen that $\{\Theta S(\overline{g}) : [\overline{g}] \in G^k/\Theta\}$ is an orthogonal set of vectors, which clearly spans P_k^Θ (where we have written $[\bar{g}]$ to denote the orbit of $\bar{g}$ under Θ, and G^k/Θ to denote the set of all such orbits in G^k). Finally, it is not hard to deduce from equation (6) and the relations defining $P(G)$ that if $\overline{g}, \overline{h} \in G^{k-1}$, then

$$\begin{aligned} & \Theta S(g_1, g_2, \dots g_{k-1}) \Theta S(h_1, h_2, \dots h_{k-1}) \\ = \; & |G|^{\lceil \frac{k}{2} \rceil - 1} \sum_{\theta'' \in \Theta} \Big(\prod_{i=2}^{\lceil \frac{k}{2} \rceil} \delta(h_1 \theta''(g_{k+1-i}), h_i) \Big) \\ & \quad \times \Theta S(h_1 \theta''(g_1), h_1 \theta''(g_2), \dots h_1 \theta''(g_{\lceil \frac{k}{2} \rceil}), h_{\lceil \frac{k}{2} \rceil + 1}, h_{\lceil \frac{k}{2} \rceil + 2} \dots h_{k-1}) \end{aligned} \tag{7}$$

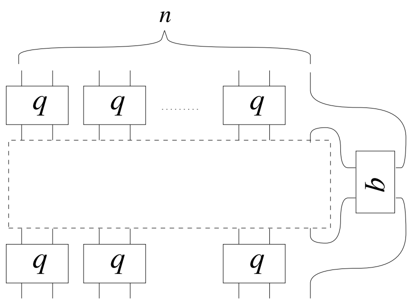
Theorem 3.3.2. *Let G, Θ be as above, and let $G \rtimes \Theta$ denote the semi-direct product associated to this group action, and let $N = R^{G \rtimes \Theta} \subset R^\Theta = M$ denote the associated subgroup-subfactor. Then*

$$P^\Theta \cong P^{N \subset M}.$$

Proof: For notational convenience let $P_k = P_k^\Theta$, $P_{1,k} = P_{1,k}^\Theta$, $Q_k = P_k^{N \subset M}$, $Q_{1,k} = P_{1,k}^{N \subset M}$. We shall write the elements of $G \rtimes \Theta$ as ordered pairs (g, θ) with the usual multiplication $(g_1, \theta_1)(g_2, \theta_2) = (g_1 \theta_1(g_2), \theta_1 \theta_2)$. We begin by recalling some of the work in [3] which will allow us to describe the standard invariant of $P^{N\subset M}$. We let $\boxed{q} = \frac{1}{|\Theta|}\sum_{\theta\in\Theta} \boxed{(I,\theta)}$ be the projection corresponding to the subgroup $\Theta \subset G \rtimes \Theta$. Define $B_n \in \mathcal{A}_{n,n}$ to be the following annular map in $P(G \rtimes \Theta)$:



for n even,



for n odd.

Define the natural inclusion map

$$i : B_n(P_n) \to B_{n+1}(P_{n+1})$$

given by

$$t \mapsto B_{n+1}(t).$$

We denote this inclusion by $\subset_i$.

Corollary 4.5 of [3] then states that the tower

$$\begin{array}{cccccccccc}
Q_0 & \subset & Q_1 & \subset & Q_2 & \subset & \dots & & & \\
 & & \cup & & & & & & & \\
 & & Q_{1,1} & \subset & Q_{1,2} & \subset & \dots & & & \\
 & & & & & & \dots & \subset & Q_n & \dots \\
 & & & & & & & & \cup & \\
 & & & & & & \dots & \subset & Q_{1,n} & \dots
\end{array}$$

is isomorphic to the tower:

$$\begin{array}{ccccccc} B_0(P_0(G \rtimes \Theta)) & \subset_i & B_1(P_1(G \rtimes \Theta)) & \subset_i & B_2(P_2(G \rtimes \Theta)) & \ldots & \\ & & \cup & & & & \\ & & B_1(P_{1,1}(G \rtimes \Theta)) & \subset_i & B_2(P_{1,2}(G \rtimes \Theta)) & \ldots & \end{array}$$

$$\begin{array}{cccc} \ldots & \subset_i & B_n(P_n(G \rtimes \Theta)) & \ldots \\ & & \cup & \\ \ldots & \subset_i & B_n(P_{1,n}(G \rtimes \Theta)) & \ldots \end{array}$$

Using the relations in $P(G \rtimes \Theta)$ we find that

$$B_n(S((g_1, \theta_1), (g_2, \theta_2), \ldots (g_{n-1}, \theta_{n-1})))$$

$$= \frac{1}{|\Theta|^n} \sum_{\theta \in \Theta,\ \overline{\gamma} \in \Theta^{2n-1}} S((\theta(g_1), \gamma_1), (\theta(g_2), \gamma_2), \ldots, (\theta(g_{n-1}), \gamma_{n-1}))$$

which *only* depends on the orbit of $(g_1, g_2, \ldots g_{n-1})$ under Θ. We shall denote this sum of elements - i.e., $|\Theta|^n$ times the right side of the above equation - by $U(\overline{g})$; again, note that $U(\overline{g})$ depends only on the orbit $[\overline{g}]$ of $\overline{g}$ under Θ. It follows then that $\{U(\overline{g})\}_{[\overline{g}] \in G^{n-1}/\Theta}$ is an orthogonal basis for $B_n(P_n(G \rtimes \Theta)$. (It is a complete set since it is the image under B_n of a basis for $P_n(G \rtimes \Theta)$. It is an orthogonal set because distinct elements are linear combinations of disjoint subsets of an orthonormal basis of $P_n(G \rtimes \Theta)$.)

Analogous to equation (6), we find - arguing this time in the group $G \rtimes \Theta$ - that if $\overline{g}, \overline{h} \in G^{k-1}$, then

$$\begin{aligned} &U(g_1, g_2, \ldots g_{k-1}) U(h_1, h_2, \ldots h_{k-1}) \\ &= |G|^{\lceil \frac{k}{2} \rceil - 1} |\Theta|^{k-1} \sum_{\theta'' \in \Theta} \Big(\prod_{i=2}^{\lceil \frac{k}{2} \rceil} \delta(h_1 \theta''(g_{k+1-i}), h_i) \Big) \\ &\qquad \times U(h_1 \theta''(g_1), h_1 \theta''(g_2), \ldots h_1 \theta''(g_{\lceil \frac{k}{2} \rceil}), h_{\lceil \frac{k}{2} \rceil + 1}, h_{\lceil \frac{k}{2} \rceil + 2} \ldots h_{k-1}) \end{aligned} \tag{8}$$

Define $\beta_k : P_k \to B_k(P_k(G \rtimes \Theta))$ by

$$\beta_k(\Theta(S(\overline{g}))) = |\Theta|^{1-k} U([\overline{g}])\ ,\ \overline{g} \in G^{k-1}\ . \tag{9}$$

By the foregoing remarks, β_k maps an orthogonal basis of P_k onto an orthogonal basis of $B_k(P_k(G \rtimes \Theta))$ and is thus a well defined bijection of vector spaces. To establish the isomorphism of towers we only need to verify that, for all k,

1. β_k is a homomorphism,
2. $\beta_k(P_{1,k}) = B_k(P_{1,k}(G \rtimes \Theta))$,
3. $\beta_k|_{P_{k-1}} = i \circ \beta_{k-1}$.

It is a straightforward consequence of equations (6), (7) and the carefully chosen constant in Definition (9) that β_k indeed preserves multiplication and is thus a homomorphism.

The second assertion above is a consequence of Remark 3.3.1(d), and the fact that $\{U(\overline{g}) : g_1 = 1, [\overline{g}] \in G^{k-1}/\Theta\}$ is a basis for $B_k(P_{1,k}(G \rtimes \Theta))$ (the proof of which fact is analogous to that of Remark 3.3.1(d)).

Finally, the third assertion above follows from Remark 3.3.1(e).

□

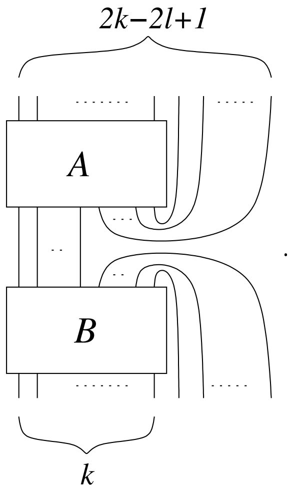
In the sequel, we shall economise on parentheses and write $P^{\Theta;n}$ for $(P^{\Theta})^n$.

COROLLARY 3.3.3. *Suppose there exists $\overline{g}^0 \in G^{l-1}$ such that the mapping*

$$\Theta \ni \theta \mapsto \theta(\overline{g}^0) \in G^{l-1}$$

is injective. Then $P^{\Theta} = P^{\Theta;2l}$.

Proof: If $k \geq 2l-1$ and $A, B \in P_k$, define $\Pi(A,B)$ to be the element of $P_{2k-2l+1}$ given by the following tangle:



It follows from the group relations that

$$\begin{aligned} &\Pi(T(g_1,\ldots g_{k-1}),T(h_1,\ldots h_{k-1})) \\ &= |G|^{l-1}(\prod_{i=1}^{l-1}\delta(g_{k-i}h_i,1))T(g_1,\ldots g_{k-l},h_l,\ldots h_{k-1}) \ . \end{aligned} \tag{10}$$

Let us write (as in Remark 3.3.1(g))

$$\Theta T(\overline{g}) = \sum_{\theta\in\Theta} T(\theta(\overline{g})) \ .$$

Then, for arbitrary $\overline{g} \in G^{2k-2l+1}$, we find, using (9), that

$$\begin{aligned} &\Pi(\Theta T(g_1,\ldots g_{k-l},g_1^0,\ldots,g_{l-1}^0),\Theta T((g_{l-1}^0)^{-1},(g_{l-2}^0)^{-1}\ldots,(g_1^0)^{-1},g_{k-l+1},\ldots g_{2k-2l+1})) \\ &= |G|^{l-1}\sum_{\theta,\theta'}\left(\prod_{i=1}^{l}\delta(\theta(g_{l-i}^0)\theta'((g_{l-i}^0)^{-1}),1)\right) \\ &\quad\times T(\theta(g_1),\theta(g_2),\ldots\theta(g_{k-l}),\theta'(g_{k-l+1}),\ldots,\theta'(g_{2k-2l+1})) \ . \end{aligned} \tag{11}$$

Since $\theta \mapsto \theta(\overline{g}^0)$ is 1-1, it follows that

$$\prod_{i=1}^{l}\delta(\theta(g_{l-i}^0)\theta'((g_{l-i}^0)^{-1}),1) = \delta(\theta(\overline{g}^0),\theta'(\overline{g}^0)) = \delta(\theta,\theta') \ ,$$

and thus the right side of (10) simplifies to

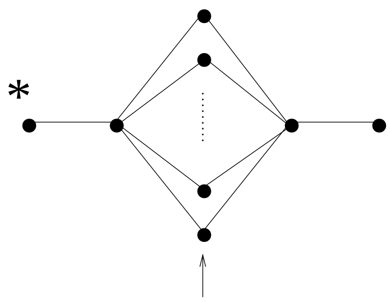
$$|G|^{l-1}\sum_{\theta} T(\theta(g_1),\ldots\theta(g_{2k-2l+1}) = |G|^{l-1}\Theta T(\overline{g}).$$

Since $\{\Theta T(\overline{g})\}$ is a basis for $P_{2k-2l+1}$, we have shown $P^{\Theta;2l} = P^{\Theta;2l+1} = P^{\Theta;2l+3} = P^{\Theta;2l+7} = \ldots = P^{\Theta}$.

Example 3.3.4. We consider a few examples.

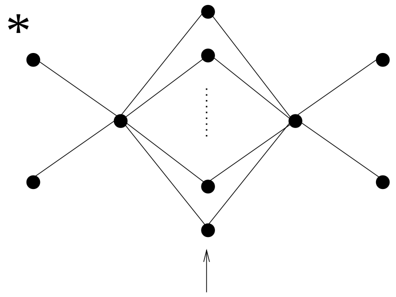
(a) Let $\Theta = \mathbb{Z}_2$ act on $\mathbb{Z}_n$ 'by inversion'; thus, the involutory automorphism corresponding to the non-trivial element of Θ is given by $\tau(x) = -x \ \forall x \in \mathbb{Z}_n$. It follows from 'the Mackey machine' - see [4] or [8], for instance - that the principal graph (for P^{Θ} and hence for the subgroup-subfactor corresponding to the inclusion $\mathbb{Z}_2 \subset \mathbb{Z}_n \rtimes \mathbb{Z}_2$) is given thus:

Case (i): n is odd



(n−1)/2 vertices

Case (ii): n is even



(n−2)/2 vertices

In both cases, it is clear that $P^{\Theta;1} = TL \neq P^{\Theta;2}$, while it follows from Proposition 2.2.1 that $P^{\Theta;3} = P^{\Theta;4} = P^{\Theta}$ if n is odd. (Of course, the case $n = 3$ must be discussed separately, since we get $P^{\Theta} = TL$ in this case.) Further, an argument given in [9] (see the pictorial identity within the example $G = D_{2n+1}$ towards the end) shows that $P^{\Theta;2} = P^{\Theta;3}$ for all n. In particular, the planar depth of P^{Θ} is two, if n is odd.

On the other hand, it turns out that for even n, the planar depth of P^{Θ} is four, as we show now. In view of the remarks of the last paragraph, it will suffice to show that $P^{\Theta;2} \neq P^{\Theta}$ in this case. Since P^{Θ} has finite depth, it will suffice to show that $P^{\Theta;2}$ is the free product of $P(\mathbb{Z}_2)$ and $P(\mathbb{Z}_{n/2})$ (since free products necessarily have infinite depth). (See [2] for the definition and these facts about free products - or free compositions, as they are called there - of subfactors; also see [6] for free products in the planar algebra context.)

We know from Theorem 3.3.2 that $P^{\Theta} = P^{N \subset M}$, where $N = R^{\mathbb{Z}_n \rtimes \mathbb{Z}_2}$ and $M = R^{\mathbb{Z}_2}$. Then, it follows from the analysis of [3] that the free product $P^{N\subset Q} * P^{Q\subset M}$ is contained in $P^{N\subset M}$. Let H denote the centre of $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, so that $H = \mathbb{Z}_2 \times \mathbb{Z}_2$; and let $Q = R^H$, so that $N \subset Q \subset M$ is an intermediate subfactor. Since the subgroup H is normal in $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, and since

$$(R^A \subset R^B) \cong (R^{A/B} \subset R)$$

whenever B is a normal subgroup of a finite group A, we find that $P^{N\subset Q} \cong P((\mathbb{Z}_n \rtimes \mathbb{Z}_2)/(\mathbb{Z}_2 \rtimes \mathbb{Z}_2)) \cong P(\mathbb{Z}_{n/2})$ and that $P^{Q\subset M} \cong P(\mathbb{Z}_2)$. We may conclude that

$$\begin{aligned} dim\ P_2^{N\subset M} &= dim\ P_2^{\Theta;2} \\ &= \text{no. of orbits of } \Theta \text{ in } \mathbb{Z}_n \\ &= \frac{n-2}{2} + 2 \\ &= n/2 + 2 - 1 \\ &= (dim\ P_2^{N\subset Q}) + (dim\ P_2^{Q\subset M}) - 1\ . \end{aligned}$$

It is proved in [3] that this equality allows us to conclude that $P^{\Theta;2} = P^{N\subset Q} * P^{Q\subset M}$, as desired.

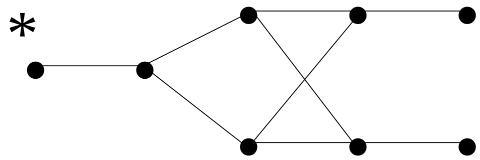
(b) Let $\Theta = \mathbb{Z}_3$ act on $\mathbb{Z}_2 \times \mathbb{Z}_2$, with the generator of $\mathbb{Z}_3$ cyclically permuting the three non-trivial elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. It again follows from the Mackey machine that the principal graph is given by the extended Coxeter diagram $\tilde{E}_6$; and we may conclude from Corollary 2.2.2 that

$$P^{\Theta;k} = \begin{cases} TL & if\ k \leq 2 \\ P^{\Theta} & if\ k > 2 \end{cases}$$

(c) The action of $\mathbb{Z}_3$ on $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Example (b) above, can be extended to an action of S_3 (with the transpositions interchanging pairs of non-trivial elements); in this case, P^{Θ} has principal graph given by the extended Coxeter diagram $\tilde{E}_7$; and we may conclude from Corollary 2.2.2 that

$$P^{\Theta;k} = \begin{cases} TL & if\ k \leq 3 \\ P^{\Theta} & if\ k > 3 \end{cases}$$

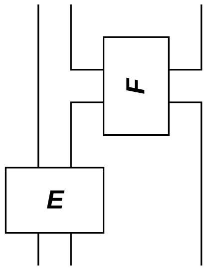
(d) Consider $\Theta = \mathbb{Z}_3$ acting on $G = \mathbb{Z}_7$ with the generator of $\mathbb{Z}_3$ acting as the map $x \mapsto x^2$. In this case, we find that the principal graph is given by:



We can deduce from Proposition 2.2.1 that $P^{\Theta;3} = P^{\Theta;4} = P^{\Theta}$. We can easily see that $P^{\Theta;1} = TL \neq P^{\Theta;2}$; we shall now proceed to show that $P^{\Theta;2} = P^{\Theta;3}$.

Let X (resp., Y) denote the sum of the 2-boxes labelled by the members of $[1] = \{1,2,4\}$ (resp., $[3] = \{3,6,5\}$). If we let Z denote the 2-box labelled by 0, then it is clear that $\{Z, X, Y\}$ is a basis for $P_2^{\Theta;2}$.

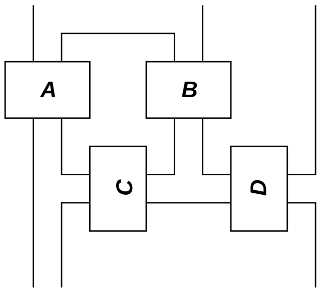
If $z, w \in G$, let us write $((z,w))$ for the element (of $P(G)_3$) obtained if we substitute z and w respectively, for E and F in the picture given by

$((E,F)) =$ 

and $[z,w] = \sum_{\theta\in\Theta}((\theta(z),\theta(w)))$. The definitions imply - by considering the Θ-orbits in $G \times G$ - that P_3^{Θ} is linearly spanned by the set

$$\{[0,0],[0,1],[0,3]\} \cup \{[1,w] : w \in G\} \cup \{[3,w] : w \in G\} \ .$$

Next we write (A,B,C,D) for the value of the following picture (where $A,B,C,D \in P_2^{\Theta;2}$):



and we have the following identities:

$$\begin{aligned}
(Z,Z,Z,Z) &= \sqrt{7}\ [0,0] \\
(Z,Z,Z,X) &= \sqrt{7}\ [0,1] \\
(Z,Z,Z,Y) &= \sqrt{7}\ [0,3]
\end{aligned}$$

$$\begin{aligned}
(X,Z,Y,Z) &= \sqrt{7}\,[1,0] \\
(X,X,Z,Z) &= \sqrt{7}\,[1,1] \\
(X,X,X,Z) &= \sqrt{7}\,[1,2] \\
(X,Y,X,Z) &= \sqrt{7}\,([1,3]+[1,5]) \\
(X,X,Y,Z) &= \sqrt{7}\,[1,4] \\
(X,Y,X,X) &= \sqrt{7}\,(2[1,0]+[1,2]+[1,4]+[1,5]+[1,6]) \\
(X,Y,Y,Z) &= \sqrt{7}\,[1,6]\ .
\end{aligned}$$

These identities show that $[0,x],[1,x] \in P_3^{\Theta;2}$ for all $x \in G$. Similar computations show that $[3,x] \in P_3^{\Theta;2}$ for all $x \in G$. Thus, we have shown that $P_3^{\Theta} \subset P^{\Theta;2}$, so $P^{\Theta;3} \subset P^{\Theta;2}$, and $P^{\Theta;3} = P^{\Theta;2}$, as desired.

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