Exchange Relation Planar Algebras

Zeph A. Landau

Mathematical Sciences Research Institute
1000 Centenial Drive, # 5070
Berkeley, California 94720-5070
landau@msri.org

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Abstract

An inclusion of II_1 factors $N \subset M$ of finite index has as an invariant, a double sequence of finite dimensional algebras known as the standard invariant. Planar algebras were introduced by V. Jones as a geometric tool for computing stantdard invariants of existing subfactors as well as generating standard invariants for new subfactors.

In this paper we define a class of planar algebras, termed exchange relation planar algebras, that provide a general framework for understanding several classes of known subfactor inclusions: the Fuss-Catalan algebras (i.e. those coming from the presence of intermediate subfactors) and all depth two subfactors.

In addition, we present a new class of planar algebras (and thus a new class of subfactors) coming from automorphism subgroups of finite groups.

Keywords: operator algebras, Von Neumann algebras, subfactors, planar algebras, exchange relation planar algebras, standard invariant, Fuss-Catalan algebras, depth two inclusions.

AMS classification (1991): 46L37.

1 Introduction

One of the major areas of study within subfactor theory has been the classification of subfactors $N \subset M$. In [Jo1], V. Jones defined the notion of index for type II_1 subfactor inclusions $N \subset M$. In addition he introduced a canonical construction of a tower of subfactors

$$N \subset M \subset M_1 \subset M_2 \subset \dots$$

known as the *basic construction*. For $N \subset M$ an inclusion of type II_1 subfactors with finite index, the double sequence of finite dimensional algebras

$$\begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset \dots \\ & & \cup & & \cup & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset \dots \end{array}$$

known as the *standard invariant*, became an extremely important invariant in the study of subfactors (see for example [GHJ, JS, Po1, Po2]). In fact S. Popa determined that the standard invariant completely classifies certain kinds of subfactor inclusions [Po1].

In [Po2], Popa provided an abstract algebraic characterization of the standard invariant called a *λ-lattice*. Subsequently, Jones developed an equivalence between λ-lattices and objects he termed *planar algebras* [Jo3]. In the planar algebra framework, many of the seemingly complicated algebraic conditions associated with a λ-lattice can be simply described in terms of the geometry of the plane.

One of the methods for constructing planar algebras is similar to the use of generators and relations to construct groups. In this paper we define a type of relation, termed an exchange relation, that ensures finite dimensionality of the graded algebras of the resulting planar algebra. We call such planar algebras *exchange relation* planar algebras. We show that the exchange relation planar algebra provides a general framework for understanding several classes of known subfactor inclusions: the Fuss-Catalan algebras (i.e. those coming from the presence of intermediate subfactors) and all depth two subfactors. We then present a large class of new planar algebras coming from automorphisms of finite groups. We introduce the notion, due to D. Bisch and V. Jones, of a free product of planar algebras and use this notion and other techniques to compute several examples within this new class of planar algebras.

The significance of the work here is two-fold. The first is the discovery of the new class of examples of planar algebras presented in section 9. The second is the notion of an exchange relation planar algebra. It is arguably the simplest class of planar algebras in the sense that the proof of Theorem 1 reduces to a simple geometric argument. The simplicity of this class is especially remarkable given the rich and complex structure contained in several of its members, namely the Fuss-Catalan algebras and the Hopf-* algebras.

2 Notation

All subfactors refered to in this paper will be of type II_1. Given an inclusion of II_1 factors $N \subset M$, the Jones tower corresponding to the basic construction will be denoted by

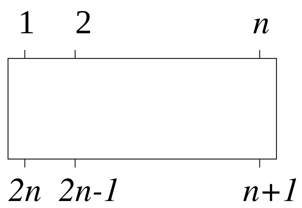
$$N \subset M \subset M_1 \subset M_2 \ldots,$$

with corresponding Jones projections $e_{M_{i-1}}^{M_i} : L^2(M_i) \to L^2(M_{i-1})$. When there is no ambiguity, we will use the shorthand of $e_1 = e_N^M,\ e_i = e_{M_{i-1}}^{M_i}$ for $i \geq 2$.

The connection between subfactors and planar algebras (discussed in the next section) is a correspondence between the standard invariant of a subfactor and a planar algebra. We shall denote by $P^{N \subset M}$, the planar algebra corresponding to the standard invariant for the subfactor inclusion $N \subset M$. In addition a *subfactor corresponding to a planar algebra* P will be a subfactor $N \subset M$ such that $P = P^{N \subset M}$.

3 Planar Algebras

We briefly outline some of the definitions and main ideas presented in [Jo3]. This presentation is modeled after that given in [Bh]. By a *standard n-box*, B_n, we shall mean a rectangle in $\mathbb{R}^2$ with n distinguished points on the top and n distinguished points on the bottom. We label the points from 1 to $2n$ clockwise beginning at the top left.

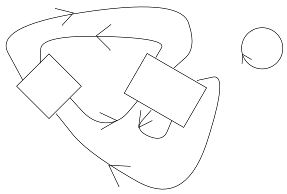


Definition 1 *A* <u>*planar network*</u>*,* $\mathcal{N}$*, will be a picture in* $\mathbb{R}^2$ *consisting of a finite number of n-boxes,* B_n *(n varying), along with a finite number of oriented non-intersecting open or closed curves, called* <u>*strings*</u>*, such that:*

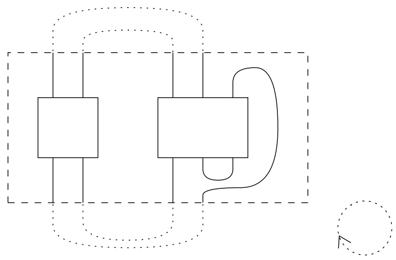
1. *The strings lie in* $\mathbb{R}^2 \setminus \cup B_n$.
2. *Those strings that are not closed begin at odd marked points of the* B_k *and end at even marked points.*
3. *All marked points on boxes have strings attached to them.*
4. *Each connected component of the complement of the network has an orientation compatible with its boundary.*

Planar networks are defined up to smooth orientation preserving diffeomorphism of $\mathbb{R}^2$.

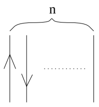
We give an example of a planar network:



Let $L_i, i = 1, 2, 3, \ldots$ be sets and set $L = \coprod L_i$ be their disjoint union which we shall call the set of *labels*. By a *labeled planar network*, we shall mean a planar network with an assignment of a label from the set L_k to each standard k-box appearing in the network. A *planar k-tangle* $\mathcal{T}$ $(k = 0, 1, 2 \ldots)$ is the intersection of a planar network $\mathcal{N}$ with the standard $k-$ *box*, B_k, with the condition that the boundary of B_k meet $\mathcal{N}$ transversely solely at the marked points of B_k, and these points are non-endpoints of strings of $\mathcal{N}$. In addition, we insist that the leftmost region of the k-box be positively oriented. Planar k-tangles are defined up to smooth isotopy acting identically on the boundary. For example, after diffeomorphism, our example pictured above may be intersected with B_4 to produce the following planar 4-tangle which we shall call A. (The dotted lines pictured are <u>not</u> part of the 4-tangle, but merely serve to assist the reader in seeing what happened.)



A *labeled planar k-tangle* is defined in the obvious way. The set of labeled planar k-tangles on a labeling set L shall be denoted by $\mathfrak{T}_k(L)$. We note that elements of $\mathfrak{T}_0(L)$ are the labeled planar networks on L with the unbounded region positively oriented. By the algebra $\mathcal{P}_n(L)$, we shall mean the vector space over $\mathbb{C}$ of elements in $\mathfrak{T}_n(L)$, with multiplication by vertical concatenation: stacking the first box on top of the second and fusing the strings meeting the labeled points $2n, 2n-1, \ldots n+1$ of the top box to the strings meeting the labeled points $1, 2, \ldots, n$ of the bottom box.

$\mathcal{P}_n(L)$ is then an associative algebra with unit 

. $\mathcal{P}_n(L)$ embeds unitally into $\mathcal{P}_{n+1}(L)$ by adding a vertical string of the appropriate orientation on the right. We view this embedding as an inclusion and define the *universal planar algebra* $\mathcal{P}(L)$ to be the filtered algebra $\bigcup_{n=0}^{\infty} \mathcal{P}_n(L)$. A *planar algebra* on L will be a filtered quotient of $\mathcal{P}(L)$ that is invariant under an *annular* action which we define as follows.

A *j-k tangle* will be the complement of the interior of a j-box B_j inside a k-tangle (that contains a j-box). For example, considering the above 4-tangle A with the interior of the 3-box removed gives an annular 3-4 tangle. We denote by $\mathcal{A}_{j,k}$ the set of all j-k tangles up to boundary preserving isotopy. We denote by $\mathcal{A}(L) = \bigcup_{j,k} \mathcal{A}_{j,k}(L)$ the set of all annular j-k tangles with boxes labeled by elements in L (up to boundary preserving isotopy). Given

$A \in \mathcal{A}_{k,l}(L), B \in \mathcal{A}_{j,k}(L)$, the product AB is defined to be the labeled annular j-l tangle obtained by filling in the "hole" in A with B. In a similar way, A induces a map

$$\Pi_A : \mathfrak{T}_k(L) \to \mathfrak{T}_l(L),$$

that clearly satisfies $\Pi_A \circ \Pi_B = \Pi_{AB}$. A *general planar algebra* will be a filtered algebra $P = \cup_n P_n$ along with a surjective homomorphism $\Phi : \mathcal{P}(L) \to P$ for some labeling set L such that

- $\Phi(\mathcal{P}_n(L)) = P_n$
- $\ker \Phi$ is invariant under $\mathcal{A}(L)$.

We say that Φ presents P.

Given a planar tangle $\mathcal{T}$ containing n-boxes of size $j_1, j_2, \ldots j_k$ and k labeled tangles $\mathcal{T}_i$, with $\mathcal{T}_i$ a labeled j_i tangle, we let $\Pi_{\mathcal{T}}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$ be the tangle obtained by filling in the i-th box of $\mathcal{T}$ with the tangle $\mathcal{T}_i$. If P is a general planar algebra presented by Φ, then Φ carries $\Pi_{\mathcal{T}}$ to a multilinear map

$$\Pi_{\mathcal{T},\Phi} : P_{j_1} \times P_{j_2} \times \cdots \times P_{j_k} \to P.$$

These maps give a presentation of the planar algebra P on the very large labeling set, namely itself as follows. Set $L' = \coprod L'_i$ with $L'_i = P_i$. Define the presenting map $\phi_P : \mathcal{P}(L') \to P$ on a labeled tangle $\mathcal{T}$ with labels $R_1, \ldots R_n$ by

$$\phi_P(\mathcal{T}) = \Pi_{\mathcal{T},\Phi}(R_1, \ldots R_n)$$

where $R_1, \ldots R_n \in L'$ on the right hand side are thought of as elements of P. In this way the standard j-box labeled by $R_j \in L'_j = P_j$ is mapped to the element $R_j \in P_j$. We say that Φ_P *presents* P *on itself.*

Given a planar algebra presented on itself by Φ_P we define a subplanar algebra as follows. Let $S = \coprod S_i$ be a disjoint union of $S_i \subset P_i(L)$. We define the subplanar algebra $P(S) \subset P(L)$ generated by S, to be $\Phi_P(\mathcal{P}(S))$. It should be noted that for a given subset S, determining whether $P(S) \neq P(L)$ is often a difficult task.

Define $\mathcal{P}_{1,n}(L)$, $n \geq 1$, to be those n-tangles in which the points on the boundary marked by 1 and $2n$ are connected together by a single vertical string. Let $P_{1,n}$ be the image of $\mathcal{P}_{1,n}(L)$ under Φ. A general planar algebra is said to be *connected* if dim P_0 =dim$P_{1,1}$ $= 1$. In this case we have a map, Z, called the partition function, from labeled networks to $\mathbb{C}$ given by $\Phi(\mathcal{N}) = Z(\mathcal{N})\mathrm{id}$, for $\mathcal{N}$ with positive orientation of the unbounded region and $\Phi(\,|\,\mathcal{N}) = Z(\mathcal{N})\mathrm{id}$, for $\mathcal{N}$ with negative orientation of the unbounded region.

A *planar algebra* is a connected general planar algebra with its partition function multiplicative on connected components, and with the two scalars $Z(\quad)$ and $Z(\quad)$ both nonzero.

We describe two useful methods for constructing planar algebras.

1. Let Z' be a $\mathbb{C}$-valued planar isotopy invariant of labeled planar networks for some labeling set L, with Z' multiplicative on connected components and $Z'(\quad)$, $Z'(\quad)$ both nonzero. Define $\mathcal{J}_k = \{\mathcal{T} \in \mathcal{P}_k(L) : Z'(A(\mathcal{T})) = 0$ for all $A \in \mathcal{A}_{k,0}\}$. Setting $\Phi : \mathcal{P}_k(L) \to \mathcal{P}_k(L)/\mathcal{J}_k = P_k$ gives a planar algebra $P = \cup P_k$ presented by Φ with partition function Z'.

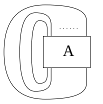
2. Given a label set L and a subset $R \subset \mathcal{P}(L)$, let $\mathcal{J}_j(R)$ be the linear span of
$$\bigcup_{\mathcal{T} \in R \cap \mathcal{P}_k(L)} \mathcal{A}_{k,j}(L)(\mathcal{T}).$$
Then let $P_n(L,R) = \mathcal{P}_n(L)/\mathcal{J}_n(R)$. This gives a planar algebra $P(L,R) = \cup P_n(L,R)$. We shall say this planar algebra is given by *generators L and relations R.*

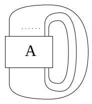
   Occasionally we will write that the set of relations includes "$A = B$" for some particular $A, B \in \mathcal{P}_n(L)$. This will mean that $A - B$ is in the set of relations.

Given an involution $l \to l^*$ on the label set L, we may construct an involution on $\mathcal{P}(L)$ as follows. Given a labeled tangle $\mathcal{T} \in \mathfrak{T}(L)$, define $\mathcal{T}^*$ to be the horizontal reflection of $\mathcal{T}$ with a) the boundary of each reflected k-box renumbered so that the reflected boundary labels $1, 2 \dots 2n$ are replaced with the numbers $2n, 2n-1, \dots 1$, b) each k-box label l replaced by l^*, and c) the orientation of each strand reversed.

We extend this involution sesquilinearly to $\mathcal{P}(L)$. A *-algebra P presented by Φ on L is a *general planar* *-algebra if Φ is *-homomorphism.

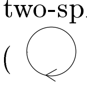
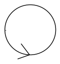
Let P be a general *-algebra, and present it on itself by Φ_P as described above. There are then two natural traces given by:

$Tr_L(A) = Z($ 

$)$ and $Tr_R(A) = Z($ 

$)$

We call P *finite dimensional* if P_n is finite dimensional for each n. If P is a connected, finite dimensional planar *-algebra with a positive definite trace, then each P_n is semi-simple and has a unique norm that makes it a C^*-algebra. We call such an algebra a *C^*-planar algebra.*

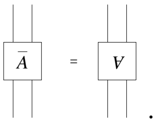
P is *spherical* if the partition function is an invariant of the two-sphere (equivalently $Tr_L = Tr_R$). In this case we denote by $\delta = Z($  $) =$ $Z($  $)$ and define $tr = \delta^n Tr_L$ on P_n. With this definition, tr respects the inclusion maps $P_n \subset P_{n+1}$ and we have $tr(1) = 1$.

The theorem connecting planar algebras to subfactors appears in section 4 of [Jo3] which is summarized as, "the centralizer tower [standard invariant] for an extremal finite index type II_1 subfactor admits the structure of a spherical C^*-planar algebra, and vice versa."[Jo3] This connection establishes for each subfactor $N \subset M$ the existence of a corresponding planar algebra $P^{N\subset M}$ as well as for each planar algebra P the existence of at least one subfactor $N \subset M$ with $P = P^{N \subset M}$. The underlying connection is the isomorphisms $N' \cap M_k \cong P_{k+1}^{N\subset M}$ and $M' \cap M_k \cong P_{1,k+1}^{N\subset M}$.

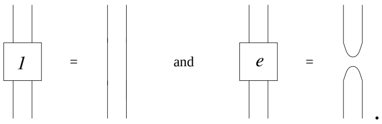
Remark on terminology: For the rest of this paper, by *planar algebra* we shall mean *spherical C^*-planar algebra.*

4 The Exchange Relation

For purposes of notation, we define the 2-box labeled by $\overline{A}$, $A \in L_2$, to be the upside down 2-box labeled by A:

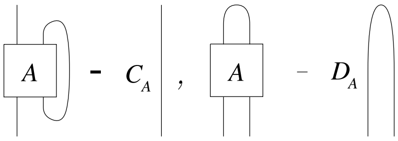


In addition we define the 2-boxes labeled by 1 and e as follows:

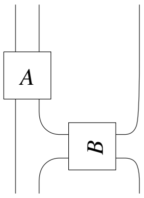


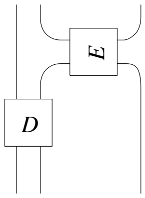
(We note that neither 1 nor e need be in L_2 since they are both always present in any planar algebra.)

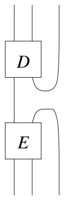
Definition 2 *A subset $R \subset P(L)$ will be called an* <u>*exchange relation*</u> *if R contains the following elements:*

1. 

for all $A \in L_2 \cup \overline{L_2}$.

2. 

$- \sum_{D,E \in L_2 \cup \overline{L_2} \cup \{1,e\}} (C^{AB}_{DE}$


$+ C'^{AB}_{DE}$


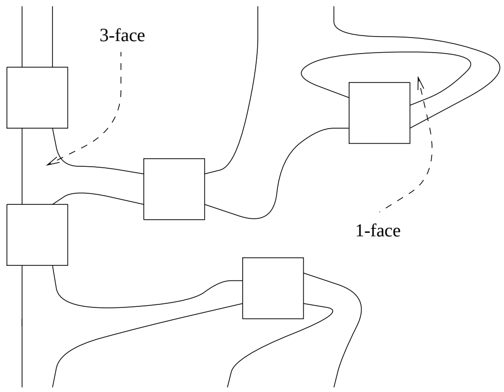
$)$.

where $C_A, D_A, C^{AB}_{DE}, C'^{AB}_{CD}$ are all scalars.

A planar algebra that can be constructed as $P(L, R)$ with R an exchange relation will be called an <u>*exchange relation*</u> *planar algebra.*

A note on the terminology: the use of the words exchange relation is suggested by the case when C'^{AB}_{CD} is always zero. In this case, the second relation consists of moving the left 2-box down and the right one up (a kind of exchanging).

Definition 3 *Given an element in $\mathfrak{T}_n(L)$, we will call a region that is not connected to the boundary an* <u>*internal face*</u>*. An* <u>*internal n-face*</u> *will be an internal face that has n basic boxes on its boundary. We provide a picture to clarify:*



With this notation we prove the following about exchange relations:

Theorem 1 *For $L = L_2$ a finite label set and R an exchange relation, we have*

$$dim(P_n(L,R)) < \infty$$

for all n.

Proof: Let ϕ_R be the presenting map of $P(L,R)$ on L. Set $T_n(L) = \phi_R(\mathfrak{T}_n(L))$. With fixed n we first prove that any element of $P_n(L,R)$ can be written as a linear combination of elements in $T_n(L)$, none of which contain an internal face.

We proceed by contradiction. Of all $x \in T_n(L)$ such that x cannot be written as a linear combination of labeled n-tangles with no internal faces, consider those x with:

- x having a minimal number of basic 2-boxes.

Of all choices of x with the above property, pick an x with its smallest internal n-face of minimal size.

If x has an internal 1-face we can apply relations of type 1 (in Definition 2) to contradict minimality of the basic number of 2-boxes. Thus x must have its minimal internal n-face of size $n \geq 2$. But then pick two neighboring 2-boxes along the boundary of a minimal internal n-face. By applying the appropriate relation of type 2 (in Definition 2), we can rewrite x as a linear combination of diagrams with its smallest internal face of size strictly less than n. By choice of x, therefore, we know that each of these diagrams are linear combinations of elements in $T_n(L)$ with no internal faces and thus so is x. This of course contradicts the choice of x.

We have established that $P_n(L,R)$ is linearly spanned by elements $x \in T_n(L)$ with no internal faces. This immediately gives us $P_0(L,R) \subset \mathbb{C}$. Consider an element $x \in T_n(L),\ n \geq 1$ with no internal faces. If we shrink the 2- boxes down to points, we get a graph to which we can apply Euler's formula. Set v to be the number of vertices and e the number of edges. Since we have no faces (a face would only occur if x had an internal face), and every connected component has at least two single valent vertices (otherwise that component would have an internal face) we have $1 \leq v - e \leq n$. If we denote by b the number of 2-boxes in x (and thus the number of 4-valent vertices when the 2-boxes were shrunk down to points), we have $v = 2n + b$, $e = \frac{4b+2n}{2} = 2b + n$. Thus $1 \leq n - b \leq n$ and so $0 \leq b \leq n - 1$. Thus we have established that $P_n(L,R)$ is spanned by elements of $T_n(L)$ with no internal faces and no more than $n-1$ basic 2-boxes. Clearly there are only a finite number of such elements. $\square$

The above result suggests that many different C^*-planar algebras can be generated from exchange relations. However, an arbitrary choice of coefficient does not guarantee that $P(L,R) \neq 0$. There are quite a few examples where we do have $P(L,R) \neq 0$, some of which we will discuss in the next few sections.

5 The Intermediate Subfactor Relation

An abstract characterization of intermediate subfactors was given in [Bi1]. Subject to the constraint that $N' \cap M = \mathbb{C}$, it is:

Theorem 2 (Bi1) *Consider a subfactor $N \subset M$, $N' \cap M = \mathbb{C}$, with N_1, M_1 such that $N_1 \subset N \subset M \subset M_1$ is the basic construction. Every projection $q \in N' \cap M_1$ with*

1. $qe_N^M = e_N^M$,

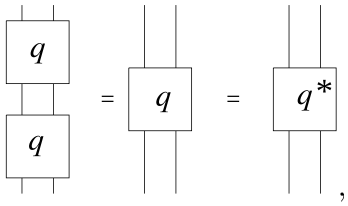
2. $E_M^{M_1}(q) \in \mathbb{C}$,

3. $m = E_M^{M_1}(qe_{N_1}^N e_N^M)$ *is a scalar multiple of a projection,*

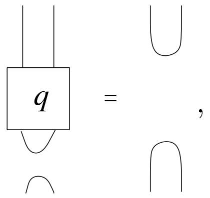
corresponds to an intermediate subfactor $Q = \{q\}' \cap M$ *with* q *implementing the trace preserving conditional expectation from* M *to* Q*.*

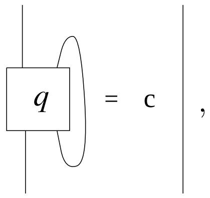
In the planar algebra picture, the above theorem becomes:

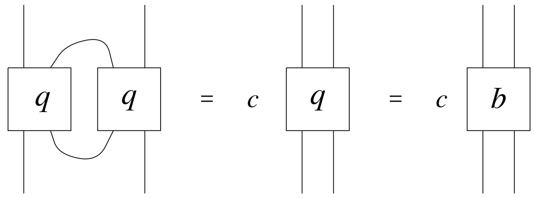
Every 2-tangle q that, for some scalar c, satisfies

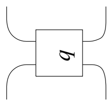
0) 

(i.e. q a projection)

1) 

2) 

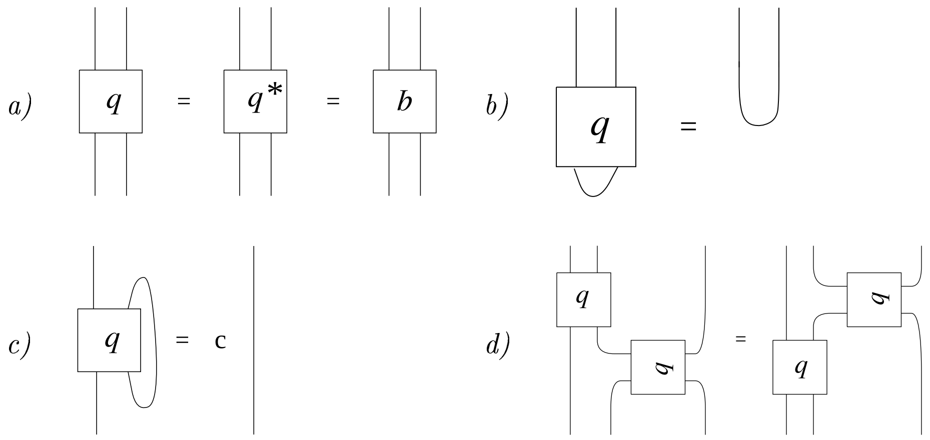
3) 

(i.e. 

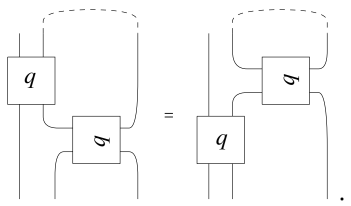
a multiple of a projection),

corresponds to an intermediate subfactor.

Corollary 2.1 *The above conditions are equivalent to*



Proof: It is clear that $b) \Leftrightarrow 1)$, $c) \Leftrightarrow 2)$. Using relation d) and looping together the top middle and top left lines gives:



Using this, along with a) and c) implies 3). Looping together the bottom left and middle lines of d), along with a) and b), implies 0).

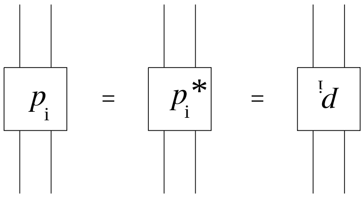
All that remains is to show that 0-4 imply d). Let A be the left hand side of d), B the right hand side. Now apply the trace to $(A-B)(A-B)^*$. It is left to the reader to apply 0-4 to the resulting pictures to compute $tr((A-B)(A-B)^*) = 0$. Since the trace is positive definite we have $A = B$. $\square$

Remark: It is interesting to note that we needed the positive definiteness of the trace to go from 0-4 to a) -d) but not to go the other way. To some extent we have removed some of the "hidden" conditions imposed by the faithfulness of the trace.

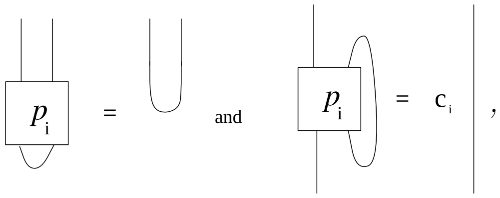
From the above corollary, we see that the planar algebra generated by one intermediate subfactor is an exchange relation planar algebra. In fact, we can say more. Here we give the planar algebra corresponding to Fuss-Catalan algebras (see [BiJo] and [La2]), the algebras related to a chain of intermediate subfactors.

Theorem 3 *The Fuss-Catalan algebras $FC_{k,n}(\alpha_0, \alpha_1, \ldots \alpha_k)$ are isomorphic to the planar algebra with $L = L_2 = \{p_1, p_2, \ldots p_k\}$, with involution $p_i^* = p_i$ and the following relations:*

0)

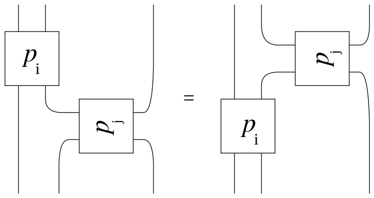


1)



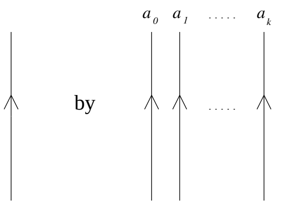
$c_i = \frac{\alpha_0 \alpha_1 \ldots \alpha_{k-i}}{\alpha_k \alpha_{k-1} \ldots \alpha_{k-i+1}}.$

2)

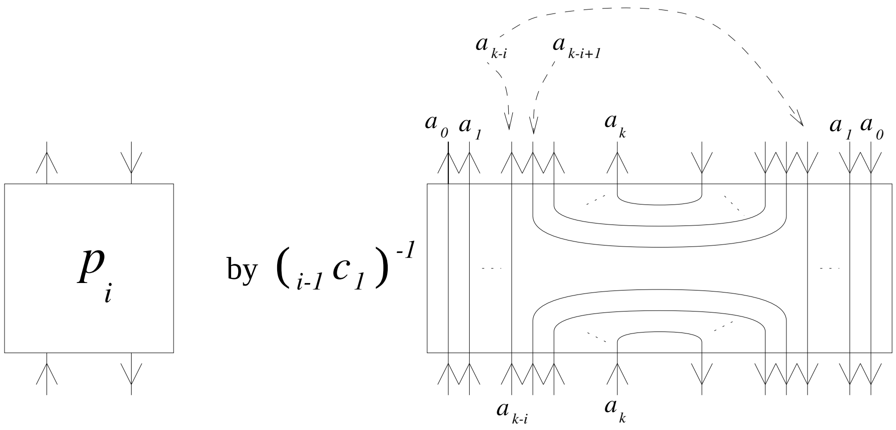


for $i \leq j$.

Proof: The proof follows from the map noted in [Jo3] that replaces



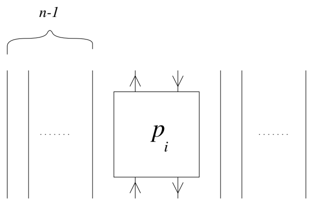
and



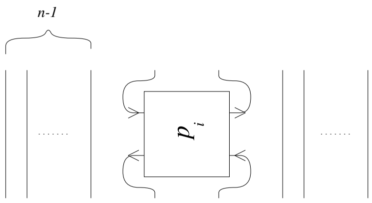
with

$$ {}_a c_i = \begin{cases} \alpha_0 \alpha_1 \cdots \alpha_a & \text{for } i \text{ even} \\ \alpha_k \cdots \alpha_{k-a} & \text{for } i \text{ odd.} \end{cases} \quad \text{and } {}_{-1}c_i = 1. $$

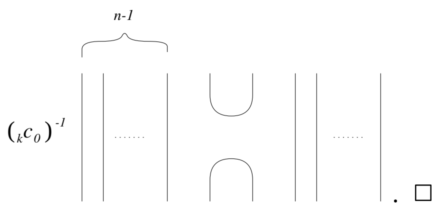
and where a loop labeled by a_i is replaced by a factor of α_i. One can quickly check that this is a homomorphism. The fact that it is an isomorphism can be shown by noticing that the inverse map is given by sending ${}_{i-1}E_n$, $1 \leq i \leq k$, in the abstract presentation given in [La2] to the picture



if n is odd, and to ${}_{i-1}c_1 / {}_{k-i}c_2$ times



if n is even, as well as sending ${}_kE_n$ to the picture

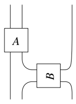


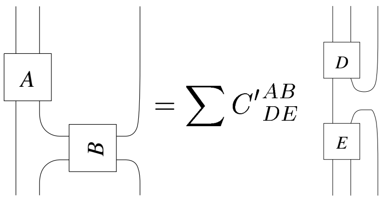
Thus the Fuss-Catalan algebras are another example of an exchange relation planar algebra. We should note that though the theorem only gives an exchange relation of type 2 (Definition 2) for $A = p_i$ and $B = p_j$ for $i \leq j$, a ninety degree counter-clockwise rotation gives a collection of relations of type 2 for $A = p_j$ and $B = p_i$, $i \leq j$.□

6 Depth 2 Subfactors

A subfactor is said to have depth 2 if the principal graph (see [GHJ] for discussion of principal graph) consists of vertices of distance less than or equal to two from the root. Depth 2 irreducible (i.e. $N' \cap M = \mathbb{C}$) subfactors have been completely characterized via actions of finite quantum groups [Sz]. (Also see [NiVa] for the non irreducible case.) Here we examine depth 2 irreducible subfactors from a planar algebra point of view.

Theorem 4 *Planar algebras representing irreducible depth 2 subfactors are exchange relation planar algebras.*

Proof: $N \subset M$ of depth 2 implies that $N' \cap M_2 = \mathrm{span}(N' \cap M_1 e_2 N' \cap M)$ where $N \subset M \subset M_1 \subset M_2 \subset \ldots$ is the basic construction tower with e_i implementing the conditional expectation from M_i to M_{i-1}. Applying this fact to the 3-box  of $P_3^{N \subset M} \cong N' \cap M_2$ gives:



where the right hand side is a linear combination of elements in $(N' \cap M_1)e_2(N' \cap M_1)$. This is all we need since irreducibility implies every 1-box is a scalar multiple of the identity and thus we have the relation of type 1 required in Definition 2. Furthermore we note that because we have depth 2, any basic n-box with $n > 2$ can be written as a linear combination of planar n-tangles only containing 2-boxes. $\square$

<u>Remark</u> The algebra of 3-boxes in the above planar algebra gives a pictorial interpretation of the Hopf-*-algebra (or quantum group) structure. We will explore this connection in detail in a future paper.

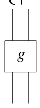
7 Group Action

The simplest example of an irreducible depth 2-subfactor is given by an outer action of a finite group (see for example [JS]). In this situation, we are given a factor M and an outer action $\{\alpha_g\}_{g\in G}$ of a group G on M implemented by conjugation by unitaries $\{u_g\}_{g\in G}$, $u_g \in B(L^2(M))$ (i.e., $\alpha_g(m) = u_g m u_g^*$). Let $N = M^G$ be the fixed point algebra of the action of G. It is known that the basic construction of $N \subset M$ gives $M_1 = M \rtimes G = \{M, u_g\}''$. In addition we have $N' \cap M = \mathbb{C}$, $N' \cap M_1 = \{u_g\}''$, $e_1 = \frac{1}{|G|}\sum_{g\in G} u_g$, $M' \cap M_2 = \{u_g e_2 u_g^*\} \cong \mathbb{C}^{|G|}$ (see [GHJ]).

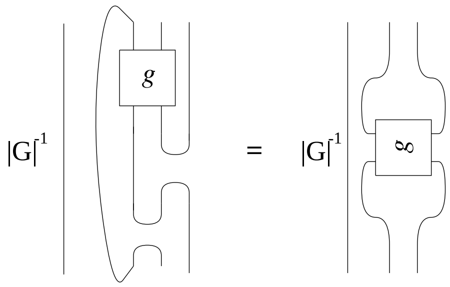
We now show $|G|E_{M'\cap M_2}(u_g e_2 e_1) = u_g e_2 u_g^*$. To do so we show $|G|tr_{M'\cap M_2}(u_g e_2 e_1 u_h e_2 u_h^*) = tr_{M'\cap M_2}(u_g e_2 u_g^* u_h e_2 u_h^*)$. The right hand side is $\delta(g,h)\frac{1}{|G|}$ since $\{u_g e_2 u_g^*\}_{g\in G}$ are orthogonal projections. Now

$$u_g e_2 e_1 u_h e_2 u_h^* = u_g e_2 e_1 e_2 u_h^* = \frac{1}{|G|} u_g e_2 u_h^*.$$

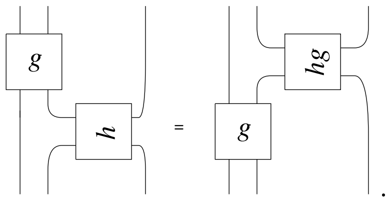
Again $tr_{M'\cap M_2}(u_g e_2 u_h^*) = \delta(g,h)\frac{1}{|G|}$ and we are done.

Thus we have $M' \cap M_2 = \{|G|E_{M'\cap M_2}(u_g e_2 e_1) : g \in G\}''$ where $\{|G|E_{M'\cap M_2}(u_g e_2 e_1)\}_{g\in G}$ are orthogonal projections summing to 1. If we denote by 

the 2 box corresponding to u_g, we see that we have shown that

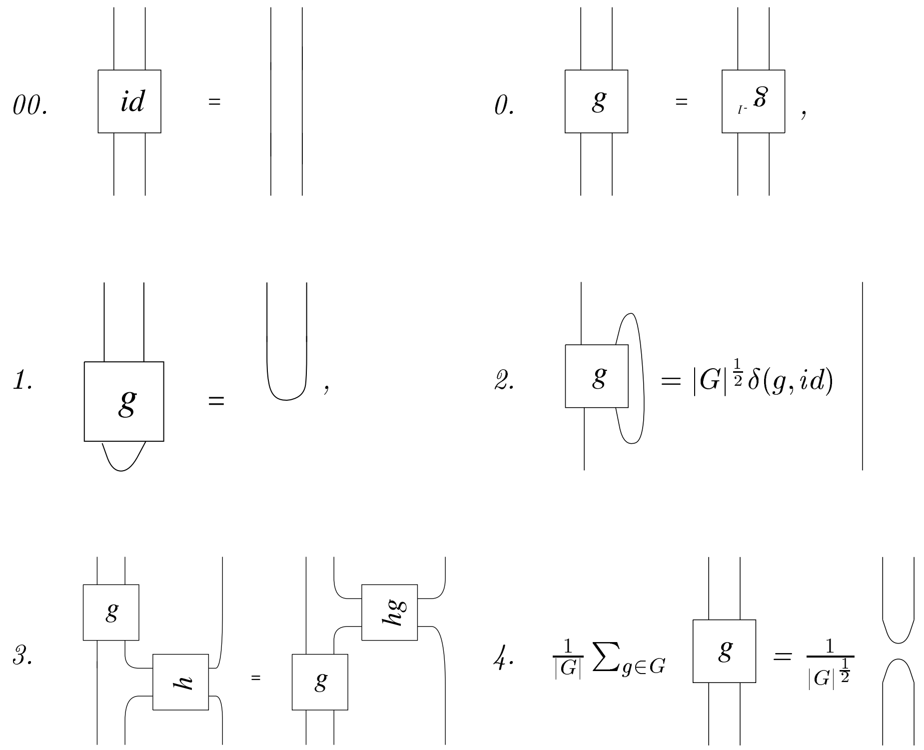


are an orthogonal set of projections. Finally, $u_g u_h e_2 u_h^* = u_{gh} e_2 u_{gh}^* u_g$ establishes the relation



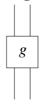
We are led to the following:

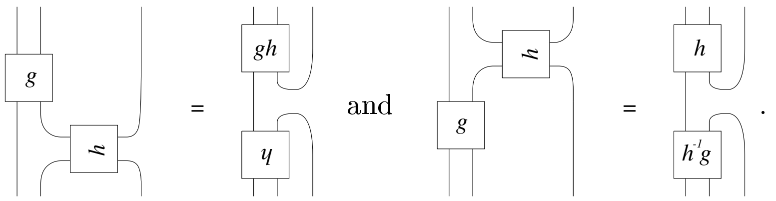
Theorem 5 *The planar algebra given by an outer group action G is described as follows: the labeling set is $L = L_2 = \{g\}_{g \in G}$ with $g^* = g^{-1}$ and*



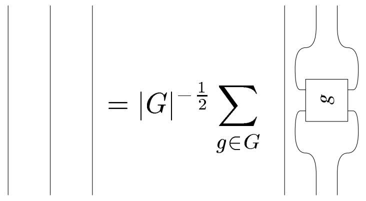
Proof:

Relation 00. is just the fact that $u_{id} = 1$. Relation 0. follows from the definition of the involution * and the fact that $|G|E_{M'\cap M_2}(u_g e_2 e_1)$ is self adjoint (we showed it to be the projection $u_g e_2 u_g*$). Relation 1. follows from $u_g e_1 = e_1$. Relation 2. is the fact that $E_M(u_g) = \delta(g, id)$, and relation 3. was shown above. Relation 4. is just $e_1 = \frac{1}{|G|}\sum_g u_g$.

It remains to show that no further relations are needed. Consider the planar algebra generated by relations 00.-4. By theorem 1, we have that the 2-tangles are spanned by all pictures with no internal faces, i.e. $P_2(L, R)$ =span $\{$  $\}_{g\in G}$ (since the only additional 2-tangle is $e_1 = \frac{1}{|G|}\sum_{g\in G} u_g$). Thus dim $P_2(L, R) \leq |G|$. In addition, a rotated version of relation 3. establishes



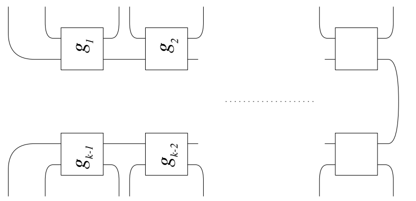
This combined with the fact that



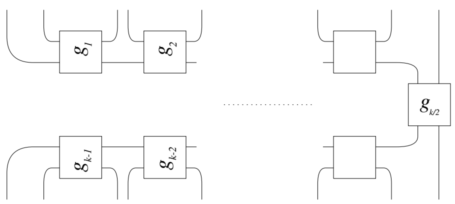
establishes that $P_3(L,R) = \operatorname{span}\{P_2(L,R)e_2P_2(L,R)\}$. Thus $P(L,R)$ represents a depth two subfactor and since dim($M^G \cap M \rtimes G$)= $|G|$ $\geq$dim ($P_2(L,R)$) we have that relations 0-4 are sufficient for describing the planar algebra $M^G \subset M$. $\square$

Definition 4 *Let $P(G)$ be the planar algebra $P(L,R)$ appearing in Theorem 5.*

Theorem 6 *An orthogonal basis for $P_k(G)$, $k > 2$ is given by the pictures:*



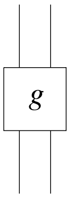
for k odd, and



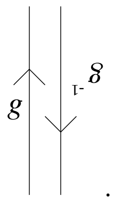
for k even.

Proof: For any two distinct elements x, y listed above, we leave it to the reader to check that $tr(y^*x) = 0$. This establishes orthogonality. The fact that they are a spanning set follows from the knowledge that $\dim(N' \cap M_k) = |G|^k = \dim\ (P_{k+1}(G))$. (See GHJ). $\square$

We describe the partition function of the above planar algebra. Given an element $x \in \mathcal{P}_0(L)$, we define a functional $Z' : \mathcal{P}_0(L) \to \mathbb{C}$ as follows.

1. Replace every 2-box 

with the picture



.

2. On each loop l pick an arbitrary distinguished point. Let $w(l)$ be the conjugacy class of the word in G obtained by starting at the distinguished point for l, traveling around in the direction of the loop orientation and reading off sequentially the letters in G (if no letters of G are present we set $w(l) = id$). Though the word obtained in this way depends on the starting point, the conjugacy class is independent of the starting point.

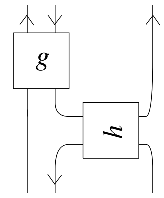
3. If for any loop l, $w(l) \neq [id]$, the conjugacy class of the identity, set $Z'(X) = 0$. Otherwise let

$$Z'(x) = \left(\frac{1}{|G|^{\frac{1}{2}}}\right)^{\#\text{ of loops}}$$

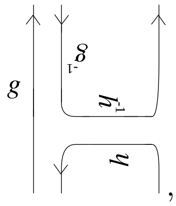
4. Extend Z' to all of $\mathcal{P}_0(L)$ by linearity.

Theorem 7 *The functional Z' as described above is the partition function for $P(G)$.*

Proof: It is straightforward to check that Z' is invariant under relations 00.-3. of Theorem 5. For clarity we demonstrate 3:

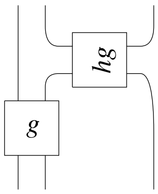


becomes

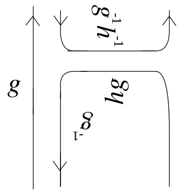


,

while

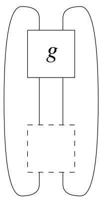


becomes



.

To show 4. requires a little more work. We first note that since the relations 0-3 are a complete set of exchange relations, that by the proof of Theorem 1 we have that every $A_{2,0}$ annulus acting on $P_2(G)$ can be written as a linear combination of $A_{2,0}$ annuli with no internal faces. A spanning set of such annuli is pictured below



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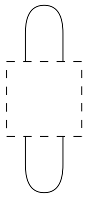


.

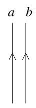
It is now straightforward to show that plugging in the right hand side and the left hand side of relation 4. does not change the value of Z'.

Thus we have the invariance of Z' under the four relations. Since these relations can be used to reduce any 0-tangle to a scalar, we must have that Z' is the desired partition function. □

8 Free Product

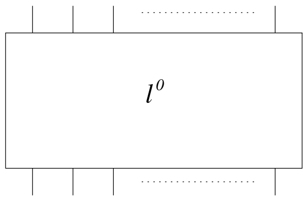
In [BhLa] it is shown how to get information about the standard invariant of $N \subset P$ and $P \subset M$ from that of $N \subset M$. Here we address the issue of how to gain information about the standard invariant of $N \subset M$ given that of $N \subset P$ and $P \subset M$. As noted in [BiHa], there is more information in $N \subset M$ than just that of $N \subset P$ and $P \subset M$. It is true however, that the knowledge of $N \subset P$ and $P \subset M$ gives some (and sometimes all) of the structure of $N \subset M$. We now define the free product of two planar algebras. The definition is due to D. Bisch and V. Jones who were the first to know the result of Theorem 9.

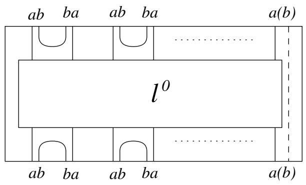
Definition 5 (Bisch and Jones) *Given two planar algebras $P^0 = P(L^0)$, $P^1 = P(L^1)$ define the free product planar algebra $P(L) = P^0 * P^1$ as follows. Let the labeling set be $\overline{L = L^0 \coprod L^1}$. Given an element in $X \in \mathfrak{T}_0(L)$, define the partition function Z of $P(L)$ in the following manner:*

1. *Replace all single strands* *with two strands labeled a and b* 

.

2. *Replace all basic boxes labeled by $l^0 \in L^0$ as follows:*

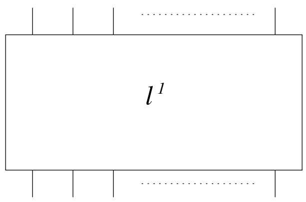


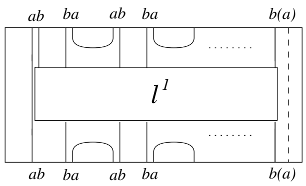
by


,

where the parenthetical string of type "b" is drawn if l^0 is an n box with n odd.

3. *Replace all basic boxes labeled by $l^1 \in L^1$ as follows:*



by


,

where the parenthetical string of type "a" is drawn if l^1 is an n box with n even.

4. *The resulting picture is an overlay of two 0-tangles: an element of $\mathcal{P}_0(L^0)$ (the tangle connected by strands labeled by "a") and an element of $\mathcal{P}_1(L^1)$ (the tangle connected by strands labeled by "b"). Call these two 0-tangles S and T respectively. Define the partition function $Z(X) = Z_0(S)Z_1(T)$ where Z_0 (resp. Z_1) is the partition function for $P(L^0)$ (resp. $P(L^1)$).*

We now describe the principal graph Γ of $P(L^0) * P(L^1)$ in terms of the principal graphs Γ_0 and Γ_1 of $P(L^0)$ and $P(L^1)$ respectively. We note that the fact that the algebraic structure of any planar algebra can be expressed by a principal graph (see [GHJ]) relies on the identification of the planar algebra with subfactors. Let $N_k^i, i = 0, 1$ be the set of vertices of Γ_i of distance k from the root, and let N_k be those vertices of distance k from the root of Γ. For a, b labeling nodes in a principal graph, we shall let $\langle a, b\rangle$ denote the number of lines connecting the two nodes. Let p_n, $n \in N_k^i$ be the projection in $P_k(L^i)$ onto the simple summand indexed by the node n. We have the following:

Theorem 8 *The principal graph Γ can be described as follows:*

1. *N_0 has one vertex, the root, which is connected to all the elements of N_1.*

2. *N_k is indexed by words of the form:*

$$n^0_{i_0} n^1_{i_1} n^2_{i_2} \dots n^l_{i_l}$$

where $a_0^{i_0} a_1^{i_1} a_0^{i_2} \dots a^{i_l}_{l \bmod 2}$ is a middle pattern that first appears in $FC_{1,k}$ (see [BiJo1] or [La2]), and $n^j_{i_j} \in N^{j \bmod 2}_{i_j}$.

3. *If $i_l = 1$, the vertex in N_k labeled by*

$$n^0_{i_0} n^1_{i_1} n^2_{i_2} \dots n^l_{i_l}$$

is connected to the vertices labeled

a)

$$n^0_{i_0} n^1_{i_1} n^2_{i_2} \dots n^{l-1}_{i_{l-1}} m^l_2 m^{l+1}_1$$

with $m_2^l \in N_2^{l \bmod 2}$, $m_1^{l+1} \in N_1^{l+1 \bmod 2}$, *where the multiplicity of the connection is* $\langle n_1^l, m_2^l \rangle$.

b)

$$n_{i_0}^0 n_{i_1}^1 n_{i_2}^2 \dots n_{i_{l-2}}^{l-2} m_{i_{l-1}+1}^{l-1} m_1^l$$

with $m_{i_{l-1}+1}^{l-1} \in N_{i_{l-1}+1}^{l-1 \bmod 2}$, $m_1^l \in N_1^{l \bmod 2}$, *where the multiplicity of the connection is* $\langle n_{i_{l-1}}^{l-1}, m_{i_{l-1}+1}^{l-1} \rangle$.

4. *If* $i_l \neq 1$, *the vertex in* N_k *labeled by*

$$n_{i_0}^0 n_{i_1}^1 n_{i_2}^2 \dots n_{i_l}^l$$

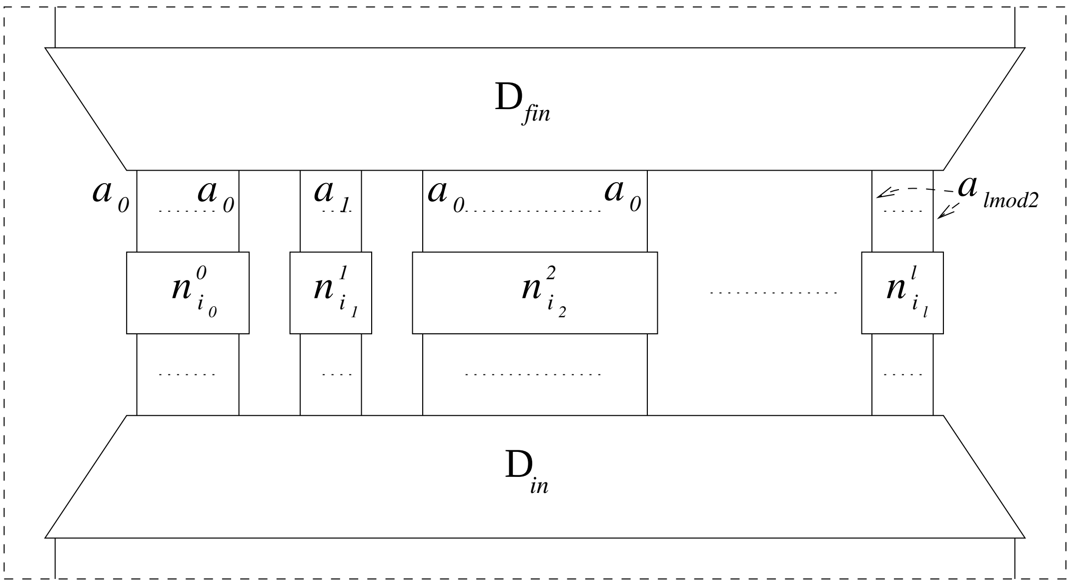
is connected to the vertices labeled

$$n_{i_0}^0 n_{i_1}^1 n_{i_2}^2 \dots n_{i_{l-1}}^{l-1} m_{i_l+1}^l m_1^{l+1}$$

with $m_{i_l+1}^l \in N_{i_l+1}^{l \bmod 2}$, $m_1^{l+1} \in N_1^{l+1 \bmod 2}$, *where the multiplicity of the connection is* $\langle n_{i_l}^l, m_{i_l+1}^l \rangle$.

Proof:

In what follows we have replaced the strand labels "a" and "b" that occur in Definition 5 by the strand labels "a_0" and "a_1". We shall use the notation and structural results of [BiJo1]. By the definition of the planar algebra $P = P^0 * P^1$, we see that the following pictures span P_k



where D ranges over basis elements of $FC_{1,k}$ and n_i^j ranges over elements in $P_i^{j mod 2}$. Of course, for this diagram to make sense we must have that $a_0^{i_0}a_1^{i_1}a_0^{i_2}\ldots a_{l\mathrm{mod}2}^{i_l}$ is the middle pattern of D. Because of the connection to subfactors, we know that the principal graph Γ at the kth level is just the algebraic structure of $P_k/(P_{k-1}e_{k-1}P_{k-1})$. (See description of principal graph in [GHJ].) Careful analysis of these pictures allows us to conclude that pictures of the following type are in $P_{k-1}e_{k-1}P_{k-1}$:

- Those where $a_0^{i_0}a_1^{i_1}a_0^{i_2}\ldots a_{l\mathrm{mod}2}^{i_l}$ appears as a middle pattern for an element in $FC_{1,k'}$ for some $k' < k$.

and

- Those where for some j, $n_{i_j}^j$ is orthogonal to the summands p_n for all $n \in N_k^{j\,\mathrm{mod}2}$. (That is, some $n_{i_j}^j$ without a component of "new stuff" inside $P_{i_j}^{j\,\mathrm{mod}2}$.)

With the above knowledge, we see that the projections onto the simple summands of $P_k/(P_{k-1}e_{k-1}P_{k-1})$ are given by the above pictures subject to the restrictions that

1. $a_0^{i_0}a_1^{i_1}a_0^{i_2}\ldots a_{l\mathrm{mod}2}^{i_l}$ is a middle pattern that first appears for an element in $FC_{1,k}$.
2. D is the unique projection in $FC_{1,k}$ with this middle pattern (see [BiJo1] for details as to uniqueness).
3. each $n_{i_j}^j = p_{n_{i_j}}$ for some $n_{i_j} \in N_{i_j}^{j\,\mathrm{mod}2}$.

The result follows. □

<u>Remark</u>

In the case when $P_1(L^0) = P_1(L^1) = \mathbb{C}$, an equivalent, and in some ways, simpler description of the principal graph Γ is a follows. Denote by Γ_0', (resp. Γ_1') the graph of Γ_0 (resp. Γ_1) with the root vertex removed.

1. Draw a copy of Γ_0, let the root of this copy of Γ_0 be the root of Γ.
2. Fuse a copy of Γ_1' to every vertex of odd distance from the root.

3. To every vertex of newly added copies of Γ'_1 that are of even distance from the root, fuse a copy of Γ'_0.

4. To every vertex of the newly added copies of Γ'_0 that are of odd distance from the root, fuse a copy of Γ'_1.

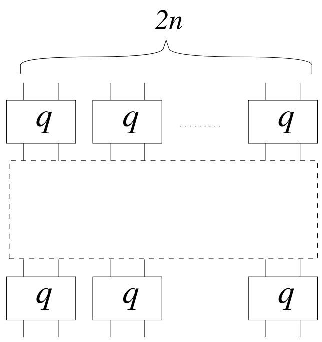
5. Return to step 3.

Given a subfactor with an intermediate subfactor $N \subset Q \subset M$, the free product provides some of the underlying structure for the planar algebra of $N \subset M$:

Theorem 9 *The free product of the planar algebras for $N \subset Q$ and $Q \subset M$ is a subplanar algebra of the planar algebra for $N \subset M$.*

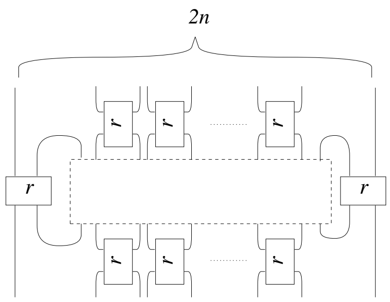
Proof:

Denote by ϕ the presenting map of $P^{N\subset M}$ on itself with partition function Z. We shall use the description of the planar algebras of $P^{N\subset Q}$ and $P^{Q\subset M}$ given in [BhLa]. Denote by $\boxed{q}$ the two box corresponding to the intermediate subfactor Q and set $\boxed{r} = (\frac{[M:Q]}{[Q:N]})^{\frac{1}{2}} \boxed{q}$. Let $B_{2n},\ B'_{2n} \in \mathcal{A}_{2n,2n}$ be the elements



,

and

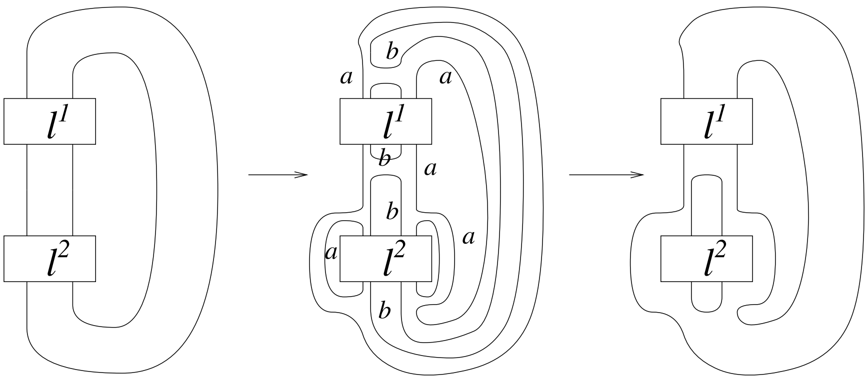


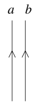
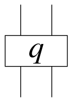
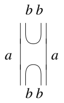
respectively. Set $L^1_{2i} = \{l \in P^{N\subset M}_{2i} : B_{2i}(l) = l\}$, $L^2_{2i} = \{l \in P^{N\subset M}_{2i} : B'_{2i}(l) = l\}$. Then Theorem 4.3 in [BhLa] gives a presentation of $P^{N\subset Q}$ and $P^{Q\subset M}$ with labeling sets $L^1 = \coprod L^1_{2i}$ and $L^2 = \coprod L^2_{2i}$ respectively. Denote by ϕ_* the presenting map for $P^{N\subset Q} * P^{Q\subset M}$ on the labeling set $L^1 \coprod L^2$. Let Z_* denote the corresponding partition function. We now establish the result by showing $Z(T) = Z_*(T)$ for any $T \in \mathcal{T}_0(L^1 \coprod L^2)$. This will suffice since annular invariance of planar algebras ensures that two planar algebras with the same partition function are isomorphic.

It follows directly from the definitions of $P^{N\subset Q}$, $P^{Q\subset M}$, and the free product, that $Z(T) = Z_*(T)$ for zero tangles T that have boxes labeled from L^1 or L^2 but not both. For an arbitrary zero tangle T, define $T_{(a,b)}$ to be the diagram obtained by performing steps 1-3 in Definition 5. Create a new zero tangle using the same standard boxes as those appearing in T as follows:

1. Delete all closed loops of $T_{(a,b)}$.
2. Delete all strands labeled by "b" that connect to diagrams with labels in L^1. (By construction these strands always join adjacent points of the same box.)
3. Delete all strands labeled by "a" that connect to diagrams with labels in L^2.
4. Ignore the "a" and "b" coloring of the rest of the strands.

What remains is a zero tangle which we shall call T'. We provide an example to clarify; here the first drawing is T, the middle is $T_{(a,b)}$, and the last is T'.



By construction, each connected component of T' contains boxes with labels from one of L^1 or L^2 but not both. Since both Z and Z_* are multiplicative on connected components, we have $Z(T') = Z_*(T')$. Let $\mathcal{T}$ (resp. $\mathcal{T}'$) be the unlabeled planar tangle corresponding to T (resp. T') with the labels of the boxes removed. We then have $\Pi_{\mathcal{T},\Phi_*} = [M:Q]^{c_1}[Q:N]^{c_2}\Pi_{\mathcal{T}',\Phi_*}$. This follows from applying steps 1-3 in Definition 5 to both $\mathcal{T}$ and $\mathcal{T}'$ and seeing that they differ only by loops of types "a" and "b" no matter what the labels of the boxes are. In addition, because the map that takes to  and  to $[M:Q]^{-\frac{1}{2}}$  is a planar isomorphism (see Theorem 3), we have that $\Pi_{\mathcal{T},\Phi} = [M:Q]^{c_1}[Q:N]^{c_2}\Pi_{\mathcal{T}',\Phi}$. Applying this to the labels of T and T', we have

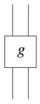
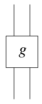
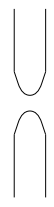
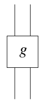
$$Z(T) = [M:Q]^{c_1}[Q:N]^{c_2}Z(T') = [M:Q]^{c_1}[Q:N]^{c_2}Z_*(T') = Z_*(T).\Box$$

9 New Planar Algebras

Given θ an automorphism of G, we will also denote by θ the action on $\mathbb{C}G$ given by the linear extension of $\theta(u_g) = u_{\theta(g)}$. The following result produces a large class of new planar algebras:

Theorem 10 *Let Θ be a collection of automorphisms of G. Let $F^{\Theta} \subset P_2(G) = \mathbb{C}G$ be the set of elements in $\mathbb{C}G$ fixed by all automorphisms in*

Θ *(i.e.,* $F^{\Theta} = \{x \in \mathbb{C}G : \theta(x) = x$ *for all* $\theta \in \Theta\}$*). Let* $P^{\Theta}(G)$ *be the subplanar algebra of* $P(G)$ *generated by* $L' = F^{\Theta}$*. We then have* $P_2^{\Theta}(G) \cong F^{\Theta}$.

Proof: We begin by examining a 2-tangle $\mathcal{T}$ containing k 2-boxes. Let Φ be the presenting map of $P(L, G)$ onto itself. Consider $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k)$. Because the group planar algebra is an exchange relation planar algebra on the label set G, we can use relations 00. $-$ 3. in Theorem 5 to reduce $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k)$ to a linear combination of labeled 2-tangles with no internal faces, namely a linear combination of elements of the form  ,  . In actuality, since relation 3. (Theorem 5) only involves one term, rather than a linear combination of terms, we have that $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k)$ can be reduced to a scalar times a single  or  . Though this fact is not essential, it will simplify notation. For convenience suppose $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k) = C$  , the same argument will hold for $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k) = C$  . We are interested in examining the form of C. The process of reducing $\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k)$ using 0. - 3. indicates that

$$C = \prod_{w \in W} \frac{\delta(w, id)}{|G|} \prod_{y \in Y} \chi(y)$$

where W, Y are certain sets of words in $\{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_k^{\pm 1}\}$ and $\chi(y) = 1$ for all words y (and is extended by linearity to all of $\mathbb{C}G$). The $w \in W$ (resp. $y \in Y$) correspond to those words labeling the 2-boxes that were involved in the reduction using relation 2. (resp. relation 1.). Furthermore the above g is also a word in $\{g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_k^{\pm 1}\}$ which we shall write as $g = \prod_{i=1}^{l} g_{f(i)}^{\sigma(i)}$ where $f(i) \in \{1, 2, \ldots k\}$ and $\sigma(i) \in \{\pm 1\}$. Now

$$\theta(\Pi_{\mathcal{T},\Phi}(g_1, \ldots, g_k)) = \theta(C u_g) = C \prod_{i=1}^{l} \theta(g_{f(i)}^{\sigma(i)}).$$

On the other hand

$$\Pi_{\mathcal{T},\Phi}(\theta(g_1),\ldots,\theta(g_k)) = C'\prod_{i=1}^{l}\theta(g_{f(i)}^{\sigma(i)}),$$

with

$$C' = \prod_{w\in W}\frac{\delta(\theta(w),id)}{|G|}\prod_{y\in Y}\chi(\theta(y)) = C$$

since θ is an automorphism. Thus

$$\Pi_{\mathcal{T},\Phi}(\theta(g_1),\ldots,\theta(g_k)) = \theta(\Pi_{\mathcal{T},\Phi}(g_1,\ldots,g_k)).$$

It follows since θ and $\Pi_{\mathcal{T},\Phi}$ are linear that if we replace the g_i with elements $m_i \in F^{\Theta}$ (i.e., invariant under θ) then

$$\Pi_{\mathcal{T},\Phi}(m_1,\ldots,m_k) = \Pi_{\mathcal{T},\Phi}(\theta(m_1),\ldots,\theta(m_k)) = \theta(\Pi_{\mathcal{T},\Phi}(m_1,\ldots,m_k))$$

Thus $\Pi_{\mathcal{T},\Phi}(m_1,\ldots,m_k) \in F^{\Theta}$ and the result follows. □

One of the significant consequences of the above theorem comes from translating the result into the language of subfactors. The above tells us that for every collection of automorphisms Θ of a finite group G, there exists a subfactor $N \subset M$ with $[M:N] = |G|$ and $N' \cap M_1 \cong F^{\Theta}$, the fixed point subalgebra of $\mathbb{C}G$ under all the automorphisms in Θ.

We note that an element in $\mathbb{C}G$ fixed under θ_1 and θ_2 is also fixed under θ_1^{-1} and $\theta_1 \circ \theta_2$ and therefore we need only look at Θ such that Θ is a subgroup of the group of automorphisms of G.

Let us take Θ to be a subgroup of the group of inner automorphisms, i.e. $\Theta = \{\theta_h : \theta_h(g) = hgh^{-1}, h \in H \text{ a subgroup of } G\}$. For such a set Θ we have $F^{\Theta} = \{\mathbb{C}H\}' \cap \mathbb{C}G$ and thus

Corollary 10.1 *For all finite groups G and subgroups H, there exist subfactors $N \subset M$ with $[M:N] = |G|$ and $N' \cap M_1 \cong \{\mathbb{C}H\}' \cap \mathbb{C}G$.*

One interesting question in subfactor theory is classifying the types of lattices that can occur as lattices of intermediate subfactors [Wa]. For subfactors $N \subset M$ with $N' \cap M = \mathbb{C}$, the lattice is known to be finite, but relatively little else is known about which, if any, lattices cannot appear as intermediate subfactor lattices. Theorem 10 leads to a class of lattices that do exist:

Theorem 11 *For Θ a subgroup of the automorphism group of G, let $\Theta_g = \{\theta(g) : \theta \in \Theta\}$ denote the orbit of g under the elements of Θ. Let $S = \{H \subset G : H$ a subgroup of $G, H = \bigcup_{h\in H} \Theta_h\}$. Then there exists an intermediate subfactor lattice that is the lattice of subgroups in S.*

Proof:

We first note that $\{id\}$ and G are always in S. We examine the intermediate subfactor lattice for an inclusion of subfactors with planar algebra $P^{\Theta}(G)$. By Corollary 2.1 we are interested in certain planar 2-tangles of $P^{\Theta}(G)$. Since $P^{\Theta}(G)$ is a subplanar algebra of $P(G)$, we only need to test whether

$$\frac{1}{|H|}\sum_{h\in H} u_h$$

is in $P^{\Theta}(G)$ for all subgroups H of G. (Since the only 2-tangle projections corresponding to intermediate subfactors of $M^G \subset M$ are given in this way.) Thus we need $\frac{1}{|H|}\sum_{h\in H} u_h \in F^{\Theta}$ which is equivalent to $H = \bigcup_{h\in H} \Theta_h$ and we are done. $\square$

We include the resulting corollary because of its ease to state.

Corollary 11.1 *The lattice of all normal subgroups of a finite group G is in the set of intermediate subfactor lattices.*

Proof: Apply Theorem 11 to the case when Θ is the set of inner automorphisms of G. $\square$

10 Examples: S_3, H, D_8, D_{2k+1}

In this section, we focus on $P^{\Theta}(G)$ for $G = S_3$, H, D_8, D_{2k+1}, and Θ the collection of all inner automorphisms of G.

Example $G = S_3$:

Denote the generators of S_3 by x and y with relations $x^2 = y^3 = 1$ and $xy = y^{-1}x$. We have $\mathbb{C}G \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ and thus by Theorem 11, $P_2^{\Theta}(G) \cong (\mathbb{C}G)' \cap \mathbb{C}G \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. The orbits under Θ are then $\Theta_1 = \{1\}$,

$\Theta_x = \{x, xy, xy^2\}$, $\Theta_y = \{y, y^2\}$. The projection

$$\frac{1}{3}(u_1 + u_y + u_{y^2})$$

is in $P_2^{\Theta}(G)$ and corresponds to an intermediate subfactor. Thus $FC_{1,n} \subset P_n^{\Theta}(G)$. But since $\dim(FC_{1,2}) = 3 = \dim(P_2^{\Theta}(G))$ we have $P_2^{\Theta}(G) = FC_{1,2}$. Since $P^{\Theta}(G)$ is generated by a labeling set consisting of only 2-boxes (i.e. $L_2 = F_2^{\Theta}(G)$), and $FC_{1,2}$ has all these 2-boxes, we must have that $P_n^{\Theta}(S_3) = FC_{1,n}$ the Fuss-Catalan algebra on one 2-box. □

We introduce some additional notation:

Given a planar algebra $P^{\Theta} = P^{N \subset M}$, suppose there is a projection $q \in P_2^{\Theta}$ corresponding to an intermediate subfactor Q, $N \subset Q \subset M$. We define $P^{\Theta,1}$ and $P^{\Theta,2}$ to be the planar algebras corresponding to the subfactor inclusions $N \subset Q$ and $Q \subset M$.

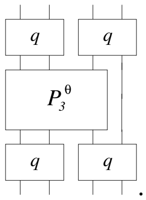
In addition we denote by $P^{Tl}(\alpha)$ the planar algebra with $P_i^{Tl}(\alpha) \cong \mathrm{FC}_{0,i}(\alpha)$, the Temperley-Lieb algebras with parameter α.

The technique for analyzing the planar algebras arising from $\Theta = H, D_8, D_{2n+1}$ will be the same. We first locate a projection $q \in P_2^{\Theta}$ corresponding to an intermediate subfactor (Corollary 2.1). We then analyze the structure of $P^{\Theta,1}$ and $P^{\Theta,2}$ using work in [BhLa]. Theorem 9 ensures that $P^{\Theta,1} * P^{\Theta,2}$ is a subplanar algebra of P^{Θ}. We show that in these cases it is in fact all of P^{Θ}.

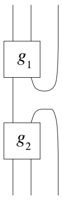
<u>Example $G = H$:</u>

Let the quaternion group have generators $\{\pm id, \pm i, \pm j, \pm k\}$ with relations $i^2 = j^2 = k^2 = -id$ and $ij = k$, $jk = i$, $ki = j$. Letting $\Theta = H$ we have $\Theta_{id} = \{id\}$, $\Theta_{-id} = \{-id\}$, $\Theta_i = \{\pm i\}$, $\Theta_j = \{\pm j\}$, $\Theta_k = \{\pm k\}$. Setting $a_i = \frac{1}{2}(-i) \oplus \frac{1}{2}(i)$, $a_j = \frac{1}{2}(-j) \oplus \frac{1}{2}(j)$, $a_k = \frac{1}{2}(-k) \oplus \frac{1}{2}(k)$, we have P^{Θ} is the subplanar algebra of $P(H)$ generated from the labeled two boxes $\{\pm id, a_i, a_j, a_k\}$ and thus $\dim(P_2^{\Theta}) = 5$. We have that $q = \frac{1}{2}(-id) \oplus \frac{1}{2}(id)$ satisfies the conditions of Corollary 2.1 and thus q corresponds to an intermediate subfactor. Using Corollary 3.5 of [BhLa], we have $P_2^{\Theta,1} \cong \langle q\, id\, q,\ q(-id)q,\ qa_iq,\ qa_jq,\ qa_kq\rangle = \langle q,\ a_i,\ a_j,\ a_k\rangle$ where the angled brackets (<>) are used to denote the subalgebra generated by the elements within them. It follows that $P_2^{\Theta,1} \cong \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

We now look at $P_3^{\Theta,1}$ which by Corollary 3.5 of [BhLa] is isomorphic to

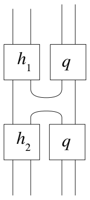


Now $P_3^{\Theta} \subset P_3(H)$, and from Theorem 6 we have $P_3(H)$ is linearly spanned by



$g_1, g_2 \in H.$

It follows that $P_3^{\Theta,1}$ is isomorphic to an algebra contained in the linear span of



$h_1, h_2 \in \{q, a_i, a_j, a_k\}.$

Thus $\dim(P_3^{\Theta,1}) \leq 16$. But the basic construction ensures $\dim(P_3^{\Theta,1}) \geq 16$ and thus $\dim(P_3^{\Theta,1}) = 16$ and $P^{\Theta,1}$ is a planar algebra coming from the depth two inclusion $M^{G'} \subset M$ with $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $P^{\Theta,1} = P(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Although with a little work we can show that $P^{\Theta,2} = P^{Tl}(\sqrt{2})$ we shall actually be able to conclude this without analyzing the structure of $P^{\Theta,2}$. Theorem 9 gives that $P^{\Theta,1} * P^{\Theta,2}$ is a subplanar algebra of $P^{\Theta}(H)$. $P^{Tl}(\sqrt{2})$ is a sub-planar algebra of $P^{\Theta,2}$ (P^{Tl} is a subplanar algebra of any nontrivial planar algebra P). We thus have $P^{\Theta,1} * P^{Tl}(\sqrt{2})$ is a sub-planar algebra of P^{Θ}. By Theorem 8

$$\dim(P^{\Theta,1} * P^{Tl}(\sqrt{2}))_2 = \dim(P_2^{\Theta,1}) + \dim(P_2^{Tl}(\sqrt{2})) - 1 = 4 + 2 - 1 = 5.$$

Since P^{Θ} is generated by a labeling set consisting only of 2- boxes, we can conclude $P^{\Theta} = P^{\Theta,1} * P^{Tl}(\sqrt{2})$ and $P^{Tl}(\sqrt{2}) = P^{\Theta,2}$.

<u>Example $G = D_8$:</u>

The same technique can be used to compute P^{Θ} for $\Theta = D_8$. Let D_8 be given by generators x and y with relations $x^2 = y^4 = 1$, $xy = y^{-1}x$. We have $\Theta_1 = \{1\}$, $\Theta_y = \{y, y^{-1}\}$, $\Theta_x = \{x, xy^2\}$, $\Theta_{y^2} = \{y^2\}$, $\Theta_{xy} = \{xy, xy^{-1}\}$. Setting $q = \frac{1}{2}(1) \oplus \frac{1}{2}(y^2) \in P_2^{\Theta}$, we have that q satisfies the conditions in Corollary 2.1. Surprisingly, the identical analysis to the case $\Theta = H$ (above) gives $P^{\Theta,1} = P(\mathbb{Z}_2 \times \mathbb{Z}_2)$. (Here

$$qP_2^{\Theta}q = \mathbb{C}\langle q,\ \frac{1}{2}(y) \oplus \frac{1}{2}(y^{-1}),\ \frac{1}{2}(x) \oplus \frac{1}{2}(xy^2),\ \frac{1}{2}(xy) \oplus \frac{1}{2}(xy^{-1})\rangle \cong \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2).)$$

Using the same dimension considerations as in the case $\Theta = H$ we conclude

$$P^{\Theta} = P(\mathbb{Z}_2 \times \mathbb{Z}_2) * P^{Tl}(\sqrt{2}) \text{ and } P^{\Theta,2} = P^{Tl}(\sqrt{2}).$$

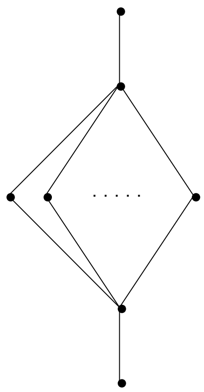
<u>Example $G = D_{2n+1}$:</u>

Let D_{2n+1} be given by generators x and y and relations $x^2 = y^{2n+1} = 1$, $xy = y^{-1}x$. Setting Θ to be all inner automorphisms of G we have $\Theta_1 = \{1\}$, $\Theta_{y^a} = \{y^a, y^{-a}\}$ $(1 \leq a \leq n)$, $\Theta_x = \{x, xy, xy^2, \ldots, xy^{2n}\}$. Thus $\dim(P_2^{\Theta}) = n + 2$. Setting $q = \frac{1}{2n+1}(1) \oplus \frac{1}{2n+1}(y) \oplus \cdots \oplus \frac{1}{2n+1}(y^{2n})$, we have $\dim(P_2^{\Theta,1}) = 2$, $\dim\ (P_2^{\Theta,2}) = n + 1$. Thus the identical dimensional considerations as in the above examples allow us to conclude that

$$P^{\Theta} = P^{\Theta,1} * P^{\Theta,2}$$

with $P^{\Theta,1} = P^{Tl}(\sqrt{2})$. It remains to analyze the structure of $P^{\Theta,2}$.

Lemma 1 *The planar algebra $P^{\Theta,2}$ has principal graph*



Proof: Recall that P^{Θ} is the subplanar algebra of the group algebra for $G = D_{2n+1}$ generated by the two boxes labeled by $\{1, x \oplus xy \oplus xy^2 \oplus \cdots \oplus xy^{2n}, y^a \oplus y^{-a} : 1 \leq a \leq n\}$. Let α be the action on G given by conjugation by x (i.e., $\alpha(y^a) = y^{-a}$, $\alpha(x) = x \ldots$). Set $S = \{1, y, y^2, \ldots y^{2n}, x \oplus xy \oplus xy^2 \oplus \cdots \oplus xy^{2n}\}$. Define Q to be the subplanar algebra of P^{Θ} generated by elements of the form

$$\Pi_{\mathcal{A},\Phi}(s_1, \ldots, s_k) + \Pi_{\mathcal{A},\Phi}(\alpha(s_1), \ldots, \alpha(s_k))$$

where

- Φ is the presenting map of the full group planar algebra $P(D_{2n+1})$.
- $\mathcal{A}$ ranges over all abstract planar n tangles that have only unlabeled 2 boxes ($n = 0, 1, 2 \ldots$).
- $s_1, s_2 \ldots s_k \in S$.

Because α is an algebra homomorphism, we have the following fact:

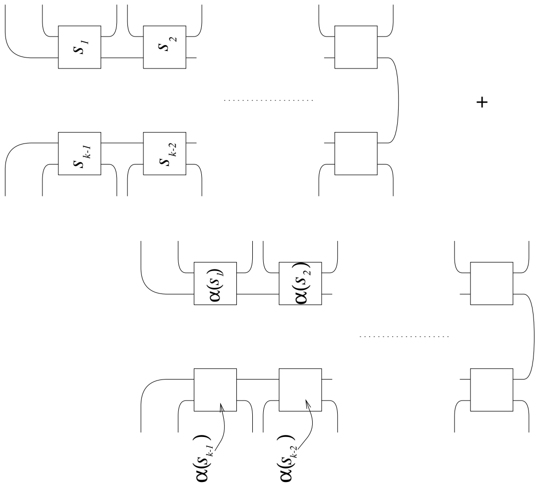
If using relations 00-4 in Theorem 5 establishes

$$\Pi_{\mathcal{A},\Phi}(g_1, \ldots, g_k) = \Pi_{\mathcal{A},\Phi}(g_1', \ldots, g_k')$$

then

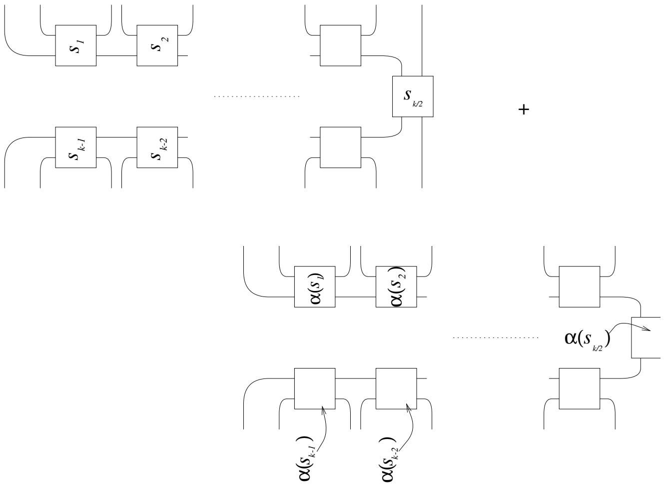
$$\Pi_{\mathcal{A},\Phi}(\alpha(g_1), \ldots, \alpha(g_k)) = \Pi_{\mathcal{A},\Phi}(\alpha(g_1'), \ldots, \alpha(g_k')).$$

Since the linear span of S is closed under multiplication, we have that Q_n ($n > 2$) is contained inside the linear span of elements of the form (see Theorem 6):



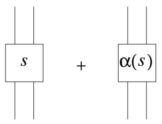
(*)

for n odd and



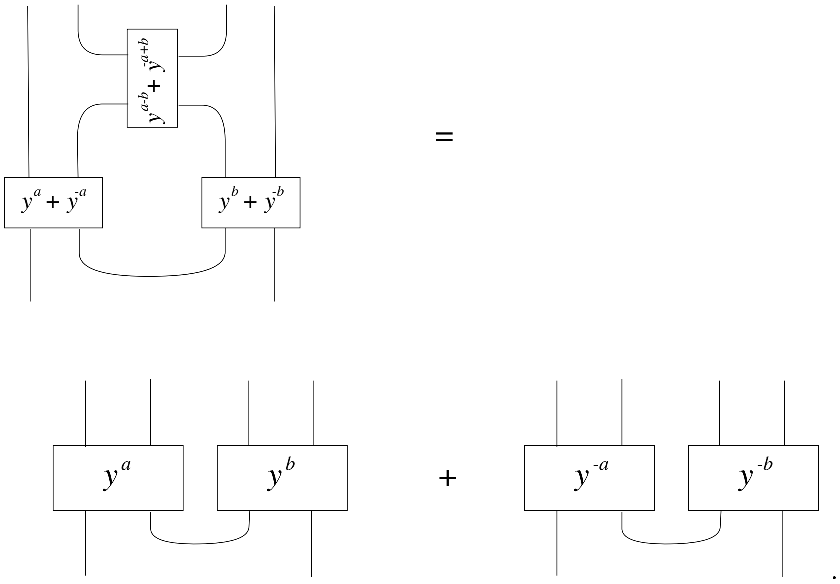
(**)

for n even.

In addition we have Q_2 is the linear span of  $s \in S$.

Recall that P^{Θ} is defined to be the planar algebra generated by the 2 boxes of Q and thus P^{Θ} is a subplanar algebra of Q. We now show that $P^{\Theta} = Q$ and that a spanning set for P_n^{Θ} is given by (*) and (**).

Using Theorem 5, it is straightforward to verify that



Applying this relation inductively on n we have that the elements in (*) and (**) are in P^{Θ} and thus $P^{\Theta} = Q$ with spanning sets for P_n^{Θ} given as above.

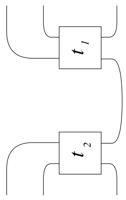
We can now use Corollary 3.5 to conclude that $P_n^{\Theta,2}$ is isomorphic to the algebra (with vertical concatenation as multiplication) spanned by the elements of the form (*) and (**) (previous page) *with* the added restriction that $s_i \in \{1, y, y^2, \dots, y^{2n}\}$.

Note that every choice of $\{s_1, s_2, \dots s_k\}$ in (*) and (**) yields the same element as the choice $\{\alpha(s_1), \alpha(s_2), \dots \alpha(s_k)\}$. These two choices are distinct with the exception of $s_1 = s_2 = \cdots = s_k = 1$ and thus

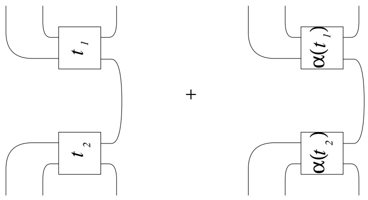
$$dim(P_n^{\Theta,2}) = \frac{1}{2}((2n+1)^{n-1} - 1) + 1, \qquad n \geq 1.$$

(We note that we used independence of the elements pictured in Theorem 6 to yield independence of the elements pictured above.)

We will now do explicit analysis of the algebras $P_2^{\Theta,2}$ and $P_3^{\Theta,2}$ and then use dimensional considerations to compute the full principal graph. Since $P_2^{\Theta,2}$ is abelian we have $P_2^{\Theta,2} \cong \mathbb{C}^{n+1}$. Analyzing the structure of $P_3^{\Theta,2}$, we see $P_3^{\Theta,2} \cong M_{n+1} \oplus M_n$ with the matrix units for M_{n+1} given by

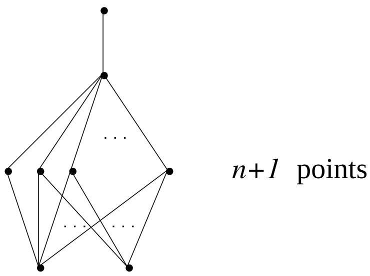


$t_1, t_2 \in \{1, (y^a + y^{-a})/2 : 1 \leq a \leq n\}$ and the matrix units for M_n given by



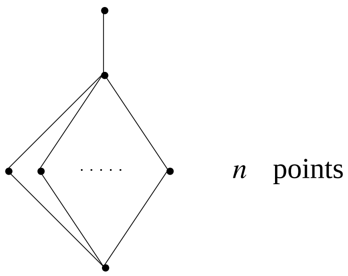
$t_1, t_2 \in \{y, y^2, \ldots y^n\}$.

We can conclude that the Bratteli diagram for $P_0^{\Theta,2} \subset P_1^{\Theta,2} \subset P_2^{\Theta,2} \subset P_3^{\Theta,2}$ is given by



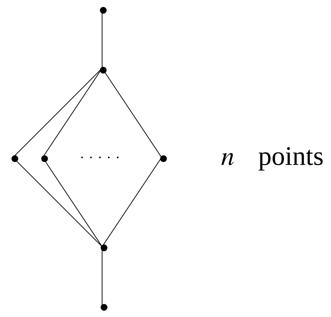
.

Thus the first levels of the principal graph are



.

This partial principal graph gives that the "old stuff" (see [GHJ] for a discussion of the structure of the principal graph) of $P_4^{\Theta,2}$ has dimension one less than the dimension of $P_4^{\Theta,2}$, and thus the only possible principal graph up to the fourth level is :



This in turn gives that the "old stuff" in $P_5^{\Theta,2}$ has dimension equal the dimension of all of $P_5^{\Theta,2}$. Thus $P^{\Theta,2}$ comes from a depth 4 subfactor inclusion and we have our result. $\Box$

11 Future Work

We conclude by briefly highlighting several areas for future work:

Examples

Theorem 10 gives a construction of a planar algebra given a group G and a subgroup, Θ, of automorphisms of G. Section 10 is clearly just the beginning of the analysis of these planar algebras. It would be nice to know the planar algebra structure for other groups of small order as well as for different subgroups of automorphisms.

Structure

Part of the proof of Lemma 1, (namely the proof that $P^{\Theta} = Q$) suggests a technique for giving the dimension as well as part of the structure for other planar algebras given by Theorem 10. The author hopes similar analysis in a future paper will yield more general structural results.

Exchange Relation Planar Algebras

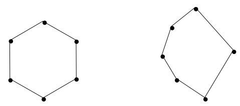
We have shown that the Fuss-Catalan algebras as well as the planar algebras arising from depth two subfactors are exchange relation planar algebras. It would be interesting to know of other examples.

Hopf *- Algebras and Exchange Relation Planar Algebras

As shown in Theorem 4, all depth two subfactor inclusions yield exchange relation planar algebras. Depth two subfactors are isomorphic to $M^H \subset M$ for H a Hopf *-algebra. In future work, the author will make explicit the relation between the exchange relation in the planar algebra and the comultiplication in H.

Intermediate Subfactor Lattices

As discussed in [Wa], it is an open question as to what kinds of lattices of intermediate subfactors can an irreducible (i.e. $N' \cap M = \mathbb{C}$) inclusion $N \subset M$ have. Though Theorem 11 adds many lattices to the list of those that do occur, to the author's knowledge it is still an open question whether, for instance, the two lattices



can occur as intermediate subfactor lattices.

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