Deficits and Excesses of Frames

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The excess of a sequence in a Hilbert space is the greatest number of elements that can be removed yet leave a set with the same closed span. We study the excess and the dual concept of the deficit of Bessel sequences and frames. In particular, we characterize those frames for which there exist infinitely many elements that can be removed from the frame yet still leave a frame, and we show that all overcomplete Weyl-Heisenberg and wavelet frames have this property.

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1. Introduction

Let H be a separable Hilbert space and I a countable index set. A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ of elements of H is a *frame* for H if there exist constants A, $B > 0$ such that

$$\forall\, h \in H, \quad A\,\|h\|^2 \;\le\; \sum_{i\in I} |\langle h, f_i\rangle|^2 \;\le\; B\,\|h\|^2. \tag{1}$$

The numbers A, B are called *lower* and *upper frame bounds*, respectively (the largest A and smallest B for which (1) holds are the *optimal* frame bounds). Frames were first introduced by Duffin and Schaeffer [5] in the context of non-

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harmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering. We refer to the monograph of Daubechies [4] or the research-tutorial [8] for basic properties of frames.

Each frame $\mathcal{F}$ is complete in H, i.e., the finite linear span of $\mathcal{F}$ is dense in H. Moreover, a frame provides basis-like representations of the elements of H. Specifically, there exist vectors $\tilde{f}_i$ such that

$$\forall\, h \in H, \quad h \;=\; \sum_{i \in I} \langle h, f_i \rangle\, \tilde{f}_i \;=\; \sum_{i \in I} \langle h, \tilde{f}_i \rangle\, f_i, \tag{2}$$

with unconditional convergence of these series. In general, however, a frame need not be a basis, and the representations in (2) need not be unique. Frames which are not bases are overcomplete, i.e., there exist proper subsets of the frame which are complete [5]. The *excess* of the frame is the greatest integer n such that n elements can be deleted from the frame and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed. In the former case, it can be shown that the frame is simply a Riesz basis to which finitely many elements have been adjoined [9]. Such frames are called "near Riesz bases" and behave in many respects like Riesz bases. A frame with infinite excess need not contain a Riesz basis as a subset; an example was constructed in [3] and is discussed in Example 5.1.

In this paper we will study the excess of frames and of more general systems, and the dual concept of the deficit of a system (the minimum number of elements that must be adjoined to obtain a complete set). Our motivation was the particular case of *Weyl–Heisenberg* or *Gabor* frames. These are frames for the Hilbert space $L^2(\mathbf{R})$ of the form $\{e^{2\pi i m\beta x}\, g(x - n\alpha)\}_{m,n \in \mathbf{Z}}$, where $g \in L^2(\mathbf{R})$ and $\alpha,\, \beta > 0$. The Balian–Low Theorem states that if a Weyl–Heisenberg frame is a Riesz basis for $L^2(\mathbf{R})$, then the window function g must be poorly localized in either time or frequency, specifically, $\|tg(t)\|_2\, \|\omega \hat{g}(\omega)\|_2 = \infty$ [4]; see also [1] for an "amalgam space" variation. Thus, the most useful Weyl–Heisenberg frames are overcomplete. It can be shown that if $\alpha\beta > 1$ then any Weyl–Heisenberg system is incomplete, if $\alpha\beta = 1$ then a Weyl–Heisenberg frame is a Riesz basis, and if $\alpha\beta < 1$ then a Weyl–Heisenberg frame is overcomplete, cf. [13], [12], [4].

It was shown in [7, Prop. 7.1.3] that if g generates an overcomplete Weyl–Heisenberg frame and is compactly supported with support contained in an interval of length $1/\beta$, then the frame has infinite excess. The question of whether *every* overcomplete Weyl–Heisenberg frame has infinite excess motivated the research for this paper. We prove in this paper that this is the case, and in fact we obtain a much stronger result: in any overcomplete Weyl–Heisenberg frame it is possible to find an infinite subset that can be deleted yet leave a frame (not merely a complete set), and furthermore we can specify the frame bounds of the resulting system. Moreover, we obtain this result as a corollary of more general results on the excesses and deficits of Bessel sequences and arbitrary frames, and we also obtain as corollaries statements about wavelet frames.

The organization of our paper and a sketch of the main results is as follows. In Section 2, we present basic notation and definitions. In Section 3, we show that if $\mathcal{F}$ is any complete sequence in a Banach space which has infinite excess, meaning that for any n there exists a finite subset $\mathcal{G}_n$ of cardinality n such that $\mathcal{F} \setminus \mathcal{G}_n$ is complete, then there actually exists a countably infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. We remark that it is not true that $\mathcal{F} \setminus \cup \mathcal{G}_n$ will necessarily be complete, even if the $\mathcal{G}_n$ are nested.

In Section 4, we restrict to the case of Bessel sequences in Hilbert spaces, i.e., sequences which at least satisfy the upper frame condition. We relate the deficit and excess of a Bessel sequence to the dimension of the kernels of the analysis operator T and synthesis operator T^* associated with the Bessel sequence. We show that if there exists a pair of operators Q, L that intertwine with T, i.e., $LT = TQ$, then the structure of the point spectrum of these operators induces restrictions on the deficit and excess of the sequence. In particular, if Q has no point spectrum then the deficit is either 0 or ∞, while if L^* has no point spectrum and $\mathcal{F}$ is a frame, then the excess is either 0 or ∞.

In Section 5, we further restrict to the case of frames in Hilbert spaces. It was proved by Duffin and Schaeffer [5] that if $\mathcal{F}$ is a frame for H and $f \in \mathcal{F}$ is such that $\mathcal{F} \setminus \{f\}$ is complete in H, then $\mathcal{F} \setminus \{f\}$ is a frame for H. We prove that if there exist infinitely many elements $g_n \in \mathcal{F}$ such that $\mathcal{F} \setminus \{g_n\}$ is complete for each individual n and if there is a uniform lower frame bound L for each frame $\mathcal{F} \setminus \{g_n\}$, then for each $\varepsilon > 0$ there exists an infinite subset $\mathcal{G}_\varepsilon$ of $\{g_n\}_{n \in \mathbf{N}}$ such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L - \varepsilon$. Moreover, we show that the existence of such elements g_n is necessary as well as sufficient in order that an infinite set may be deleted yet leave a frame, and we provide an example of a frame with infinite excess where such a collection of elements g_n yielding a uniform lower frame bound for each $\mathcal{F} \setminus \{g_n\}$ does not exist. We further show that the existence of such elements g_n can be determined from the values of the inner products of the frame elements with the standard dual frame elements.

Finally, in Section 6 we apply our results to the specific cases of Weyl–Heisenberg and wavelet systems. We prove that any Weyl–Heisenberg or wavelet system that is an overcomplete frame for its closed linear span contains an infinite subset that can be deleted yet still leave a frame for the same space. We extend these results to the case of Weyl–Heisenberg multisystems whose generating parameters are rationally related, or to wavelet multisystems whose dilation parameters are logarithmically rationally related. A sequel paper will examine the case of systems where these rationality assumptions are not satisfied.

2. Notation

$\mathbf{N}$ will denote the set of natural numbers, while I will denote a generic countable index set. $|E|$ denotes the cardinality of a set E.

Let X be a Banach space and let $\mathcal{F} = \{f_i\}_{i \in I}$ be a sequence of elements of X. The finite linear span of $\mathcal{F}$ is denoted by $\mathrm{span}(\mathcal{F})$, and $\overline{\mathrm{span}}(\mathcal{F})$ denotes the closure (in the norm-topology of X) of $\mathrm{span}(\mathcal{F})$. We say that $\mathcal{F}$ is complete if $\overline{\mathrm{span}}(\mathcal{F}) = X$.

A sequence $\mathcal{F} = \{f_i\}_{i \in I}$ in a separable Hilbert space H is a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\forall\, h \in H, \quad \sum_{i \in I} |\langle h, f_i \rangle|^2 \le B \|h\|^2.$$

Associated to any Bessel sequence are the *analysis operator* T defined by

$$\begin{aligned} T\colon H &\to \ell^2(I) \\ h &\mapsto \{\langle h, f_i \rangle\}_{i \in I} \end{aligned}$$

and the *synthesis operator* T^* defined by

$$\begin{aligned} T^*\colon \ell^2(I) &\to H \\ c \quad &\mapsto \sum_{i \in I} c_i f_i. \end{aligned}$$

These are everywhere-defined, bounded operators, each adjoint to the other. If $c \in \ell^2(I)$, then the series $\sum c_i f_i$ defining $T^* c$ converges unconditionally in the norm of H. Since $\mathrm{span}(\mathcal{F}) \subset \mathrm{ran}\, T^* \subset \overline{\mathrm{span}}(\mathcal{F})$, we have $\overline{\mathrm{span}}(\mathcal{F}) = \overline{\mathrm{ran}\, T^*}$. The elements of a Bessel sequence are uniformly bounded above in norm, specifically, $\|f_i\|^2 \le B$ for each $i \in I$.

Frames are special cases of Bessel sequences. The utility of a frame lies in the fact that there exists a dual frame $\{\tilde{f}_i\}_{i\in I}$ such that the frame expansions in (2) hold (this fails in general for Bessel sequences). The *standard dual frame* is given by $\tilde{f}_i = S^{-1}f_i$, where $S = T^*T$ is the *frame operator.* The frame operator is a positive, continuously invertible mapping of H onto itself, with $AI \le S \le BI$. A frame is *tight* if it is possible to take $A = B$ in (1), *normalized tight* if $A = B = 1$ (but note that some authors define a normalized frame to be one where $\|f_i\| = 1$ for every $i \in I$). Since S is a positive operator, it has a positive square root $S^{1/2}$. Moreover, $S^{-1/2}$ is a bounded, continuously invertible operator and $\{S^{-1/2}f_i\}_{i\in I}$ is a normalized tight frame for H [7, Cor. 6.3.5], [2, Thm. III.2]. Thus every frame is equivalent to a normalized tight frame.

A *Riesz sequence* is a sequence $\mathcal{F} = \{f_i\}_{i\in I}$ for which there exist $A,\, B > 0$ such that

$$\forall\, c \in \ell^2(I), \quad A\sum_{i\in I}|c_i|^2 \le \Bigl\|\sum_{i\in I} c_i f_i\Bigr\|^2 \le B\sum_{i\in I}|c_i|^2.$$

If a Riesz sequence is complete then it is called a *Riesz basis* for H. All Riesz bases are frames. If $\mathcal{F}$ is a Riesz basis, then for each $h \in H$ the frame expansion given in (2) is unique. A frame is a Schauder basis for H if and only if it is a Riesz basis for H.

Definition 2.1. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a sequence in a separable Banach space X.

a. The *deficit* of $\mathcal{F}$ is

$$d(\mathcal{F}) \;=\; \inf\{|\mathcal{G}| : \mathcal{G} \subset X \text{ and } \overline{\operatorname{span}}(\mathcal{F}\cup\mathcal{G}) = X\}.$$

That is, the deficit is the least cardinal $d(\mathcal{F})$ such that there exists a subset $\mathcal{G} \subset X$ of cardinality $d(\mathcal{F})$ so that $\mathcal{F}\cup\mathcal{G}$ is complete in X.

b. The *excess* of $\mathcal{F}$ is

$$e(\mathcal{F}) \;=\; \sup\{|\mathcal{G}| : \mathcal{G} \subset \mathcal{F} \text{ and } \overline{\operatorname{span}}(\mathcal{F}\setminus\mathcal{G}) = \overline{\operatorname{span}}(\mathcal{F})\}. \tag{3}$$

We will show in Lemma 4.1 that the supremum in (3) is achieved, i.e., the excess is the greatest cardinal $e(\mathcal{F})$ such that there exists a subset $\mathcal{G} \subset \mathcal{F}$ of cardinality $e(\mathcal{F})$ so that $\mathcal{F}\setminus\mathcal{G}$ is complete in $\overline{\operatorname{span}}(\mathcal{F})$.

Note that a frame for a Hilbert space H has zero deficit, whereas a Riesz sequence in H has zero excess. The converses of these statements are not true in general. However, it is true that if a frame has zero excess, then it is a Riesz basis for H [5].

3. Arbitrary Sequences

In this section we will show that if a complete sequence $\mathcal{F}$ in a Banach space X has infinite excess, then there exists a countably infinite subset $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete in X. First, however, consider the following trivial example.

Example 3.1. Let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal basis for a Hilbert space H. Then $\mathcal{F} = \{2^{-m/2} e_n\}_{m,n\in\mathbf{N}}$ is a normalized tight frame with infinite excess. Let $\mathcal{F} = \{f_n\}_{n\in\mathbf{N}}$ be any enumeration of $\mathcal{F}$, and set $\mathcal{G}_n = \{f_1, \dots, f_n\}$. Then $\mathcal{F} \setminus \mathcal{G}_n$ is complete in H for every n, yet $\mathcal{F} \setminus \cup\mathcal{G}_n = \emptyset$.

Clearly, in this example there does exist an infinite set $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. We show in the following lemma that whenever there exist increasing finite nested subsets which can be deleted from a sequence $\mathcal{F}$ yet leave a complete set, then is in fact possible to find an infinite subset that can be deleted yet leave a complete set.

Lemma 3.2. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a sequence in a Banach space X, and assume that there exists a subsequence $\{g_n\}_{n\in\mathbf{N}}$ such that $\mathcal{F} \setminus \{g_1, \dots, g_n\}$ is complete in X for each $n \in \mathbf{N}$. Then there exists an infinite subsequence $\mathcal{G}$ of $\{g_n\}_{n\in\mathbf{N}}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete in X.

Proof. Let $E = \mathcal{F} \setminus \{g_n\}_{n\in\mathbf{N}}$. Let $k_1 = 1$. Since $E \cup \{g_n\}_{n=2}^{\infty} = \mathcal{F} \setminus \{g_1\}$ is complete, there exists $k_2 > k_1$ such that

$$\operatorname{dist}\big(g_{k_1}, \overline{\operatorname{span}}(E \cup \{g_n\}_{n=2}^{k_2-1})\big) \ < \ \frac{1}{2},$$

where $\operatorname{dist}(x, Y) = \inf\{\|x - y\| : y \in Y\}$ is the distance from a vector x to a subset Y of X. Since $E \cup \{g_n\}_{n=k_2+1}^{\infty} = \mathcal{F} \setminus \{g_1, \dots, g_{k_2}\}$ is complete, there exists $k_3 > k_2$ such that both

$$\operatorname{dist}\big(g_{k_1}, \overline{\operatorname{span}}(E \cup \{g_n\}_{n=k_2+1}^{k_3-1})\big) \ < \ \frac{1}{3}$$

and

$$\operatorname{dist}\big(g_{k_2}, \overline{\operatorname{span}}(E \cup \{g_n\}_{n=k_2+1}^{k_3-1})\big) \ < \ \frac{1}{3}.$$

Continuing in this way we find $k_1 < k_2 < \cdots$ such that for each $\ell \in \mathbf{N}$ we have

$$\operatorname{dist}\bigl(g_{k_j},\, \overline{\operatorname{span}}(E \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1})\bigr) \;<\; \frac{1}{\ell+1}, \qquad j = 1, \dots, \ell. \tag{4}$$

Let $\mathcal{G} = \{g_{k_j}\}_{j=1}^{\infty}$. We claim that $\mathcal{F} \setminus \mathcal{G}$ is complete. Since $\mathcal{F}$ is complete, it suffices to show that

$$\forall\, j \in \mathbf{N}, \quad \operatorname{dist}\bigl(g_{k_j},\, \overline{\operatorname{span}}(\mathcal{F} \setminus \mathcal{G})\bigr) \;=\; 0. \tag{5}$$

Since $E \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1} \subset \mathcal{F} \setminus \mathcal{G}$, we have from (4) that for all $\ell \geq j$,

$$\operatorname{dist}\bigl(g_{k_j},\, \overline{\operatorname{span}}(\mathcal{F} \setminus \mathcal{G})\bigr) \;\leq\; \operatorname{dist}\bigl(g_{k_j},\, \overline{\operatorname{span}}(E \cup \{g_n\}_{n=k_\ell+1}^{k_{\ell+1}-1})\bigr) \;<\; \frac{1}{\ell+1}.$$

Hence (5) holds and the proof is complete. $\square$

Next, we show that it is possible to remove the hypothesis of nestedness in Lemma 3.2. Consequently, in every sequence with infinite excess there exists an infinite subsequence that can be deleted yet leave a complete set.

If S is a subspace of a Banach space X, then $\dim(S)$ denotes the dimension of a subspace S (either finite or ∞). The codimension of S is $\operatorname{codim}(S) = \dim(T)$ where T is any algebraic complement of S, i.e., any subspace such that $S+T = X$ and $S \cap T = \{0\}$. The codimension of S is independent of the choice of subspace T.

Theorem 3.3. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a complete sequence in a Banach space X with infinite excess. Then there exists an infinite subsequence $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete in X.

Proof. We claim that there must exist a subsequence $\{g_n\}_{n \in \mathbf{N}}$ of $\mathcal{F}$ such that $\mathcal{F} \setminus \{g_1, \dots, g_n\}$ is complete in X for each $n \in \mathbf{N}$. Once this is shown, the result then follows immediately from Lemma 3.2.

If no such subsequence existed, there would exist at least one maximal finite subset $G = \{g_1, \dots, g_n\}$ of $\mathcal{F}$ such that $\mathcal{F} \setminus G$ is complete. Since $\mathcal{F}$ has infinite excess, there must also exist a finite subset $H = \{h_1, \dots, h_m\}$ of $\mathcal{F}$ with $m \geq 2n$ such that $\mathcal{F} \setminus H$ is complete. Since G is maximal, we cannot have $G \subset H$. Hence $G \cap H$ contains at most $n-1$ elements and $H \setminus G$ contains at least $n+1$ elements.

Let $E = \mathcal{F} \setminus (G \cup H)$. Since $E \cup (G \setminus H) = \mathcal{F} \setminus H$ and $E \cup (H \setminus G) = \mathcal{F} \setminus G$ are both complete, we have that

$$\overline{\operatorname{span}}(E) + \operatorname{span}(G \setminus H) \;=\; X \tag{6}$$

and

$$\overline{\operatorname{span}}(E) + \operatorname{span}(H \setminus G) \;=\; X. \tag{7}$$

It follows from (6) that

$$\operatorname{codim}(\overline{\operatorname{span}}(E)) \;\le\; |G \setminus H| \;\le\; n.$$

Combining this with (7) implies that $\operatorname{span}(H \setminus G)$ contains an algebraic complement of $\overline{\operatorname{span}}(E)$ of dimension at most n. Since $|H \setminus G| \ge n+1$, at least one element $h \in H \setminus G$ must lie in the closed span of the union of E and the remaining elements of $H \setminus G$. But then $E \cup \big(H \setminus (G \cup \{h\})\big) = \mathcal{F} \setminus (G \cup \{h\})$ is complete, which contradicts the maximality of G. □

4. Bessel Sequences

In this section we consider the deficits and excesses of Bessel sequences in a Hilbert space. The following result connects the excess and deficit to the dimension of the kernels of the analysis and synthesis operators.

Lemma 4.1. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a Bessel sequence in a separable Hilbert space H, and let $T \colon H \to \ell^2(I)$ be the associated analysis operator.

a. $d(\mathcal{F}) = \dim(\ker T)$.

b. $e(\mathcal{F}) \ge \dim(\ker T^*)$.

c. If $\mathcal{F}$ is a frame then $e(\mathcal{F}) = \dim(\ker T^*)$.

Proof. a. This follows immediately from the fact that $\big(\overline{\operatorname{span}}(\mathcal{F})\big)^{\perp} = \big(\overline{\operatorname{ran} T^*}\big)^{\perp} = \ker T$.

b. For simplicity of notation, let $I = \mathbf{N}$. Let $y_1, \dots, y_m$ be linearly independent sequences in $\ker T^*$, and write $y_j = (y_{j,i})_{i \in \mathbf{N}}$. Then

$$T^* y_j \;=\; \sum_{i=1}^{\infty} y_{j,i} f_i \;=\; 0, \qquad j = 1, \dots, m, \tag{8}$$

or, in terms of an infinite matrix equation,

$$\begin{bmatrix} y_{1,1} & y_{1,2} & \cdots \\ \vdots & \vdots & \cdots \\ y_{m,1} & y_{m,2} & \cdots \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The matrix on the left above has row rank m, hence has column rank m by the same Gaussian elimination argument used for finite matrices. Let $F = \{k_1, \dots, k_m\}$ denote the indices of a set of m independent columns. We claim that $\{f_i\}_{i \in \mathbf{N} \setminus F}$ is complete in $\overline{\operatorname{span}}(\mathcal{F})$.

Suppose that $h \in \overline{\operatorname{span}}(\mathcal{F})$ satisfies $\langle f_i, h\rangle = 0$ for $i \in \mathbf{N} \setminus F$. Then from (8) we have

$$0 \;=\; \langle T^* y_j, h\rangle \;=\; \sum_{i=1}^{\infty} y_{j,i}\, \langle f_i, h\rangle \;=\; \sum_{i=1}^{m} y_{j,k_i}\, \langle f_{k_i}, h\rangle, \qquad j = 1, \dots, m.$$

That is,

$$\begin{bmatrix} y_{1,k_1} & \cdots & y_{1,k_m} \\ \vdots & \ddots & \vdots \\ y_{m,k_1} & \cdots & y_{m,k_m} \end{bmatrix} \begin{bmatrix} \langle f_{k_1}, h\rangle \\ \vdots \\ \langle f_{k_m}, h\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, the matrix on the left-hand side is invertible, so this implies that $\langle f_{k_j}, h\rangle = 0$ for $j = 1, \dots, m$. Hence $\langle f_i, h\rangle = 0$ for all $i \in \mathbf{N}$, so $h = 0$. Thus $\{f_i\}_{i \in \mathbf{N} \setminus F}$ is complete, so $e(\mathcal{F}) \geq m$.

c. If $\dim(\ker T^*) = \infty$, then $e(\mathcal{F}) = \infty$ by part b. If $\dim(\ker T^*) < \infty$, then the fact that $\dim(\ker T^*) = e(\mathcal{F})$ follows from Theorems 2.4 and 3.1 in [9]. □

Example 4.2. If $\mathcal{F}$ is a Bessel sequence that is not a frame, then it is possible that $e(\mathcal{F})$ can strictly exceed $\dim(\ker T^*)$. For example, let $\{e_n\}_{n \in \mathbf{N}}$ be an orthonormal basis for a Hilbert space H, and set $f = \sum_{n=1}^{\infty} e_n/n$. Then $\mathcal{F} = \{e_n/n\}_{n \in \mathbf{N}} \cup \{f\}$ is a Bessel sequence but is not a frame, and it is easy to see that $e(\mathcal{F}) = 1$ while $\dim(\ker T^*) = 0$. It is similarly possible to construct Bessel sequences where $e(\mathcal{F})$ is any specified finite value or infinity yet $\dim(\ker T^*) = 0$. In Example 6.7 we exhibit a Weyl–Heisenberg Bessel sequence which satisfies $e(\mathcal{F}) = 1$ and $\dim(\ker T^*) = 0$.

Next we will show that with some additional structural assumptions on the Bessel sequence, we can obtain more concrete information on the excess and deficit of the sequence.

Definition 4.3. Let $\mathcal{F}$ be a Bessel sequence in a Hilbert space H with associated analysis operator $T\colon H \to \ell^2(I)$. If there exists a pair (Q, L) of bounded operators $Q\colon H \to H$ and $L\colon \ell^2(I) \to \ell^2(I)$ such that

$$LT \;=\; TQ, \tag{9}$$

then we call (Q, L) an *intertwining pair* of operators for $\mathcal{F}$.

It follows immediately that if (9) holds then:

a. $\ker T$ is Q-invariant,

b. $\ker T^*$ is L^*-invariant,

c. $\overline{\operatorname{ran} T}$ is L-invariant,

d. $\overline{\operatorname{ran} T^*}$ is Q^*-invariant.

Therefore, in light of Lemma 4.1, if an intertwining pair of operators exists, then the excess and deficit of $\mathcal{F}$ are realized as dimensions of invariant subspaces associated with Q and L^*. Now, if N is an operator on H which has no point spectrum (i.e., there are no values $\lambda \in \mathbf{C}$ such that $\ker(N - \lambda I) \neq \{0\}$), then all non-trivial invariant subspaces of N must be infinite-dimensional. Indeed, suppose that E was a finite-dimensional invariant subspace. Then $N|_E$ maps the finite-dimensional space E into itself, hence must have an eigenvalue λ with eigenvector $x \in E$. But then λ is also an eigenvalue of N, contradicting the fact that N has no point spectrum. An operator with no point spectrum is said to have a purely continuous spectrum. Combining these remarks with Lemma 4.1, we obtain the following.

Theorem 4.4. Assume that there exists an intertwining pair of operators (Q, L) for a Bessel sequence $\mathcal{F}$ in a separable Hilbert space H.

a. If Q^* has no point spectrum, then either $\dim(\overline{\operatorname{span}}(\mathcal{F})) = 0$ or $\dim(\overline{\operatorname{span}}(\mathcal{F})) = \infty$.

b. If Q has no point spectrum, then either $d(\mathcal{F}) = 0$ or $d(\mathcal{F}) = \infty$.

c. If L^* has no point spectrum and $\mathcal{F}$ is a frame, then either $e(\mathcal{F}) = 0$ or $e(\mathcal{F}) = \infty$.

Proof. a. If Q^* has no point spectrum, then since $\overline{\operatorname{span}}(\mathcal{F}) = \overline{\operatorname{ran} T^*}$ is Q^*-invariant, it must be either $\{0\}$ or infinite-dimensional.

b. If Q has no point spectrum, then since $\ker T$ is Q-invariant, it must be either $\{0\}$ or infinite-dimensional. Hence $d(\mathcal{F}) = \dim(\ker T)$ is either 0 or ∞.

c. If L^* has no point spectrum, then since $\ker L^*$ is L^*-invariant, it must be either $\{0\}$ or infinite-dimensional. However, if $\mathcal{F}$ is a frame then $e(\mathcal{F}) = \dim(\ker T^*)$, so $e(\mathcal{F})$ must be either 0 or ∞. $\square$

5. Frames

In this section we consider the excess of frames in Hilbert spaces.

By Theorem 3.3, if $\mathcal{F}$ is a frame that has infinite excess, then there exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. However, the following example shows that it is possible that there may be no way to choose $\mathcal{G}$ so that $\mathcal{F} \setminus \mathcal{G}$ is a frame. This example is exactly the example constructed in [3] of a normalized tight frame which contains no subset that is a Riesz basis.

Example 5.1. Let H be a separable Hilbert space. Index an orthonormal basis for H as $\{e_j^n\}_{n\in\mathbf{N},\, j=1,\dots,n}$. Set $H_n = \operatorname{span}\{e_1^n, \dots, e_n^n\}$. Define

$$f_j^n = e_j^n - \frac{1}{n}\sum_{i=1}^{n} e_i^n, \qquad j = 1, \dots, n,$$

$$f_{n+1}^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_i^n.$$

Then $\mathcal{F}_n = \{f_1^n, \dots, f_{n+1}^n\}$ is a normalized tight frame for H_n [3, Lemma 2.5]. Since H_n is n-dimensional, at most one element can be removed from $\mathcal{F}_n$ if the remaining elements are to span H_n. Moreover f_{n+1}^n is orthogonal to $f_1^n, \dots, f_n^n$, so f_{n+1}^n cannot be removed. If one of the other elements is removed, say f_1^n, then since

$$\sum_{j=2}^{n+1} |\langle e_1^n, f_j^n\rangle|^2 = \left(\sum_{j=2}^{n} \frac{1}{n^2}\right) + \frac{1}{\sqrt{n}^2} = \frac{2}{n} - \frac{1}{n^2},$$

the lower frame bound for $\mathcal{F}_n \setminus \{f_1^n\}$ as a frame for H_n is at most $\frac{2}{n} - \frac{1}{n^2}$.

Now consider that $H \cong \left(\sum_{n=1}^{\infty} H_n\right)_{\ell^2}$ with the H_n mutually orthogonal. The sequence $\mathcal{F} = \{f_j^n\}_{n\in\mathbf{N},\, j=1,\dots,n+1}$ is a normalized tight frame for H with infinite excess. Suppose that $\mathcal{G}$ is any infinite subset of $\mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is complete. Then $\mathcal{G}$ cannot contain any elements of the form f_{n+1}^n. Hence $\mathcal{G} = \{f_{j_k}^{n_k}\}_{k\in\mathbf{N}}$ with $n_1 < n_2 < \cdots$ and $j_k \le n_k$ for every k. But then the lower frame bound for $\mathcal{F} \setminus \mathcal{G}$ can be at most $\frac{2}{n_k} - \frac{1}{n_k^2}$ for every k, which implies that $\mathcal{F} \setminus \mathcal{G}$ cannot have a positive lower frame bound and therefore is not a frame.

Note that in this example, if we fix a particular k then the subsequence $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$ formed by deleting the single element $f_{j_k}^{n_k}$ from $\mathcal{F}$ is a frame for H. However, there is no single positive number that can serve as a common lower frame bound for all of the subframes $\mathcal{F} \setminus \{f_{j_k}^{n_k}\}$. Suppose that $\mathcal{F}$ was a frame such that there did exist an infinite subsequence $\mathcal{G} = \{g_n\}_{n\in\mathbf{N}}$ so that $\mathcal{F}\setminus\mathcal{G}$ was a frame for H, say with lower frame bound L. Then for each fixed n, since $\mathcal{F}\setminus\mathcal{G}\subset\mathcal{F}\setminus\{g_n\}\subset\mathcal{F}$, we have that $\mathcal{F}\setminus\{g_n\}$ is a frame for H with lower frame bound L. Hence the existence of such a sequence $\{g_n\}_{n\in\mathbf{N}}$ with uniform lower frame bound for each $\mathcal{F}\setminus\{g_n\}$ is a necessary condition in order to be able to delete infinitely many elements from a frame and still leave a frame. Our next goal is to show that this condition is sufficient as well as necessary. Specifically, we will show that if such g_n exist, then there exists an infinite subsequence $\mathcal{G}_\varepsilon = \{g_{n_k}\}_{k\in\mathbf{N}}$ such that $\mathcal{F}\setminus\mathcal{G}_\varepsilon$ is a frame with lower frame bound $L-\varepsilon$.

First, we will prove the theorem for the special case of normalized tight frames. While this result will be superseded by Theorem 5.4 below, the proof of this special case is so elegant and enlightening that we choose to include it.

Theorem 5.2. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a normalized tight frame for a Hilbert space H, and assume that there exists an infinite subsequence $\mathcal{G} = \{g_n\}_{n\in\mathbf{N}}$ of $\mathcal{F}$ such that for each n, $\mathcal{F}\setminus\{g_n\}$ is complete in H (and hence a frame). If there exists a single constant $L>0$ that is a lower frame bound for each frame $\mathcal{F}\setminus\{g_n\}$, then for every $0<\varepsilon<L$ there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F}\setminus\mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L-\varepsilon$.

Proof. Since $A = B = 1$, the frame operator S for $\mathcal{F}$ is simply the identity. That is,

$$\forall\, f\in H,\quad f \;=\; Sf \;=\; \sum_{i\in I}\langle f, f_i\rangle\, f_i.$$

We are given that, for each $n\in\mathbf{N}$, $\mathcal{F}\setminus\{g_n\}$ is a frame with lower frame bound L. Let S_n be the frame operator for $\mathcal{F}\setminus\{g_n\}$, i.e.,

$$S_n f \;=\; \sum_{i\in I}\langle f, f_i\rangle\, f_i \;-\; \langle f, g_n\rangle\, g_n \;=\; f \;-\; \langle f, g_n\rangle\, g_n.$$

Since

$$\langle S_n f, f\rangle \;=\; \|f\|^2 - |\langle f, g_n\rangle|^2 \;\ge\; \|f\|^2 - \|f\|^2\,\|g_n\|^2 \;=\; (1-\|g_n\|^2)\,\|f\|^2,$$

we have that $1-\|g_n\|^2$ is a lower frame bound for $\mathcal{F}\setminus\{g_n\}$, and by considering the element $f = g_n$ we see that it is the optimal lower frame bound for $\mathcal{F}\setminus\{g_n\}$. Therefore we must have

$$\forall\, n\in\mathbf{N},\quad L \;\le\; 1-\|g_n\|^2.$$

Since $\{g_n\}_{n\in\mathbf{N}}$ is a subset of the frame $\mathcal{F}$, we have $\sum_k |\langle g_n, g_k\rangle|^2 \leq \|g_n\|^2 < \infty$. Therefore,

$$\forall\, n \in \mathbf{N}, \quad \lim_{k\to\infty} \langle g_n, g_k\rangle \;=\; 0.$$

Because of this fact, we can extract a subsequence $\mathcal{G}_\varepsilon = \{g_{n_k}\}_{k\in\mathbf{N}}$ with the property that

$$\sum_{\substack{j,k\in\mathbf{N}\\ k\neq j}} |\langle g_{n_k}, g_{n_j}\rangle| \;<\; \varepsilon.$$

We claim that $\mathcal{F}\setminus\mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L-\varepsilon$.

Consider the operator

$$Rf \;=\; \sum_{k=1}^{\infty} \langle f, g_{n_k}\rangle\, g_{n_k}.$$

This is a bounded operator since $\mathcal{G}_\varepsilon$ is a subset of the frame $\mathcal{F}$. We have

$$\begin{aligned}
\|Rf\|^2 \;&=\; \Bigl\langle \sum_{k=1}^{\infty} \langle f, g_{n_k}\rangle\, g_{n_k},\, \sum_{j=1}^{\infty} \langle f, g_{n_j}\rangle\, g_{n_j} \Bigr\rangle \\
&=\; \sum_{k=1}^{\infty} |\langle f, g_{n_k}\rangle|^2\, \|g_{n_k}\|^2 \;+\; \sum_{\substack{j,k\in\mathbf{N}\\ k\neq j}} \langle f, g_{n_k}\rangle\, \langle g_{n_j}, f\rangle\, \langle g_{n_k}, g_{n_j}\rangle \\
&\leq\; \Bigl(\sup_{k\in\mathbf{N}} \|g_{n_k}\|^2\Bigr)\, \langle Rf, f\rangle \;+\; \|f\|^2 \Bigl(\sup_{k\in\mathbf{N}} \|g_{n_k}\|^2\Bigr) \Biggl(\sum_{\substack{j,k\in\mathbf{N}\\ k\neq j}} |\langle g_{n_k}, g_{n_j}\rangle|\Biggr) \\
&\leq\; (1-L)\, \|Rf\|\, \|f\| \;+\; \|f\|^2\, (1-L)\, \varepsilon.
\end{aligned}$$

From this it follows that $\|R\| \leq 1 - L + \varepsilon$, and consequently

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \;-\; \sum_{k=1}^{\infty} |\langle f, g_{n_k}\rangle|^2 \;=\; \|f\|^2 - \langle Rf, f\rangle \;\geq\; (L-\varepsilon)\, \|f\|^2,$$

so $\mathcal{F}\setminus\mathcal{G}_\varepsilon$ is a frame with lower frame bound $L-\varepsilon$. □

Given a frame $\mathcal{F}$ with frame bounds A, B, let $S = T^*T$ be the frame operator. Recall then that $S^{-1/2}$ is a bounded, continuously invertible operator and that $S^{-1/2}(\mathcal{F})$ is a normalized tight frame for H. This can be used to give a generalization of Theorem 5.2 to the case of non-tight frames; however, the best conclusion we can draw via that approach is that the lower frame bound of $\mathcal{F}\setminus\mathcal{G}_\varepsilon$ is at least $L(A/B) - \varepsilon$, which we will see is not the best possible estimate. For many applications it is essential to have sharp knowledge of the frame bounds.

Theorem 5.4 below is the optimal result: by an argument more involved than the proof of Theorem 5.2 we will show that it is possible to construct $\mathcal{G}_\varepsilon$ so that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ has lower frame bound $L - \varepsilon$.

To attempt to motivate the proof of Theorem 5.4, suppose that there existed a subsequence $\{h_k\}_{k \in \mathbf{N}}$ of $\mathcal{F}$ which had the following properties:

a. for each $k \in \mathbf{N}$, $\mathcal{F} \setminus \{h_k\}$ is a frame for H with lower frame bound L,

b. $\{h_k\}_{k \in \mathbf{N}}$ is an orthogonal sequence,

c. each h_k is an eigenvector of $S^{1/2}$.

Note that it follows from a–c that

d. $\overline{\text{span}}\{h_k\}^\perp$ is invariant under $S^{1/2}$.

We will show that it easily follows from these assumptions that $\mathcal{F} \setminus \{h_k\}_{k \in \mathbf{N}}$ is a frame with lower frame bound L. Of course, these hypotheses are unlikely to be fulfilled in practice, and much of the actual proof of Theorem 5.4 consists of trying to approximate them.

Note first that

$$\|S^{1/2} f\|^2 \;=\; \langle Sf, f\rangle \;=\; \sum_{i \in I} |\langle f, f_i\rangle|^2.$$

Since $\mathcal{F}$ is a frame, we therefore have that

$$\forall\, f \in H, \quad A\,\|f\|^2 \;\le\; \|S^{1/2} f\|^2 \;\le\; B\,\|f\|_2. \tag{10}$$

Without loss of generality, let us assume that the values of A, B in (10) are the optimal frame bounds. Assume now that hypotheses a–d above are satisfied. In particular, assumption a says that

$$\forall\, k \in \mathbf{N}, \quad \forall\, f \in H, \quad \|S^{1/2} f\|^2 - |\langle f, h_k\rangle|^2 \;\ge\; L\,\|f\|^2.$$

Note that since $\mathcal{F} \setminus \{h_k\}$ is a subset of the frame $\mathcal{F}$ and A is the optimal lower frame bound for $\mathcal{F}$, we have $L \le A$.

Fix now $f \in H$, and write $f = f^c + \sum_k c_k h_k$ with $f^c \in \overline{\text{span}}\{h_k\}^\perp$. Then $S^{1/2} f \;=\; S^{1/2} f^c + \sum_k S^{1/2}(c_k h_k)$, and by the orthogonality and invariance assumptions, this implies that

$$\|f\|^2 \;=\; \|f^c\|^2 + \sum_{k=1}^{\infty} \|c_k h_k\|^2 \quad \text{and} \quad \|S^{1/2} f\|^2 \;=\; \|S^{1/2} f^c\|^2 + \sum_{k=1}^{\infty} \|S^{1/2}(c_k h_k)\|^2.$$

Then

$$\begin{aligned}
\|S^{1/2}f\|^2 - \sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2 \;&=\; \|S^{1/2}f^c\|^2 + \sum_{k=1}^{\infty} \Big(\|S^{1/2}(c_k h_k)\|^2 - |\langle f, h_k\rangle|^2\Big) \\
&=\; \|S^{1/2}f^c\|^2 + \sum_{k=1}^{\infty} \Big(\|S^{1/2}(c_k h_k)\|^2 - |\langle c_k h_k, h_k\rangle|^2\Big) \\
&\geq\; A\,\|f^c\|^2 + \sum_{k=1}^{\infty} L\,\|c_k h_k\|^2 \\
&=\; A\,\|f^c\|^2 + L\,(\|f\|^2 - \|f^c\|^2) \\
&\geq\; L\,\|f\|^2,
\end{aligned}$$

the last inequality following from the fact that $L \leq A$.

To approximate assumptions a–d in the actual proof of Theorem 5.4, we apply the Spectral Theorem to the positive operator $S^{1/2}$. This provides us with a set of mutually orthogonal subspaces on each of which $S^{1/2}$ acts approximately as a scalar. Further, the fact that $\{g_n\}_{n\in\mathbf{N}}$ is a Bessel sequence allows us to select a subsequence $\{g_{n_k}\}_{k\in\mathbf{N}}$ that is "approximately orthogonal," and by orthogonalizing we can obtain a sequence of elements $\{h_k\}_{k\in\mathbf{N}}$ that are both orthogonal and near to g_{n_k}, although they are no longer elements of the original frame. These approximations allow us to carry through the complete proof.

We will require the following elementary lemma.

Lemma 5.3. If $\{g_k\}_{k\in\mathbf{N}}$ is a Bessel sequence with upper bound B and if

$$\sum_{k=1}^{\infty} \|h_k - g_k\|^2 \;\leq\; \beta,$$

then

$$\forall\, f \in H, \quad \sum_{k=1}^{\infty} |\langle f, g_k\rangle|^2 \;\leq\; \sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2 \;+\; \gamma\,\|f\|^2,$$

where $\gamma = \beta + 2B^{1/2}\beta^{1/2}$.

Proof. Let $s = \{\langle f, g_k\rangle\}_{k\in\mathbf{N}}$ and $t = \{\langle f, h_k\rangle\}_{k\in\mathbf{N}}$. Then

$$\begin{aligned}
\|s\|_{\ell^2}^2 - \|t\|_{\ell^2}^2 \;&=\; \big(\|s\|_{\ell^2} - \|t\|_{\ell^2}\big)\,\big(\|s\|_{\ell^2} + \|t - s + s\|_{\ell^2}\big) \\
&\leq\; \|s - t\|_{\ell^2}\,\big(\|t - s\|_{\ell^2} + 2\|s\|_{\ell^2}\big) \\
&\leq\; \beta^{1/2}\,\|f\|\,(\beta^{1/2}\,\|f\| + 2B^{1/2}\,\|f\|)
\end{aligned}$$

$$= \gamma \, \|f\|^2. \quad \square$$

Theorem 5.4. Let $\mathcal{F} = \{f_i\}_{i \in I}$ be a frame for a Hilbert space H, and assume that there exists an infinite subsequence $\mathcal{G} = \{g_n\}_{n \in \mathbf{N}}$ of $\mathcal{F}$ such that for each n, $\mathcal{F} \setminus \{g_n\}$ is complete in H (and hence a frame). If there exists a single constant $L > 0$ that is a lower frame bound for each frame $\mathcal{F} \setminus \{g_n\}$, then for every $0 < \varepsilon < L$ there exists an infinite subsequence $\mathcal{G}_\varepsilon$ of $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame for H with lower frame bound $L - \varepsilon$.

Proof. Let A, B denote the optimal frame bounds for $\mathcal{F}$. Then since $\mathcal{F} \setminus \{g_n\}$ is a subset of the frame $\mathcal{F}$, we have $L \le A$.

Let $\varepsilon > 0$ be fixed. Our goal is to find a subsequence $\mathcal{G}_\varepsilon = \{g_{n_k}\}_{k \in \mathbf{N}}$ of $\mathcal{G}$ such that $\mathcal{F} \setminus \mathcal{G}_\varepsilon$ is a frame with lower frame bound $L - \varepsilon$. Since $\mathcal{F} \setminus \mathcal{G}_\varepsilon \subset \mathcal{F}$, the upper frame bound is automatic, so what we have to show is that

$$\forall\, f \in H, \quad \|S^{1/2} f\|^2 - \sum_{k=1}^{\infty} |\langle f, g_{n_k} \rangle|^2 \;\ge\; (L - \varepsilon)\, \|f\|^2, \tag{11}$$

where $S = T^*T$ is the frame operator for $\mathcal{F}$.

Step 1. Consider the spectral decomposition of $S^{1/2}$, i.e.,

$$S^{1/2} \;=\; \int_{A^{1/2}}^{B^{1/2}} \lambda \, dP_\lambda,$$

where the P_λ are the spectral projections onto $[0, \lambda]$. Fix a constant $\alpha > 0$ whose exact value will be specified later, and define

$$\delta \;=\; \frac{B^{1/2} - A^{1/2}}{N},$$

where N is chosen large enough that

$$\delta\, (2B^{1/2} - \delta) \;<\; \alpha.$$

Note that if the frame $\mathcal{F}$ is tight, then $A = B$ and so $\delta = 0$. In this case, we will set $N = 1$. Note that for a tight frame, the frame operator S is simply $S = AI$.

For the case of a tight frame, where $\delta = 0$, define

$$Q_1 \;=\; I.$$

Otherwise, for $j = 1, \dots, N$, define

$$Q_j \;=\; P_{A^{1/2} + j\delta} - P_{A^{1/2} + (j-1)\delta}.$$

Then the following facts hold.

a. Each Q_j is an orthogonal projection.

b. The ranges $Q_j(H)$ for $j = 1, \dots, N$ are mutually orthogonal.

c. $\sum_{j=1}^N Q_j = I$.

d. The operator $S^{1/2}$ acts approximately as a scalar on $Q_j(H)$, specifically,

$$\sum_{j=1}^{N} \bigl(A^{1/2} + (j-1)\delta\bigr)^2 \, \|Q_j f\|^2 \;\le\; \|S^{1/2} f\|^2 \;\le\; \sum_{j=1}^{N} \bigl(A^{1/2} + j\delta\bigr)^2 \, \|Q_j f\|^2. \tag{12}$$

The difference between the right and left-hand sides of (12) can be bounded as follows:

$$\begin{aligned} \sum_{j=1}^{N} \bigl(2\delta A^{1/2} + (2j-1)\delta^2\bigr) \, \|Q_j f\|^2 \;&\le\; \bigl(2\delta A^{1/2} + (2N-1)\delta^2\bigr) \sum_{j=1}^{N} \|Q_j f\|^2 \\ &=\; \delta(2B^{1/2} - \delta) \, \|f\|^2 \\ &<\; \alpha \, \|f\|^2. \end{aligned}$$

Consequently, the right-hand side of (12) is no more than $\alpha\|f\|^2$ of the left-hand side, i.e.,

$$\sum_{j=1}^{N} \bigl(A^{1/2}+(j-1)\delta\bigr)^2 \, \|Q_j f\|^2 \;\le\; \|S^{1/2} f\|^2 \;\le\; \sum_{j=1}^{N} \bigl(A^{1/2}+(j-1)\delta\bigr)^2 \, \|Q_j f\|^2 + \alpha \, \|f\|^2. \tag{13}$$

Step 2. We now iteratively construct the subsequence $\mathcal{G}_\varepsilon = \{g_{n_k}\}_{k \in \mathbf{N}}$. For $n \in \mathbf{N}$ and $j = 1, \dots, N$, define

$$g_n^j \;=\; Q_j g_n.$$

Note that $g_n = \sum_{j=1}^n g_n^j$, with $\{g_n^j\}_{j=1,\dots,N}$ an orthogonal sequence.

Define $n_1 = 1$. For $j = 1, \dots, N$, set

$$\mathcal{F}_1^j \;=\; \{g_{n_1}^j\} \qquad \text{and} \qquad H_1^j \;=\; \operatorname{span}(\mathcal{F}_1^j).$$

Let P_1^j be the orthogonal projection of H onto H_1^j. Let $T_1^j \colon H_1^j \to \mathbf{C}$ be the analysis operator for $\mathcal{F}_1^j$ as a frame for H_1^j. Since T_1^j is injective and bounded, it has a continuous inverse $(T_1^j)^{-1} \colon \mathbf{C} \to H_1^j$. Set

$$\varepsilon_1 \;=\; \frac{\beta}{2} \sum_{j=1}^{N} \frac{1}{\|(T_1^j)^{-1}\|^2},$$

where $\beta > 0$ is another constant whose exact value will be specified later. Since $\mathcal{G}$ is a Bessel sequence, we know that for each $j = 1, \dots, N$,

$$\lim_{n\to\infty} \langle g_n, g^j_{n_1} \rangle \;=\; 0.$$

Choose n_2 large enough that

$$\sum_{j=1}^{N} \|T^j_1(P^j_1 g^j_{n_2})\|^2 \;=\; \sum_{j=1}^{N}\sum_{k=1}^{1} |\langle g_{n_2}, g^j_{n_k}\rangle|^2 \;<\; \varepsilon_1.$$

Now continue the process. Set

$$\mathcal{F}^j_2 \;=\; \{g^j_{n_1}, g^j_{n_2}\} \qquad \text{and} \qquad H^j_2 \;=\; \operatorname{span}(\mathcal{F}^j_2).$$

Let P^j_2 be the orthogonal projection onto H^j_2. The analysis operator $T^j_2 \colon H^j_2 \to \mathbf{C}^2$ is continuous and injective. Set

$$\varepsilon_2 \;=\; \frac{\beta}{2^2} \sum_{j=1}^{N} \frac{1}{\|(T^j_2)^{-1}\|^2}.$$

Then choose n_3 large enough that

$$\sum_{j=1}^{N} \|T^j_2(P^j_2 g^j_{n_3})\|^2 \;=\; \sum_{j=1}^{N}\sum_{k=1}^{2} |\langle g_{n_3}, g^j_{n_k}\rangle|^2 \;<\; \varepsilon_2,$$

and so forth, to obtain the sequence $\mathcal{G}_\varepsilon = \{g_{n_k}\}_{k\in\mathbf{N}}$.

Step 3. Next we orthogonalize the vectors $g^j_{n_k}$. Define

$$h^j_k \;=\; g^j_{n_k} - P^j_{k-1} g^j_{n_k},$$

where $P^j_0 = 0$. Since $g^j_n = Q_j g_n$, we have that

$$h^j_k \;\in\; Q_j(H).$$

Further, the subspaces $Q_j(H)$ are mutually orthogonal, so we conclude that $\{h^j_k\}_{k\in\mathbf{N},\, j=1,\dots,N}$ is an orthogonal sequence.

Step 4. Define

$$h_k \;=\; \sum_{j=1}^{N} h^j_k \;=\; g_{n_k} - \sum_{j=1}^{N} P^j_{k-1} g^j_{n_k}.$$

We observe that h_k is close to g_{n_k}, specifically,

$$\sum_{k=1}^{\infty} \|h_k - g_{n_k}\|^2 \;\le\; \sum_{k=2}^{\infty} \biggl(\sum_{j=1}^{N} \|P^j_{k-1} g^j_{n_k}\| \biggr)^2$$

$$\begin{aligned}
&\le \sum_{k=2}^{\infty}\Bigl(\sum_{j=1}^{N}\|(T_{k-1}^j)^{-1}\|\,\|T_{k-1}^j(P_{k-1}^j g_{n_k}^j)\|\Bigr)^2 \\
&\le \sum_{k=2}^{\infty}\Bigl(\sum_{j=1}^{N}\|(T_{k-1}^j)^{-1}\|^2\Bigr)\Bigl(\sum_{j=1}^{N}\|T_{k-1}^j(P_{k-1}^j g_{n_k}^j)\|^2\Bigr) \\
&\le \sum_{k=2}^{\infty}\frac{\beta}{2^{k-1}} \\
&= \beta. \qquad (14)
\end{aligned}$$

Step 5. Fix $f \in H$. Recall that $\{h_k^j\}_{k\in\mathbf{N},\, j=1,\dots,N}$ is an orthogonal sequence and write

$$f \;=\; f^c + \sum_{k=1}^{\infty}\sum_{j=1}^{N} c_k^j h_k^j \;=\; f^c + \sum_{k=1}^{\infty} p_k,$$

where $c_k^j\|h_k^j\|^2 = \langle f, h_k^j\rangle$, so that $f^c \in \overline{\mathrm{span}}\{h_k\}^\perp$. The functions p_k are mutually orthogonal and are orthogonal to f^c, so

$$\|f\|^2 \;=\; \|f^c\|^2 + \sum_{k=1}^{\infty}\|p_k\|^2. \qquad (15)$$

Recall that $h_k = \sum_{j=1}^N h_k^j$. Therefore

$$\langle f, h_k\rangle \;=\; \sum_{j=1}^{N}\langle f, h_k^j\rangle \;=\; \sum_{j=1}^{N} c_k^j\,\langle h_k^j, h_k^j\rangle \;=\; \sum_{j=1}^{N} c_k^j\,\langle h_k^j, h_k\rangle \;=\; \langle p_k, h_k\rangle.$$

Now, since the Q_j are orthogonal projections with orthogonal ranges and since $h_k^j \in Q_j(H)$, we have that

$$Q_j f \;=\; Q_j f^c + \sum_{k=1}^{\infty} Q_j p_k$$

is an orthogonal decomposition. In fact, $Q_j p_k = c_k^j h_k^j$, and, more importantly,

$$\|Q_j f\|^2 \;=\; \|Q_j f^c\|^2 + \sum_{k=1}^{\infty}\|Q_j p_k\|^2. \qquad (16)$$

Recall that our goal is to show that (11) is satisfied. Using (13), (16), (15), and (10), we have that

$$\|S^{1/2} f\|^2 \;\ge\; \sum_{j=1}^{N}\bigl(A^{1/2} + (j-1)\delta\bigr)^2\,\|Q_j f\|^2$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \bigl(A^{1/2} + (j-1)\delta\bigr)^2 \, \|Q_j f^c\|^2 \;+\; \sum_{k=1}^{\infty} \sum_{j=1}^{N} \bigl(A^{1/2} + (j-1)\delta\bigr)^2 \, \|Q_j p_k\|^2 \\
&\geq \Bigl(\|S^{1/2} f^c\|^2 - \alpha \, \|f^c\|^2\Bigr) \;+\; \sum_{k=1}^{\infty} \Bigl(\|S^{1/2} p_k\|^2 - \alpha \, \|p_k\|^2\Bigr) \\
&= \|S^{1/2} f^c\|^2 - \alpha \, \|f^c\|^2 \;+\; \sum_{k=1}^{\infty} \|S^{1/2} p_k\|^2 \;-\; \alpha \, (\|f\|^2 - \|f^c\|^2) \\
&\geq A \, \|f^c\|^2 \;+\; \sum_{k=1}^{\infty} \|S^{1/2} p_k\|^2 \;-\; \alpha \, \|f\|^2. \qquad (17)
\end{aligned}
$$

Further, by (14) and Lemma 5.3, we have

$$
\sum_{k=1}^{\infty} |\langle f, g_{n_k}\rangle|^2 \;\leq\; \sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2 + \gamma \, \|f\|^2 \;=\; \sum_{k=1}^{\infty} |\langle p_k, h_k\rangle|^2 + \gamma \, \|f\|^2, \qquad (18)
$$

where $\gamma = \beta + 2B^{1/2}\beta^{1/2}$. Hence, combining (17) and (18),

$$
\|S^{1/2} f\|^2 - \sum_{k=1}^{\infty} |\langle f, g_{n_k}\rangle|^2 \;\geq\; \sum_{k=1}^{\infty} \Bigl(\|S^{1/2} p_k\|^2 - |\langle p_k, h_k\rangle|^2\Bigr) \;+\; A \, \|f^c\|^2 - (\alpha + \gamma) \, \|f\|^2. \qquad (19)
$$

Now, by hypothesis, for each k we know that $\mathcal{F} \setminus \{g_{n_k}\}$ is a frame with lower frame bound L. That is,

$$
\forall\, k \in \mathbf{N}, \quad \forall\, h \in H, \quad \|S^{1/2} h\|^2 - |\langle h, g_{n_k}\rangle|^2 \;\geq\; L \, \|h\|^2.
$$

Since $\|h_k - g_{n_k}\|^2 \leq \beta$ and since $\|g_{n_k}\|^2 \;\leq\; B$, the same type of argument as in the proof of Lemma 5.3 yields the estimate

$$
|\langle h, h_k\rangle|^2 - |\langle h, g_{n_k}\rangle|^2 \;\leq\; \gamma \, \|h\|^2.
$$

In particular, applying this to the function $h = p_k$ and combining it with (19), we conclude that

$$
\begin{aligned}
\|S^{1/2} f\|^2 - \sum_{k=1}^{\infty} |\langle f, g_{n_k}\rangle|^2 \;&\geq\; \sum_{k=1}^{\infty} \Bigl(\|S^{1/2} p_k\|^2 - |\langle p_k, g_{n_k}\rangle|^2 - \gamma \, \|p_k\|^2\Bigr) \\
&\qquad + A \, \|f^c\|^2 - (\alpha + \gamma) \, \|f\|^2 \\
&\geq\; \sum_{k=1}^{\infty} (L - \gamma) \, \|p_k\|^2 + A \, \|f^c\|^2 - (\alpha + \gamma) \, \|f\|^2 \\
&=\; (L - \gamma) \, (\|f\|^2 - \|f^c\|^2) + A \, \|f^c\|^2 - (\alpha + \gamma) \, \|f\|^2 \\
&=\; (L - \alpha - 2\gamma) \, \|f\|^2 + (A - L + \gamma) \, \|f^c\|^2
\end{aligned}
$$

$$\geq \ (L - \alpha - 2\gamma)\, \|f\|^2,$$

the last inequality following from the fact that $L \leq A$.

Finally, by choosing the constants α and β small enough, we can obtain $\alpha + 2\gamma < \varepsilon$, which completes the proof. $\square$

The next proposition shows that the excess can be realized in terms of certain inner products. We will use this to obtain a condition in Corollary 5.7 below that is both necessary and sufficient for the hypotheses of Theorem 5.4 to hold.

Recall that the standard dual of a frame $\mathcal{F} = \{f_i\}_{i\in I}$ is the frame $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$ where $\tilde{f}_i = S^{-1} f_i$. Therefore $\langle f_i, \tilde{f}_i\rangle = \|S^{-1/2} f_i\|^2 \geq 0$. Moreover, $S^{-1/2}(\mathcal{F})$ is a normalized tight frame, each element of which can have norm at most 1, so $\langle f_i, \tilde{f}_i\rangle = \|S^{-1/2} f_i\|^2 \leq 1$.

Proposition 5.5. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame in a Hilbert space H with standard dual $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$. Then the excess of $\mathcal{F}$ is

$$e(\mathcal{F}) \ = \ \sum_{i\in I} \bigl(1 - \langle f_i, \tilde{f}_i\rangle\bigr).$$

Proof. By Lemma 4.1, we have $e(\mathcal{F}) = \dim(\ker T^*)$. The orthogonal projection of H onto $\ker T^*$ is given by $P = I - TS^{-1}T^*$. Letting $\{\delta_i\}_{i\in I}$ denote the standard basis for $\ell^2(I)$, we therefore have

$$e(\mathcal{F}) \ = \ \dim(\ker T^*) \ = \ \mathrm{trace}(P) \ = \ \sum_{i\in I} \langle \delta_i, P\delta_i\rangle \ = \ \sum_{i\in I} \bigl(1 - \langle f_i, \tilde{f}_i\rangle\bigr). \quad \square$$

We will require the following lemma [7, Lemma 6.3.2] in order to obtain an equivalent form of the hypotheses of Theorem 5.4.

Lemma 5.6. Let $\mathcal{F}$ be a frame for a Hilbert space H with frame bounds A, B. If $U\colon H \to H$ is continuously invertible, then $U(\mathcal{F})$ is a frame for H with frame bounds $A\|U^{-1}\|^{-2}$, $B\|U\|^2$.

Corollary 5.7. Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a frame in a Hilbert space H with standard dual $\tilde{\mathcal{F}} = \{\tilde{f}_i\}_{i\in I}$. Let $\mathcal{G} = \{g_n\}_{n\in\mathbf{N}}$ be a subsequence of $\mathcal{F}$. Then the following two statements are equivalent.

a. There exists a constant $L > 0$ such that for each $n \in \mathbf{N}$, $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound L.

b. $\sup_{n \in \mathbf{N}} \langle g_n, \tilde{g}_n \rangle \; < \; 1.$

Proof. a $\Rightarrow$ b. Assume that statement a holds. Since $S^{-1/2}$ is a continuously invertible operator with $\|S^{1/2}\|^2 \le B$, it follows from applying Lemma 5.6 to the frame $\mathcal{F} \setminus \{g_n\}$ that $S^{-1/2}(\mathcal{F} \setminus \{g_n\})$ is a frame with lower frame bound L/B. However, since $S^{-1/2}(\mathcal{F})$ is a normalized tight frame, we can also compute the frame bound of $S^{-1/2}(\mathcal{F} \setminus \{g_n\})$ as follows:

$$\begin{aligned} \sum_{i \in I} |\langle f, S^{-1/2} f_i \rangle|^2 \; - \; |\langle f, S^{-1/2} g_n \rangle|^2 \; &\ge \; \|f\|^2 - \|S^{-1/2} g_n\|^2 \, \|f\|^2 \\ &= \; (1 - \|S^{-1/2} g_n\|^2) \, \|f\|^2. \end{aligned} \tag{20}$$

Thus $1 - \|S^{-1/2} g_n\|^2$ is a lower frame bound for $S^{-1/2}(\mathcal{F} \setminus \{g_n\})$, and by considering the element $f = S^{-1/2} g_n$ we see that it is the optimal lower frame bound. Therefore we must have $L/B \le 1 - \|S^{-1/2} g_n\|^2$, so

$$\langle g_n, \tilde{g}_n \rangle \; = \; \|S^{-1/2} g_n\|^2 \; \le \; 1 - L/B.$$

b $\Rightarrow$ a. Assume that $D = \sup_n \langle g_n, \tilde{g}_n \rangle < 1$. Then $1 - \|S^{1/2} g_n\|^2 \ge 1 - D > 0$. As in (20), we therefore have that $S^{-1/2}(\mathcal{F} \setminus \{g_n\})$ is a frame for H with lower frame bound $1 - D$. Since $S^{1/2}$ is a continuously invertible operator with $\|S^{-1/2}\|^2 \le 1/A$, it follows from Lemma 5.6 that $\mathcal{F} \setminus \{g_n\}$ is a frame for H with lower frame bound $L = A(1 - D)$. $\square$

6. Weyl-Heisenberg and Wavelet Systems

In this section, we apply our previous results to the specific case of Weyl–Heisenberg and wavelet frames. For simplicity, we will consider only the one-dimensional setting, but the results given here can be easily extended to higher dimensions.

Definition 6.1. Given a nonzero function $g \in L^2(\mathbf{R})$, called a window function, and given α, $\beta > 0$, the *Weyl-Heisenberg* or *Gabor system* determined by g, α, β is

$$(g; \alpha, \beta)_{\mathrm{WH}} \; = \; \{g_{m,n;\alpha,\beta}\}_{m,n \in \mathbf{Z}},$$

where

$$g_{m,n;\alpha,\beta}(x) \; = \; e^{2\pi i m \beta x} g(x - n\alpha).$$

A Weyl–Heisenberg multisystem is a union of such Weyl–Heisenberg systems, namely,

$$(g^1, \dots, g^r; \alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)_{\mathrm{WH}} \;=\; (g^1; \alpha_1, \beta_1)_{\mathrm{WH}} \cup \cdots \cup (g^r; \alpha_r, \beta_r)_{\mathrm{WH}}.$$

Definition 6.2. Given a nonzero function $\Psi \in L^2(\mathbf{R})$, called a wavelet, and given $a > 1$ and $b > 0$, the *wavelet system* generated by Ψ, a, b is

$$(\Psi; a, b)_{\mathrm{Wa}} \;=\; \{\Psi_{m,n;a,b}\}_{m,n \in \mathbf{Z}},$$

where

$$\Psi_{m,n;a,b}(x) \;=\; a^{m/2}\Psi(a^m x - nb).$$

A *wavelet multisystem* has the form

$$(\Psi^1, \dots, \Psi^r; a_1, \dots, a_r; b_1, \dots, b_r)_{\mathrm{Wa}} \;=\; (\Psi^1; a_1, b_1)_{\mathrm{Wa}} \cup \cdots \cup (\Psi^r; a_r, b_r)_{\mathrm{Wa}}.$$

For α, $\beta \in \mathbf{R}$ and $a > 0$, define the following operators:

$$\begin{aligned}
T_\alpha &: L^2(\mathbf{R}) \to L^2(\mathbf{R}), & T_\alpha f(x) &= f(x - \alpha),\\
V_{\alpha\beta} &: \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2), & V_{\alpha\beta} c &= \{e^{-2\pi i \alpha\beta m} c_{m,n-1}\}_{m,n \in \mathbf{Z}},\\
M_\beta &: L^2(\mathbf{R}) \to L^2(\mathbf{R}), & M_\beta f(x) &= e^{2\pi i \beta x} f(x),\\
U &: \ell^2(\mathbf{Z}^2) \to \ell^2(\mathbf{Z}^2), & U c &= \{c_{m-1,n}\}_{m,n \in \mathbf{Z}},\\
D_a &: L^2(\mathbf{R}) \to L^2(\mathbf{R}), & D_a f(x) &= a^{1/2} f(ax).
\end{aligned}$$

In particular, note that

$$g_{m,n;\alpha,\beta} = M_{m\beta} T_{n\alpha} g \qquad \text{and} \qquad \Psi_{m,n;a,b} = D_{a^m} T_{nb} \Psi.$$

The next lemma follows from elementary calculations.

Lemma 6.3. a. T_α, $V_{\alpha\beta}$, M_β, U, and D_a have no point spectrum if α, $\beta \neq 0$ and $a \neq 1$.

b. If $(g; \alpha, \beta)_{\mathrm{WH}}$ is a Bessel sequence then $(T_\alpha, V_{\alpha\beta})$ and (M_β, U) are each intertwining pairs of operators for $(g; \alpha, \beta)_{\mathrm{WH}}$.

c. If $(\Psi; a, b)_{\mathrm{Wa}}$ is a Bessel sequence then (D_a, U) is an intertwining pair of operators for $(\Psi; a, b)_{\mathrm{Wa}}$.

Consequently, conclusions about the deficit and excess of Weyl–Heisenberg and wavelet systems follow immediately from Theorem 4.4.

Corollary 6.4. Let $g \in L^2(\mathbf{R}^d)$ and $\alpha, \beta > 0$ be such that $(g; \alpha, \beta)_{\mathrm{WH}}$ is a Bessel sequence in $L^2(\mathbf{R}^d)$. Then the following statements hold.

a. $\mathrm{span}(g; \alpha, \beta)_{\mathrm{WH}}$ is either $\{0\}$ or is an infinite-dimensional subspace of $L^2(\mathbf{R}^d)$.

b. The deficit of $(g; \alpha, \beta)_{\mathrm{WH}}$ is either zero or infinite.

c. If $(g; \alpha, \beta)_{\mathrm{WH}}$ is a frame for its closed linear span, then its excess is either zero or infinite.

Corollary 6.5. Let $\Psi \in L^2(\mathbf{R}^d)$ and $a > 1$, $b > 0$ be such that $(\Psi; \alpha, \beta)_{\mathrm{Wa}}$ is a Bessel sequence in $L^2(\mathbf{R}^d)$. Then the following statements hold.

a. $\mathrm{span}(\Psi; \alpha, \beta)_{\mathrm{Wa}}$ is either $\{0\}$ or is an infinite-dimensional subspace of $L^2(\mathbf{R}^d)$.

b. The deficit of $(\Psi; \alpha, \beta)_{\mathrm{Wa}}$ is either zero or infinite.

c. If $(\Psi; \alpha, \beta)_{\mathrm{Wa}}$ is a frame for its closed linear span, then its excess is either zero or infinite.

We can extend the statements above regarding the excess of Weyl–Heisenberg and wavelet frames to the case of multisystems by making use of the results from Section 5. At the same time we conclude that infinitely many elements can be deleted from any overcomplete Weyl–Heisenberg or wavelet system yet leave a frame. To obtain this result we must impose a condition on the relationship among the generating parameters. We say that r-tuple of numbers $(a_1, \dots, a_r)$ are *rationally related* if there are r integers $k_1, \dots, k_r$ such that $k_1 a_1 = \cdots = k_r a_r$.

Theorem 6.6. Let $\mathcal{F} = (g^1, \dots, g^r; \alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_r)_{\mathrm{WH}}$ be a Weyl–Heisenberg multisystem that is an overcomplete frame for its closed linear span $\mathcal{H}$ in $L^2(\mathbf{R})$. If either $(\alpha_1, \dots, \alpha_r)$ or $(\beta_1, \dots, \beta_r)$ are rationally related, then there exists an infinite subset $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{F} \setminus \mathcal{G}$ is a frame for $\mathcal{H}$.

Proof. Suppose that $(\beta_1, \dots, \beta_r)$ are rationally related, say $\beta = k_1 \beta_1 = \cdots = k_r \beta_r$. Since $\mathcal{F}$ is overcomplete, there is some element, say $g^i_{m_0, n_0; \alpha_i, \beta_i}$ such that

$\mathcal{F}\backslash\{g^i_{m_0,n_0;\alpha_i,\beta_i}\}$ is a frame for $\mathcal{H}$. Note that for each $m, n, p \in \mathbf{Z}$ and $j = 1, \dots, r$, we have

$$M_{\beta p} g^j_{m,n;\alpha_j,\beta_j} = M_{k_j\beta_j p} M_{m\beta_j} T_{n\alpha_j} g^j = M_{(m+k_jp)\beta_j} T_{n\alpha_j} g^j = g^j_{(m+k_jp),n;\alpha_j,\beta_j}.$$

Hence for each j, we have that $(g^j;\alpha_j,\beta_j)_{\mathrm{WH}}$ maps onto itself under $M_{\beta p}$, and consequently,

$$M_{\beta p}(\mathcal{F} \setminus \{g^i_{m_0,n_0;\alpha_i,\beta_i}\}) = \mathcal{F} \setminus \{g^i_{(m_0+k_jp),n_0;\alpha_i,\beta_i}\}, \qquad p \in \mathbf{Z}. \tag{21}$$

Since $M_{\beta p}$ is a unitary operator mapping $\mathcal{H}$ onto itself, each of the subsequences in (21) is a frame for $\mathcal{H}$, all with the same frame bounds. Consequently, the result follows from Theorem 5.4. If instead $(\alpha_1, \dots, \alpha_r)$ are rationally related, then a similar proof can be given using $T_{\alpha p}$ instead of $M_{\beta p}$. $\square$

The following example shows that the frame hypothesis in Theorem 6.6 cannot be relaxed, i.e., there exist Weyl–Heisenberg systems that are Bessel sequences yet have positive but finite excess.

Example 6.7. Consider the Weyl–Heisenberg system $\mathcal{F} = (g;1,1)_{\mathrm{WH}}$ in $L^2(\mathbf{R})$ generated by the Gaussian function $g(x) = e^{-x^2}$ with $\alpha = \beta = 1$. For simplicity, write $g_{m,n}(x) = g_{m,n;1,1}(x) = e^{2\pi i m x} g(x-n)$. Because $\alpha = \beta = 1$, it can be shown that if this system was a frame for $L^2(\mathbf{R})$ then it would be a Riesz basis, e.g., see [8, Thm. 4.3.3]. This would contradict the Balian–Low Theorem, so this system cannot be a frame. Let $Q = [0,1) \times [0,1)$. The Zak transform is the isometric isomorphism $Z\colon L^2(\mathbf{R}) \to L^2(Q)$ defined by

$$Zf(x,\omega) = \sum_{k\in\mathbf{Z}} e^{2\pi i k\omega} f(x+k).$$

We refer to [4] or [8] for details on the Zak transform. It can be shown that Zg is a continuous and bounded function on Q and has a single zero in Q. This shows that $(g;1,1)_{\mathrm{WH}}$ is a Bessel sequence but is not a frame for $L^2(\mathbf{R})$.

The synthesis operator for $(g;1,1)_{\mathrm{WH}}$ is the mapping $T^*\colon \ell^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$ defined by

$$T^*c = \sum_{m,n} c_{m,n} g_{m,n;1,1} \quad \text{for } c = \{c_{m,n}\}_{m,n\in\mathbf{Z}} \in \ell^2(\mathbf{Z}^2).$$

Suppose that $T^*c = 0$ for some $c \in \ell^2(\mathbf{Z}^2)$. Then, using basic properties of the Zak transform,

$$0 \;=\; ZT^*c \;=\; \sum_{m,n} c_{m,n} Zg_{m,n} \;=\; \sum_{m,n} c_{m,n} e_{m,n} Zg,$$

where $e_{m,n}(x,\omega) = e^{2\pi imx}e^{2\pi in\omega}$. Since $c \in \ell^2(\mathbf{Z}^2)$ and $\{e_{m,n}\}_{m,n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(Q)$, we have that $H = \sum_{m,n} c_{m,n}e_{m,n}$ is a well-defined function in $L^2(Q)$. Therefore, since Zg is bounded we have that $0 = ZT^*c = H\cdot Zg$. However, Zg is nonzero a.e., so this implies that $H = 0$ a.e., and therefore $c = 0$. Thus $\ker T^* = \{0\}$.

A similar argument, using the fact that $1/Zg \notin L^2(Q)$, shows that $e(\mathcal{F}) = 1$. This was first proved in [11]. Thus $(g;1,1)_{\mathrm{WH}}$ provides an example of a Weyl–Heisenberg system that is a Bessel sequence but not a frame and such that $\dim(\ker T^*) < e(\mathcal{F})$. This shows that even for Weyl–Heisenberg systems, the inequality in Lemma 4.1b can be strict (see also Example 4.2).

The excess in this example is exactly 1. In particular, $(g;1,1)_{\mathrm{WH}}\backslash\{g\} = \{g_{m,n}\}_{(m,n)\neq(0,0)}$ is complete, but no proper subset of $(g;1,1)_{\mathrm{WH}}\backslash\{g\}$ is complete. However, $(g;1,1)_{\mathrm{WH}}\backslash\{g\}$ is not a Schauder basis for $L^2(\mathbf{R})$ [6, p. 168]. In fact, while g can be approximated arbitrarily closely by finite linear combinations of elements of $(g;1,1)_{\mathrm{WH}}\backslash\{g\}$, no series of the form $\sum_{(m,n)\neq(0,0)} c_{m,n}g_{m,n}$ can converge to g, even in a weak sense, cf. [10] and [14, Thm. 1]. We refer to [14] for a detailed study of convergence questions involving Weyl–Heisenberg systems at the critical density $\alpha = \beta = 1$.

A technique similar to the one used in Theorem 6.6 can be applied to the wavelet case. We say that $(a_1,\dots,a_r)$ are *logarithmically rationally related* if there are r integers $k_1,\dots,k_r$ such that $a_1^{k_1} = \cdots = a_r^{k_r}$.

Theorem 6.8. Let $\mathcal{F} = (\Psi^1,\dots,\Psi^r;a_1,\dots,a_r;b_1,\dots,b_r)_{\mathrm{Wa}}$ be a wavelet multisystem that is an overcomplete frame for its closed linear span $\mathcal{H}$ in $L^2(\mathbf{R})$. If $(a_1,\dots,a_r)$ are logarithmically rationally related, then there exists an infinite subset $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{F}\setminus\mathcal{G}$ is a frame for $\mathcal{H}$.

Proof. The proof is similar to the proof of Theorem 6.6, using the fact that if $a = a_1^{k_1} = \cdots = a_r^{k_r}$ and $p \in \mathbf{Z}$, then D_{a^p} is a unitary operator such that

$$D_{a^p}\Psi^j_{m,n;a_j,b_j} \;=\; D_{a_j^{k_jp}}D_{a_j^m}T_{nb_j}\Psi^j \;=\; D_{a_j^{m+k_jp}}T_{nb_j}\Psi^j \;=\; \Psi^j_{(m+k_jp),n;a_j,b_j}. \qquad \square$$

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