INTERMEDIATE STANDARD INVARIANTS AND INTERMEDIATE PLANAR ALGEBRAS

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Abstract. Given an inclusion of II_1 subfactors $N \subset M$ of finite index, the basic construction yields a canonical infinite chain of subfactors $N \subset M \subset M_1 \cdots$. In this paper, we show how to derive the canonical infinite chain for $N \subset Q$ for an intermediate subfactor Q, $N \subset Q \subset M$, from the canonical chain for $N \subset M$. This extends a result by D. Bisch which shows how to get the odd elements of the chain for $N \subset Q$. In addition, we show how to derive the planar algebra of $N \subset Q$ from that of $N \subset M$. We consider several types of examples which demonstrate our results and, in some cases, shed new light on the original subfactor $N \subset M$.

1. Introduction

One of the major areas of study within subfactor theory has been the classification of subfactors $N \subset M$. In [Jo1], V. Jones defined the notion of index for type II_1 subfactor inclusions $N \subset M$. In addition he introduced a canonical construction of a tower of subfactors

$$N \subset M \subset M_1 \subset M_2 \subset \dots$$

known as the *basic construction.* For $N \subset M$ an inclusion of type II_1 subfactors with finite index, the double sequence of finite dimensional algebras

$$\begin{array}{ccccccc} N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \dots \\ & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & \dots \end{array}$$

known as the *standard invariant*, became an extremely important invariant in the study of subfactors (see for example [GoHaJo, JoSu, Po2, Po1]). In fact S. Popa determined that the standard invariant completely classifies certain kinds of subfactor inclusions [Po2]. The standard invariant can be partially described by two possibly infinite graphs called the principal and dual graphs of the subfactor ([Jo1], [Jo4], [Oc]), which encode the structure of the two horizontal sequences of algebras in the standard invariant.

In [Po1], Popa provided an abstract algebraic characterization of the standard invariant called a λ-*lattice.* Subsequently, Jones developed an equivalence between λ-lattices and objects he termed *planar algebras* [Jo2]. In the planar algebra framework, many of the algebraic conditions associated with a λ-lattice can be simply described in terms of the geometry of the plane.

In this paper, we consider the situation in which a subfactor $N \subset M$ has an intermediate subfactor Q. It is natural to ask how invariants of $N \subset M$ are related to invariants of $N \subset Q$ and $Q \subset M$. We show in Corollary 3.4 how to obtain the standard invariants of $N \subset Q$ by "cutting down" the standard invariant of $N \subset M$. This is an extension of Bisch's results in [Bi1]. We also describe in Theorem 5.3 the planar algebras of $N \subset Q$ in terms of the planar algebra of $N \subset M$ in the case that $N \subset M$ is irreducible. The standard invariant and planar algebra of $Q \subset M$ can be obtained by applying these results to $M \subset M_1$.

These results are a useful tool for analyzing any subfactor $N \subset M$, containing an intermediate subfactor Q, $N \subset Q \subset M$. In Section 6 we apply these results to various examples. The natural starting point is the situation where everything is known about the planar algebra (standard invariant) of a subfactor $N \subset M$ containing an intermediate subfactor Q, and our results are used to compute the planar algebra (standard invariant) of $N \subset Q$ and $Q \subset M$. This is the case in the well studied example of a group subfactor (see Section 6.4). In Corollary 6.7 of Section 6.2 we apply our results to obtain an alternate description of the standard invariant of the subgroup subfactor. In general however, it is not necessary to know everything about the planar algebra (standard invariant) of the subfactor $N \subset M$ in order to compute the planar algebra (standard invariant) of the intermediate subfactors $N \subset Q$ and $Q \subset M$. This is the situation for Krishnan-Sunder subfactors (Section 6.2) and Tree subfactors (Section 6.3).

Knowledge of the planar algebras (standard invariants) of the subfactors $N \subset Q$ and $Q \subset M$ provides additional information about the planar algebra (standard invariant) of $N \subset M$. Namely, the free product construction of the planar algebras $N \subset Q$ and $Q \subset M$ is always a subplanar algebra of $N \subset M$. In Section 6.1 we verify that for a free product of two planar algebras, our construction recovers the two component planar algebras. The most satisfying application of our results is to the exchange relation subfactors arising from the groups $\mathbb{H}$ and D_8 (Section 6.5). In these examples, we begin knowing only a description of the planar algebra (standard invariant) in terms of generators and relations. Applying Theorem 5.3 gives us full descriptions of the planar algebras of the two intermediate subfactors, and simple dimension considerations allow us to conclude that the original exchange relation planar algebra is the free product of the two intermediate ones. Thus we obtain a full description of the planar algebra (standard invariant) of these two exchange relation planar algebras.

2. Notation and some basic facts

In this paper, all factors will be of type II_1, and all subfactors $N \subset M$ will be of finite index where $[M : N]$ shall denote the index. By tr_M we will mean the unique normal faithful trace defined on M. E_N^M will denote the trace preserving conditional expectation from M onto N; we shall often omit M and write E_N when doing so is unambiguous.

The following provides a list of basic well known facts that shall be used in the next section.

Fact 2.1. *Given a II_1 factor A and projections $r \in A$ and $s \in (rAr)'$, we have*

$$tr_{rArs}(rzrs) = (tr_A(r))^{-1} tr_A(rzr) \tag{1}$$

for all $z \in A$.

Proof. For a projection $r \in A$ we have

$$tr_{rAr}(rzr) = (tr_A(r))^{-1} tr_A(rzr) \quad \text{for} \quad z \in A.$$

For a projection $s' \in A'$ we have

$$tr_{As'}(zs') = tr_A(z) \quad \text{for} \quad z \in A.$$

Combining these facts gives us the result. □

Let $A \subset B$ be an inclusion of von Neumann algebras.

Fact 2.2. *$q(A' \cap B)q = (qA)' \cap qBq$ for any projection $q \in A' \cap B$.*

Fact 2.3. *With B a factor, $r(A' \cap B)r = (rA)' \cap rB$ for any projection $r \in B'$.*

In what follows we let a be an arbitrary element of A.

Proof of Fact 2.2. Given $x \in A' \cap B$ we have $qxq \in qBq$ and $[qxq, aq] = 0$ (since $q, x \in A'$). Thus $q(A' \cap B)q \subset (qA)' \cap qBq$. Given $qyq \in (qA)' \cap qBq$ we have $qyq \in B$, $[qyq, a] = 0$ (since $qyq \in (Aq)'$ and $q \in A'$). Thus $qyq = q(qyq)q \in q(A' \cap B)q$ which establishes the opposite containment. □

Proof of Fact 2.3. Given $x \in A' \cap B$, we have $[rxr, ar] = 0$ (since $r \in B' \subset A'$). Thus $r(A' \cap B)r \subset (rA)' \cap rBr$. Given $y \in (rA)' \cap rBr$ we have $y = rbr = br$ for some $b \in B$. Now $bar = brar = arbr = abr$ and since $Br \cong B$ we have $ba = ab$ and thus $b \in A'$. Therefore $(rA)' \cap rBr \subset r(A' \cap B)r$ and we are done. □

Fact 2.4. *(Definition of trace preserving expectation) $x = E_N^M(y)$ if and only if $tr_M(xn) = tr_M(yn)$ for all $n \in N$.*

3. The Intermediate Standard Invariant

We begin with an inclusion $N \subset M$ of II_1 factors containing an intermediate factor Q. D. Bisch gave a partial description of $N \subset Q$ in [Bi2], giving the standard invariant of the inclusion $N \subset Q_2$, where Q_2 is the algebra obtained from the basic construction for $N \subset Q$. Many of the same techniques used in [Bi2] are used in what follows. We denote the Jones tower built from the basic construction of $N \subset M$ as follows:

$$N \subset Q \subset M \subset P_1 \subset M_1 \subset P_2 \subset M_2 \subset P_3 \subset \ldots \subset M_{2n+1}$$

with projections:

$$e_{0,i} : L^2(M_{i-1}) \to L^2(P_{i-1}) \qquad \text{and} \qquad e_{1,i} : L^2(M_{i-1}) \to L^2(M_{i-2})$$

so that $P_i = \langle M_{i-1}, e_{0,i}\rangle$ and $M_i = \langle M_{i-1}, e_{1,i}\rangle$ (here we set $M_0 = M$, $M_{-1} = N$, $P_0 = Q$). Let $tr_{N\subset M}$ denote the unique normalized trace on the tower.

The description of the algebras generated by $e_{0,i}$ and $e_{1,i}$ are given in [BiJo1]. We accumulate a few of the relations appearing in [BiJo1] that we will use. In what follows, $[a, b] = 0$ means that a and b commute, and $[a, B] = 0$ means that a commutes with all elements of the set B.

Fact 3.1. *The following relations hold:*

(1) $e_{0,i}e_{1,i} = e_{1,i}$,
(2) $[e_{a,i}, e_{b,j}] = 0$ *for* $|i - j| \geq 2$,
(3) $[e_{0,i}, e_{0,i\pm1}] = 0$,
(4) $[e_{0,i},\ P_{i-1}] = [e_{1,i},\ M_{i-2}] = 0$,
(5) *for i even,* $e_{0,i}e_{1,2i\pm1}e_{0,i} = [Q : N]^{-1}e_{0,i}e_{0,i\pm1}$, *and* $e_{1,i}e_{0,i\pm1}e_{1,i} = [M : Q]^{-1}e_{1,i}$,
(6) *for i odd,* $e_{0,i}e_{1,i\pm1}e_{0,i} = [M : Q]^{-1}e_{0,i}e_{0,i\pm1}$, *and* $e_{1,i}e_{0,i\pm1}e_{1,i} = [Q : N]^{-1}e_{1,i}$,
(7) $e_{1,i}e_{1,i\pm1}e_{1,i} = [M : N]^{-1}e_{1,i}$.

Theorem 3.2. *Given $N \subset Q \subset M$ and the notation introduced above, set $p = e_{0,1}\ e_{0,3}\ e_{0,5} \cdots e_{0,2n-1}$. Then the chain*

$$Np \subset pMp \subset pM_1p \subset pM_2p \subset \ldots \subset pM_{2n-1}p$$

is isomorphic to the first $2n - 1$ steps of the basic construction of $N \subset Q$. The Jones projections are given by $e_{0,2i}p : L^2(pM_{2i-1}p) \to L^2(pM_{2i-2}p)$ and $e_{1,2i+1}p : L^2(pM_{2i}p) \to L^2(pM_{2i-1}p)$. The unique normalized trace on the chain, denoted $tr_{N\subset Q}$, is given by $tr_{N\subset Q}(x) = [M : Q]^n tr_{N\subset M}(x)$.

Proof. We prove the result by showing the above chain to be the basic construction and that the inclusion $Np \subset pMp$ is isomorphic to $N \subset Q$. The trace formula is immediate from the fact $tr_{N\subset M}(p) = [M : Q]^n$.

To show the chain is the basic construction it suffices to show [Po2]:

(1) $[pM_ip : pM_{i-1}p] = [Q : N]$
(2) $[e_{0,2i}p, pM_{2i-2}p] = 0 = [e_{1,2i+1}p, pM_{2i-1}p]$
(3) $E_{pM_{2i-1}p}(e_{0,2i}p) = [Q : N]^{-1}1 = E_{pM_{2i}p}(e_{1,2i+1}p)$

To show the tower is isomorphic to that of $N \subset Q$ we show:

(4) $(Np \subset pMp) \cong (N \subset Q)$.

Proof of (4): We have $pMp = pe_{0,1}Me_{0,1}p = pQe_{0,1}p = pQp = Qp$, and thus

$$(Np \subset pMp) = (Np \subset Qp) \cong (N \subset Q).$$

Proof of (1): We will make use of local index formulas (see [Jo1] or Sections 2.2-2.3 of [JoSu]). Note that $pM_{2i}p = pe_{0,2i+1}M_{2i}e_{0,2i+1}p = pP_{2i}e_{0,2i+1}p = pP_{2i}p$. Decompose p as xy where $x = e_{0,1}e_{0,3}\dots e_{0,2i-1}$ and $y = e_{0,2i+1}e_{0,2i+3}\dots e_{0,2n+1}$, and note that $x \in M_{2i-1}$ and $y \in (xP_{2i}x)'$. We have

$$[P_{2i} : M_{2i-1}] = [xP_{2i}x : xM_{2i-1}x] = [xyP_{2i}xy : xyM_{2i-1}xy] = [pM_{2i}p : pM_{2i-1}p].$$

Therefore, $[pM_{2i}p : pM_{2i-1}p] = [Q : N]$.

Similarly,

$$[xM_{2i-1}x : xP_{2i-2}x] = [pM_{2i-1}p : pP_{2i-2}p] = [pM_{2i-1}p : pM_{2i-2}p],$$

since $y \in (xM_{2i-1}x)'$. Since $x \in e_{0,2i-1}P_{2i-2}e_{0,2i-1}$ and $xe_{0,2i-1} = x$ we have

$$[e_{0,2i-1}M_{2i-1}e_{0,2i-1} : e_{0,2i-1}P_{2i-2}e_{0,2i-1}] = [xM_{2i-1}x : xP_{2i-2}x].$$

Using the fact $e_{0,2i-1} \in P'_{2i-2} \cap M_{2i-1}$, we have

$$\begin{aligned}[e_{0,2i-1}M_{2i-1}e_{0,2i-1} : e_{0,2i-1}P_{2i-2}e_{0,2i-1}] &= tr_{M_{2i-1}}(e_{0,2i-1})tr_{P'_{2i-2}}(e_{0,2i-1})[M_{2i-1} : P_{2i-2}]\\ &= [M : Q]^{-2}[M_{2i-1} : P_{2i-2}]\\ &= [Q : N].\end{aligned}$$

Therefore, $[pM_{2i-1}p : pM_{2i-2}p] = [Q : N]$.

Proof of (2): Since $[e_{0,2i}, p] = 0$, and $[e_{0,2i}, M_{2i-2}]$=0 (Fact 3.1) we have $[e_{0,2i}p, pM_{2i-2}p] = 0$. Similarly $[e_{1,2i+1}, p] = 0$ and $[e_{1,2i+1}, M_{2i-1}] = 0$ (Fact 3.1) imply $[e_{1,2i+1}p, pM_{2i-1}p] = 0$.

Proof of (3): To show $E_{pM_{2i-1}p}(e_{0,2i}p) = [Q : N]^{-1}1$ we use Fact 2.4 and show $tr_{pM_{2i}p}(e_{0,2i}p(pm_{2i-1}p)) = [Q : N]^{-1}tr_{pM_{2i}p}(pm_{2i-1}p)$ for all $m_{2i-1} \in M_{2i-1}$.

Write $p = xy$ as in the proof of (1). We note $[x, y] = 0$ (Fact 3.1), $y \in (xP_{2i}x)'$ and $pM_{2i}p = pP_{2i}p$. Therefore by Fact 2.1 (and the fact that $[e_{0,2i}, p] = 0$) we have

$$\begin{aligned} tr_{pP_{2i}p}(e_{0,2i}p(pm_{2i-1}p)) &= (tr_{P_{2i}}(x))^{-1} tr_{P_{2i}}(e_{0,2i}xm_{2i-1}x) \\ &= (tr_{P_{2i}}(x))^{-1} tr_{P_{2i}}(E_{M_{2i-1}}(e_{0,2i})xm_{2i-1}x) \\ &= [Q : N]^{-1}(tr_{P_{2i}}(x))^{-1} tr_{P_{2i}}(xm_{2i-1}x). \end{aligned}$$

A similar calculation yields

$$tr_{pP_{2i}p}(pm_{2i-1}p) = (tr_{P_{2i}}(x))^{-1} tr_{P_{2i}}(xm_{2i-1}x).$$

Combining these two results gives us

$$tr_{pM_{2i}p}(e_{0,2i}p(pm_{2i-1}p)) = [Q : N]^{-1} tr_{pM_{2i}p}(pm_{2i-1}p),$$

as desired.

To prove $E_{pM_{2i}p}(e_{1,2i+1}p) = [Q : N]^{-1}$ we proceed similarly by showing

$$tr_{pM_{2i+1}p}(e_{1,2i+1}p(pm_{2i}p)) = [Q : N]^{-1} tr_{pM_{2i+1}p}(pm_{2i}p)$$

for all $m_{2i} \in M_{2i}$. This time we write $p = \hat{x}\hat{y}$ where $\hat{x} = e_{0,1}e_{0,3}\ldots e_{0,2i+1}$ and $\hat{y} = e_{0,2i+3}e_{0,2i+5}\ldots e_{2n+1}$. Again we have $[\hat{x}, \hat{y}] = 0$, $\hat{y} \in (\hat{x}M_{2i+1}\hat{x})'$, and $[e_{1,2i+1}, \hat{x}] = 0$, thus by Fact 2.1

$$\begin{aligned} tr_{pM_{2i+1}p}(e_{1,2i+1}p(pm_{2i}p)) &= (tr_{M_{2i+1}}(\hat{x}))^{-1} tr_{M_{2i+1}}(e_{1,2i+1}(\hat{x}m_{2i}\hat{x})). \\ &= (tr_{M_{2i+1}}(\hat{x}))^{-1} tr_{M_{2i+1}}(e_{1,2i+1}(e_{0,2i+1}xm_{2i}xe_{0,2i+1})). \end{aligned} \tag{2}$$

Using the property of the trace we can move the rightmost $e_{0,2i+1}$ to the left and simplify using $e_{1,2i+1}e_{0,2i+1} = e_{0,2i+1}e_{1,2i+1} = e_{1,2i+1}$ (Fact 3.1). Then by the Markov property, we obtain

$$tr_{pM_{2i+1}p}(e_{1,2i+1}p(pm_{2i}p)) = (tr_{M_{2i+1}}(\hat{x}))^{-1}[M : N]^{-1} tr_{M_{2i+1}}(xm_{2i}x).$$

A similar calculation gives

$$tr_{pM_{2i+1}p}(pm_{2i}p) = (tr_{M_{2i+1}}(\hat{x}))^{-1}[M : Q]^{-1} tr_{M_{2i+1}}(xm_{2i}x)$$

and thus we have

$$tr_{pM_{2i+1}p}(e_{1,2i+1}p(pm_{2i}p)) = [Q : N]^{-1} tr_{pM_{2i+1}p}(pm_{2i}p)$$

as desired. □

We now have the following corollary that tells how to get the standard invariant of $N \subset Q$ given the standard invariant of $N \subset M$.

Corollary 3.3. *Given $N \subset Q \subset M$. Let*

$$N \subset Q \subset Q_1 \subset Q_2 \subset \ldots \subset Q_{2n-1},$$

be the basic construction tower for the inclusion $N \subset Q$. Then the lattice of algebras:

$$\begin{array}{ccccccccc} p(N'\cap N)p & \subset & p(N'\cap M)p & \subset & p(N'\cap M_1)p & \subset & \dots & \subset & p(N'\cap M_{2n-1})p \\ & & \cup & & \cup & & & & \cup \\ & & p(M'\cap M)p & \subset & p(M'\cap M_1)p & \subset & \dots & \subset & p(M'\cap M_{2n-1})p \end{array}.$$

is isomorphic to standard invariant of $N \subset Q$ up to the $(2n-1)^{th}$ step:

$$\begin{array}{ccccccccc} N'\cap N & \subset & N'\cap Q & \subset & N'\cap Q_1 & \subset & \dots & \subset & N'\cap Q_{2n-1} \\ & & \cup & & \cup & & & & \cup \\ & & Q'\cap Q & \subset & Q'\cap Q_1 & \subset & \dots & \subset & Q'\cap Q_{2n-1}, \end{array}$$

i.e. the cut down by p (p as in Theorem 3.2) of the standard invariant of $N \subset M$ is isomorphic to the standard invariant of $N \subset Q$. The Jones projections are $e_{0,2i}p \in p(N' \cap M_{2i})p$ and $e_{1,2i+1}p \in p(N' \cap M_{2i+1})p$. The isomorphism is trace-preserving with the trace given by $tr_{N\subset Q}(x) = [M : Q]^n tr_{N\subset M}(x)$.

Proof. By Theorem 3.2 we need only show $p(N' \cap M_i)p = (Np)' \cap pM_ip$ and $p(M' \cap M_i)p = (pMp)' \cap pM_ip$.

To begin we show $p(N'\cap M_i)p = (Np)'\cap pM_ip$; we address the cases of i odd and i even separately. Again write $p = xy$ with $x = e_{0,1}e_{0,3}\dots e_{0,2i-1}$ and $y = e_{0,2i+1}e_{0,2i+3}\dots e_{0,2n+1}$. In the odd case, $y \in M'_{2i-1}$ implies by Fact 2.3 that

$$y(N' \cap M_{2i-1})y = (Ny)' \cap (M_{2i-1}y).$$

Then by Fact 2.2, since $p \in (Ny)' \cap M_{2i-1}y$ we have

$$p(N' \cap M_{2i-1})p = (Np)' \cap pM_{2i-1}p.$$

Similarly, for the even case, since $y \in P'_{2i}$ and $p \in (Ny)' \cap P_{2i}y$, we have

$$p(N' \cap P_{2i})p = (Np)' \cap pP_{2i}p.$$

We have already shown that $pP_{2i}p = pM_{2i}p$ so we need only show

$$p(N' \cap P_{2i})p = p(N' \cap M_{2i})p.$$

The forward inclusion is immediate. For the opposite inclusion, given $w \in N' \cap M_{2i}$ we have $E_{P_{2i}}(w) \in N' \cap P_{2i}$. Since

$$pwp = pe_{0,2i+1}we_{0,2i+1}p = pE_{P_{2i}}(w)e_{0,2i+1}p = pE_{P_{2i}}(w)p,$$

we have $p(N' \cap M_{2i})p \subset p(N' \cap P_{2i})p$.

Thus we have shown that $p(N' \cap M_i)p = (Np)' \cap pM_ip$ for i both odd and even.

The identical argument gives us $p(Q' \cap M_i)p = (Qp)' \cap pM_ip$. Since $pMp = Qp$, we need only show $p(Q' \cap M_i)p = p(M' \cap M_i)p$ to have $p(M' \cap M_i)p = (pMp)' \cap pM_ip$. The inclusion $p(M' \cap M_i)p \subset p(Q' \cap M_i)p$ is immediate. Now consider $x \in Q' \cap M_i$. Let $\{m_i\}$ be a Pimsner-Popa basis for M over

Q [PiPo]. It is simple to show that $\phi(w) = \sum m_i w m_i^*$ maps $Q' \cap M_i$ to $M' \cap M_i$. (In fact a scalar multiple of this map turns out to be the $tr|_{Q'}$ preserving condition expectation from Q' to M' [Bi2], though we will not use this fact.)

Now, we have for $w \in Q' \cap M_i$,

$$e_{0,1}\phi(e_{0,1}we_{0,1})e_{0,1} = \sum_i e_{0,1}m_i e_{0,1}we_{0,1}m_i^* e_{0,1} = \sum_i e_{0,1}E_Q^M(m_i)we_{0,1}m_i^* e_{0,1}$$
$$= \sum_i e_{0,1}wE_Q^M(m_i)e_{0,1}m_i^* e_{0,1} = \sum_i e_{0,1}we_{0,1}m_i e_{0,1}m_i^* e_{0,1} = e_{0,1}we_{0,1},$$

where the last equality follows from the Pimsner-Popa basis property $\sum m_i e_{0,1} m_i^* = 1$. Thus $pwp = p\phi(e_{0,1}we_{0,1})p$ and since $\phi(e_{0,1}we_{0,1}) \in M' \cap M_i$ we have that $p(Q' \cap M_i)p \subset p(M' \cap M_i)p$. With the isomorphism established, the description of the Jones projection and the trace follow from Theorem 3.2. □

It is often convenient to express the entire standard invariant instead of the standard invariant up to the $(2n-1)^{\text{th}}$ algebra. For each n, let q_n be the projection denoted by p in Theorem 3.2. Define the inclusion map $j : q_n M_{2n-1} q_n \to q_{n+1} M_{2n+1} q_{n+1}$ by $x \mapsto q_{n+1}x$. Note that $q_{n+1}x = e_{0,2n+1}x$. Since $e_{0,2n+1}$ commutes with M_{2n-1}, j is a one-to-one $*$-homomorphism. It is inverted by the scaled conditional expectation, $[M : Q]E_{M_{2n-1}}$. We denote the inclusion map j by $\subset_j$.

Corollary 3.4. *Given $N \subset Q \subset M$. Let*

$$N \subset Q \subset Q_1 \subset Q_2 \subset \ldots \subset Q_{2n-1} \subset \ldots$$

be the basic construction tower for the inclusion $N \subset Q$. Then the lattice of algebras

$$\begin{array}{ccccccccccc} \mathbb{C} & \subset & q_1(N' \cap M)q_1 & \subset & q_1(N' \cap M_1)q_1 & \subset_j & q_2(N' \cap M_2)q_2 & \subset & q_2(N' \cap M_3)q_2 & \subset_j & \ldots \\ & & \cup & & \cup & & \cup & & \cup & & \\ & & q_1(M' \cap M)q_1 & \subset & q_1(M' \cap M_1)q_1 & \subset_j & q_2(M' \cap M_2)q_2 & \subset & q_2(M' \cap M_3)q_2 & \subset_j & \ldots \end{array}$$

is isomorphic to the standard invariant of $N \subset Q$,

$$\begin{array}{ccccccccccc} N' \cap N & \subset & N' \cap Q & \subset & N' \cap Q_1 & \subset & N' \cap Q_2 & \subset & N' \cap Q_3 & \subset & \ldots \\ & & \cup & & \cup & & \cup & & \cup & & \\ & & Q' \cap Q & \subset & Q' \cap Q_1 & \subset & Q' \cap Q_2 & \subset & Q' \cap Q_3 & \subset & \ldots \end{array}$$

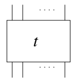
The Jones projections are $e_{0,2n-2}q_n \in q_n(N' \cap M_{2n-2})q_n$, for $n \geq 2$, and $e_{1,2n-1}q_n \in q_n(N' \cap M_{2n-1})q_n$, for $n \geq 1$. The trace on $q_n(N' \cap M_{2n-2})q_n$ and $q_n(N' \cap M_{2n-1})q_n$ is given by $tr_{N\subset Q}(x) = tr_{N\subset M}(x)[M : Q]^n$.

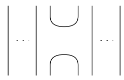
Proof. By the previous discussion, j is an isomorphism and $j(q_n(N' \cap M_i)q_n) = q_{n+1}(N' \cap M_i)q_{n+1}$ for all $i \leq 2n-1$. The result then follows from Corollary 3.3. □

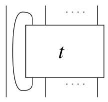
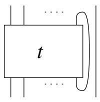
4. The Planar Algebra Perspective

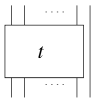
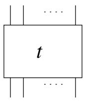
We now translate Corollary 3.3 into the planar algebra setting and define the planar algebra structure of the intermediate subfactor. V. Jones introduced his theory of planar algebras in [Jo2]. In this section we assume the reader is familiar with the following notions from [Jo2]: tangles on a given labeling set L (denoted by $\mathfrak{T}(L)$), basic n boxes, the annular actions by $\mathcal{A}_{n,m}$ from n-boxes to m-boxes, the presenting map Φ of a planar algebra on itself, and planar algebras defined by partition functions. A summary of planar algebra terminology is given in [La1].

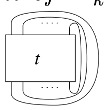
In Theorem 4.2.1 of [Jo2], Jones establishes the equivalence of Popa's axiomatization of the standard invariant (λ-lattices) and planar algebras.

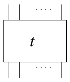
Theorem 4.1. [Jo2] *Let $N \subset M$ be an extremal* II_1 *subfactor. For each n let $P_n^{N\subset M} = N' \cap M_{n-1}$. Then $P^{N\subset M} = \cup_n P_n^{N\subset M}$ has a spherical C^*–planar algebra structure (with labelling set $P^{N\subset M}$) for which* $\Phi($  $) = t$ *and, suppressing the presenting map Φ,*

(i) *For $i = 1, 2, \ldots, n-1$,* $[M : N]^{-\frac{1}{2}}$ 
i
is the ith Jones projection.

(ii)  $= [M : N]^{\frac{1}{2}} E_{M'}(t)$,  $= [M : N]^{\frac{1}{2}} E_{M_{k-1}}(t)$.

(iii)  $=$  *(where on the right, t is considered as an element of M_{k+1}).*

(iv) $Z($  $) = [M : N]^{\frac{k}{2}} tr(t)$.

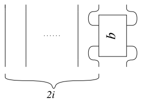
Moreover, any other spherical planar algebra structure Φ' with $\Phi'($  $) =$ *t and (i),(ii),(iii),(iv) for Φ' is equal to Φ.*

Conversely (Theorem 4.3.1 of [Jo2]*), given any spherical C^* planar algebra P, there exists an extremal inclusion of subfactors $N \subset M$ such that P is equivalent to $P^{N\subset M}$ as planar algebras. We will say that a planar algebra P and a subfactor $N \subset M$ are associated to one another if P is equivalent to $P^{N\subset M}$.*

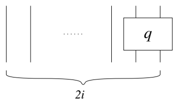
Let us assume we have a planar algebra $(P_k(L))_n$ representing an inclusion $N \subset M$ and that there exists an intermediate subfactor Q, $N \subset Q \subset M$. Let $e_{0,i}$ and $e_{1,i}$, for $i \geq 1$ be as in Section 3, and let the 2- tangle  represent $e_{0,1}$.

Lemma 4.2. *For each $i \geq 1$,*

$$e_{0,2i} = [M : Q]^{1/2}[Q : N]^{-1/2}\Phi_{N\subset M}(\ \ )$$

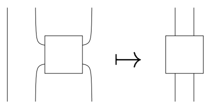


$$e_{0,2i-1} = \Phi_{N\subset M}(\ \ )$$

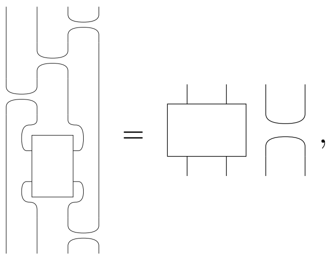


Proof. We first prove the statement holds for $e_{0,2}$. The result will then follow by induction.

First note that there is a bijection $\nu : M' \cap M_2 \to N' \cap M_1$ obtained by "turning the 2-box sideways," i.e.

$\nu :$ 

Let $y \in M' \cap M_2$. Since



it follows that $[M : N]^2 e_{1,3}e_{1,2}e_{1,1}ye_{1,3} = ([M : N]^{\frac{1}{2}}e_{1,3})\nu(y)$. Consider the case $y = e_{0,2}$.

Using Fact 3.1 we have

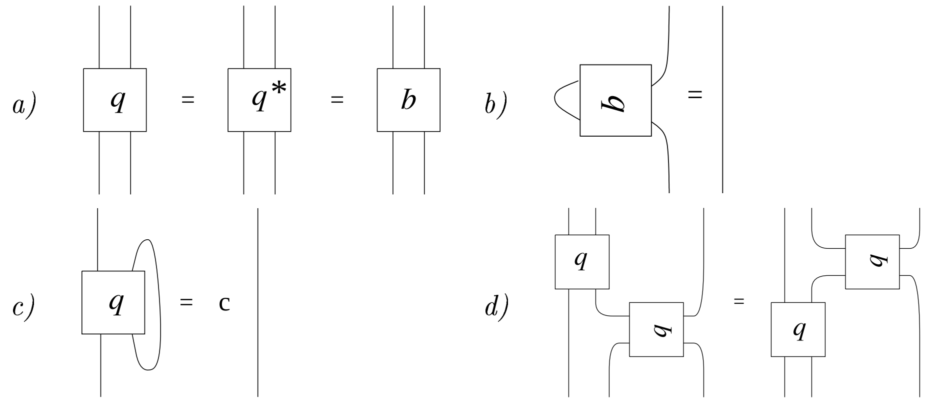
$$\begin{aligned}
[M : N]^2 e_{1,3}(e_{1,2}e_{1,1}e_{0,2})e_{1,3} &= [M : N][M : Q](e_{1,3}e_{0,1})e_{1,2}e_{1,3} \\
&= [M : N][M : Q]e_{0,1}e_{1,3}e_{1,2}e_{1,3} \\
&= [M : Q]e_{0,1}e_{1,3} \\
&= ([M : N]^{\frac{1}{2}}e_{1,3})[M : Q]^{\frac{1}{2}}[Q : N]^{-\frac{1}{2}}e_{0,1}
\end{aligned}$$

It follows that $\nu(e_{0,2}) = [M : Q]^{\frac{1}{2}}[Q : N]^{-\frac{1}{2}}e_{0,1}$, which proves the result in the base case.

In general, there is a similar bijection from $M'_{i+1} \cap M_{i+3}$ to $M'_i \cap M_{i+2}$, for $i \geq 0$, obtained by removing one string on the left and turning the 2-box on the right sideways. The proof of the induction step parallels the proof of the base case and is left as an exercise for the reader. □

The following corollary assembles some useful planar relations satisfied by a projection corresponding to an intermediate subfactor. These relations completely characterize projections corresponding to intermediate subfactors in the case $N' \cap M = \mathbb{C}$.

Corollary 4.3. *Let $N \subset M$ be an extremal II_1 subfactor. Let $P^{N\subset M}$ be the planar algebra of $N \subset M$, and let $\Phi_{N\subset M}$ be the presenting map of $P^{N\subset M}$ on itself, i.e. $\Phi_{N\subset M} : \mathcal{P}(L) \to P^{N\subset M}$ with $L = \coprod P_k^{N\subset M}$. Suppose there exists an intermediate subfactor Q, $N \subset Q \subset M$. If we let $\boxed{q}$ denote the projection $e_{0,1}$ corresponding to Q, we have*



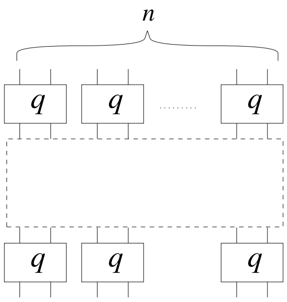
with $c = [M : N]^{1/2}[M : Q]^{-1}$. Furthermore, in the case $N' \cap M = \mathbb{C}$, the converse is also true. Namely a 2-box $\boxed{q}$ satisfying a)-d) above implies the existence of an intermediate subfactor Q, $N \subset Q \subset M$ with $\boxed{q}$ corresponding to $e_{0,1}$.

Proof. The relations a)-d) follow simply from first translating the pictures to statements about $e_{0,i}$ using Lemma 4.2 and part *ii*) of Theorem 4.1 and then verifying these statements about $e_{0,i}$ using Fact 3.1 and other well known facts about $e_{0,i}$. For example c) follows from $E_N(e_{0,1}) = [M : Q]^{-1}1$, d) follows from $[e_{0,1}, e_{0,2}] = 0$. We omit the details.

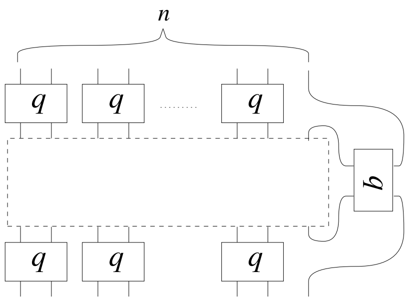
The converse statement is Corollary 4.3.1 of [La1] (see also [BiJo3]).

□

Denote by $N \subset Q \subset Q_1 \subset Q_2 \subset \ldots$ the Jones tower of $N \subset Q$. Define $B_n \in \mathcal{A}_{n,n}$ to be the following:



for n even,



for n odd.

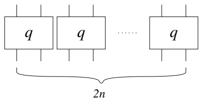
Corollary 4.4. *The lattice of algebras:*

$$\begin{array}{ccccccccccccc} B_{2n}(P_0(L)) & \subset & B_{2n}(P_1(L)) & \subset & B_{2n}(P_2(L)) & \subset & B_{2n}(P_3(L)) & \subset & \dots & \subset & B_{2n}(P_{2n}(L)) \\ & & \cup & & \cup & & \cup & & & & \cup \\ & & B_{2n}(P_{1,1}(L)) & \subset & B_{2n}(P_{1,2}(L)) & \subset & B_{2n}(P_{1,3}(L)) & \subset & \dots & \subset & B_{2n}(P_{1,2n}(L)) \end{array}$$

is isomorphic to standard invariant of $N \subset Q$ up to the $(2n-1)^{th}$ step:

$$\begin{array}{ccccccccccc} N' \cap N & \subset & N' \cap Q & \subset & N' \cap Q_1 & \subset & N' \cap Q_2 & \subset & \dots & \subset & N' \cap Q_{2n-1} \\ & & \cup & & \cup & & \cup & & & & \cup \\ & & Q' \cap Q & \subset & Q' \cap Q_1 & \subset & Q' \cap Q_2 & \subset & \dots & \subset & Q' \cap Q_{2n-1} \end{array}$$

Proof. Let p be as in Theorem 3.2. By Lemma 4.2,

$p =$ 

so the corollary follows from Corollary 3.3. □

We can express the standard invariant in a more satisfactory way if we define the natural inclusion map

$$i : B_n(P_n) \to B_{n+1}(P_{n+1})$$

given by

$$t \mapsto B_{n+1}(t).$$

We denote this inclusion by $\subset_i$.

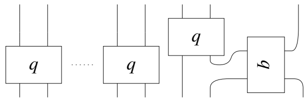
Corollary 4.5. *The lattice of algebras:*

$$\begin{array}{ccccccccccc} B_0(P_0(L)) & \subset_i & B_1(P_1(L)) & \subset_i & B_2(P_2(L)) & \subset_i & \dots & \subset_i & B_n(P_n(L)) & \subset_i & \dots \\ & & \cup & & \cup & & & & \cup & & \\ & & B_1(P_{1,1}(L)) & \subset_i & B_2(P_{1,2}(L)) & \subset_i & \dots & \subset_i & B_n(P_{1,n}(L)) & \subset_i & \dots \end{array}$$

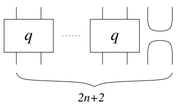
is isomorphic to the standard invariant of $N \subset Q$:

$$\begin{array}{ccccccccccc} N' \cap N & \subset & N' \cap Q & \subset & N' \cap Q_1 & \subset & \dots & \subset & N' \cap Q_{n-1} & \subset & \dots \\ & & \cup & & \cup & & & & \cup & & \\ & & Q' \cap Q & \subset & Q' \cap Q_1 & \subset & \dots & \subset & Q' \cap Q_{n-1} & \subset & \dots \end{array}$$

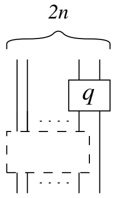
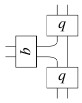
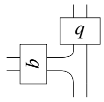
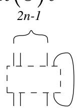
The Jones projections are

$$e_{2n}^Q = [M:Q]^{1/2}[Q:N]^{-1/2}\Phi_{N\subset M}($$  $$) \in B_{2n+1}(P_{2n+1}(L))$$

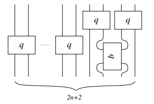
and

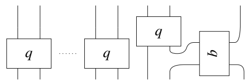
$$e_{2n+1}^Q = [M:N]^{-1/2}\Phi_{N\subset M}($$  $$) \in B_{2n+2}(P_{2n+2}(L)),$$

for $n \geq 1$. The trace on $B_{2n}(P_{2n}(L))$ and $B_{2n+1}(P_{2n+1}(L))$ is given by $tr_{N\subset Q}(x) = [M:Q]^n tr_{N\subset M}(x)$.

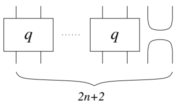
Proof. To prove the isomorphism, it suffices by Corollary 4.4 to show that $i : B_{2n-1}(P_{2n-1}(L)) \to B_{2n}(P_{2n-1}(L))$ and $i^2 : B_{2n-2}(P_{2n-2}(L)) \to B_{2n}(P_{2n-2}(L))$ are isomorphisms. The second map is obviously a homomorphism and it is easy to show that its inverse is $[M:Q]E_{M_{2n-3}}$. Now consider the first map. Denote by $\alpha \in \mathcal{A}_{2n-1,2n}$ the annular tangle  . Corollary 4.3 establishes that the tangles  and  are equal and thus $i(t) = \alpha(t)$ for all $t \in B_{2n-1}(P_{2n-1}(L))$. It follows that i is a homomorphism since for $s, t \in B_{2n-1}(P_{2n-1}(L))$ we have $i(s)i(t) = i(s)\alpha(t) = i(s)t = \alpha(s)t = \alpha(st) = i(st)$. To see i is one-to-one, let $\epsilon \in \mathcal{A}_{2n,2n-1}$ be the map  . Then by relation *c)* of Corollary 4.3, $\epsilon(\alpha(t))$ is a scalar multiple of t. Therefore, $i|_{B_{2n-1}(P_{2n-1}(L))}$ is a isomorphism. Since q is self-adjoint, this is a $*$-isomorphism.

We now prove the Jones projection formulas. Fix $n \geq 1$ and let $p = e_{0,1}\ e_{0,3}\ e_{0,5} \cdots e_{0,2n+1}$. be as in Theorem 3.2. By Corollary 3.3, the Jones projections in $B_{2n+2}(P_{2n+1}(L))$ and $B_{2n+2}(P_{2n+2}(L))$ are $e_{0,2n}p = pe_{0,2n}$ and $e_{1,2n+1}p$, respectively. Then by Lemma 4.2,

(3) $e_{0,2n}p \quad = \quad [M:Q]^{1/2}[Q:N]^{-1/2}\Phi_{N\subset M}($  $)$

(4) $\quad = \quad i([M:Q]^{1/2}[Q:N]^{-1/2}\Phi_{N\subset M}($  $))$.

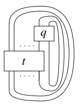
By Theorem 4.1 and relation *b)* of Corollary 4.3,

$e_{1,2n+1}p = [M:N]^{-1/2}\Phi_{N\subset M}($  $)$.

By Corollary 3.3, the trace on $B_{2n}(P_{2n}(L))$ is $tr_{N\subset Q}(x) = [M:Q]^n tr_{N\subset M}(x)$. For the odd case, given $t \in B_{2n+1}(P_{2n+1}(L))$,

$$tr_{N\subset Q}(t) = tr_{N\subset Q}(i(t)) = [M:Q]^{n+1}tr_{N\subset M}(i(t)), \tag{5}$$

the last equality following from the even case above. Now the trace formula in condition (iv) of Theorem 4.1 and relation *c)* in Corollary 4.3 imply that

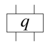
$[M:N]^{n+1}tr_{N\subset M}(i(t)) = Z($  $)$

$$= [M:N]^{1/2}[M:Q]^{-1}([M:N]^{n+1/2}tr_{N\subset M}(t)$$
$$= [M:Q]^{-1}[M:N]^{n+1}tr_{N\subset M}(t)$$

The trace result for the odd case follows by dividing the above by $[M:N]^{n+1}$ and combining with (5). $\square$

5. The Intermediate Planar Algebra

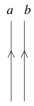
Given a subfactor $N \subset M$ with an intermediate subfactor, we described in Section 4 how to obtain the standard invariant of the intermediate subfactor from the planar algebra of $N \subset M$. We will now exhibit the planar algebras of the intermediate subfactors in the case that $N \subset M$ is irreducible ($N' \cap M = \mathbb{C}$). For the remainder of this section we will assume $N \subset M$ is irreducible.

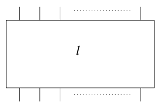
Let $N \subset M$ be an irreducible II_1 subfactor of finite index with intermediate subfactor $N \subset Q \subset M$ and corresponding projection . Let $P^{N\subset M}$ be the planar algebra corresponding to the inclusion $N \subset M$. Let $\Phi_{N\subset M}$ be the presenting map of $P^{N\subset M}$ on itself i.e., $\Phi_{N\subset M} : \mathcal{P}(L) \to P^{N\subset M}$ with $L = \coprod P_k^{N\subset M}$.

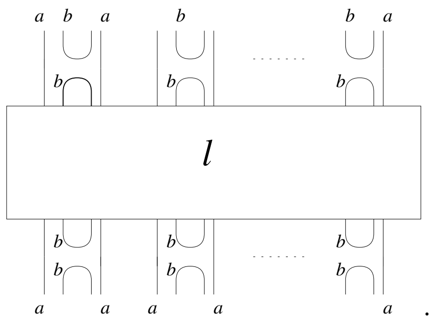
For $n > 0$ set $S_{2n-1} = \emptyset$, $S_{2n} = \{l \in L_{2n} = P_{2n}^{N\subset M} : B_{2n}(l) = l\} \subset L$.

We now construct a planar algebra on the label set S. What follows, applied to example 6.1, is the inverse construction to the free-product. Thus it may be helpful to read the rest of this section with that example in mind. Other related constructions are reduction and cabling in [Jo2]. Define a partition function $Z : \mathfrak{T}_0(S) \to \mathbb{C}$ as follows:

Given a zero tangle T with standard boxes of size $i_1, i_2, \dots i_n$, define $Z(T) = \Phi_{N\subset M}(T) \cdot [M : Q]^{\frac{1}{2}c}$ where c is determined by the following process. In T :

(1) Replace every strand with two strands labeled a and b  .

(2) Replace the standard (even) box  with



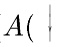
.

(3) Let d be the number of closed loops labeled by "b" in the resulting picture. Set $c = \frac{1}{2}(i_1 + i_2 + \cdots + i_n) - d$.

Note that if we ignore the strands of type "b", the tangle consisting only of strands of type "a" is identical to T.

It is straightforward to verify that Z is a multiplicative planar isotopy invariant and thus can be used to construct a planar algebra with partition function Z, which we will call $P^{N\subset Q}$. We shall denote the corresponding planar presentation map from the label S to $P^{N\subset Q}$ by Φ_Z. As our notation suggests, $P^{N\subset Q}$ is the planar algebra of $N \subset Q$, which we prove in Theorem 5.3. The proof of Theorem 5.3 relies on the following lemmas.

Lemma 5.1. *In $P^{N\subset Q}$,*

$Z(A($  $)) = Z(A($  $))$

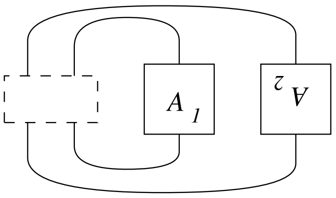
for all $A \in \mathcal{A}_{2,0}(S)$, *consequently*

$$\Phi_Z(\boxed{q}) = \Phi_Z(\,||\,) = 1$$

Proof. Since Z is multiplicative on connected components, it suffices to show the result for $A \in \mathcal{A}_{2,0}(S)$ that do not contain zero-tangles. Label the strands of the 2-box $\boxed{q}$ clockwise 1 through 4 with 1 at the upper left and 4 at the lower left. The 0-tangle $A(\boxed{q})$ can be characterized in one of the following two ways:

i) The region between strands 1 and 2 of $A(\boxed{q})$ is connected to the region between strands 3 and 4 of $A(\boxed{q})$.
ii) The region between strands 1 and 2 of $A(\boxed{q})$ is disconnected from the region between strands 3 and 4 of $A(\boxed{q})$.

Case i): Up to isotopy on the sphere, we have that A is of the form:



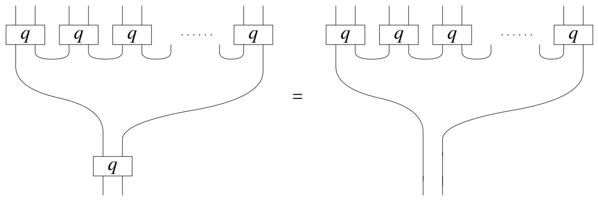
,

where A_1 and A_2 are 1-tangles in $\mathfrak{T}(S)$. Since $P_1^{N\subset M} = \mathbb{C}$ we have $\Phi_{N\subset M}(\boxed{A_i}) = \Phi_{N\subset M}(\beta_i\, |\,)$ with $\beta_i \in \mathbb{C}$, and thus $\Phi_{N\subset M}(A(\boxed{q})) = \beta_1\beta_2[Q : N]$, $\Phi_{N\subset M}(A(\,||\,)) = \beta_1\beta_2[M : N]$.

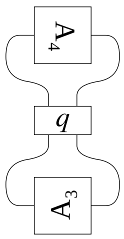
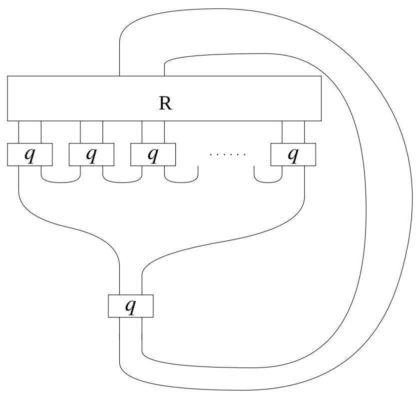
Now in the computation of Z, $A(\boxed{q})$ has 1 extra 2-box and 1 fewer "b" loop than $A(\,||\,)$ and so

$$Z(A(\boxed{q}))/Z(A(\,||\,)) = [M : Q]\Phi_{N\subset M}(A(\boxed{q}))/\Phi_{N\subset M}(A(\,||\,))$$
$$= [M : Q][Q : N]/[M : N] = 1.$$

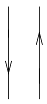
Case ii): It follows from Corollary 4.3 that the following relation holds in $P^{N\subset M}$:

(6)  .

Because we are in case ii), as an element of $P^{N\subset M}$, either $A($  $)$ looks like

(a)  or (b)  .

depending on whether strands 1 and 2 are joined together (figure (a)) or not (figure (b)). In figure (a), A_3 and A_4 are 1-tangles in $\mathfrak{T}(S)$. In figure (b), the box labeled by R is just like a tangle with the exception that the number of strands leaving the top is different from the number of strands leaving the bottom.

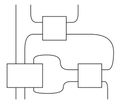
In either case (a) or (b), $\Phi_{N\subset M}(A($  $)) = \Phi_{N\subset M}(A($  $))$. In case (a) this follows by the first two relations in Corollary 4.3 and the fact that $P_1^{N\subset M} = \mathbb{C}$, and in case (b) it follows from (6). In the computation of Z for either case (a) or (b), $A($  $)$ has one extra 2-box and one extra "b" labeled loop than $A($  $)$. These cancel each other in the computation of Z and we have $Z(A($  $)) = Z(A($  $))$. $\square$

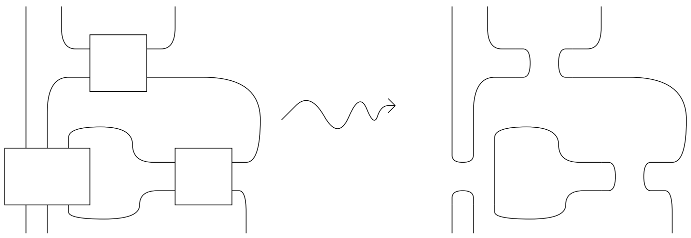
Recall that Φ_Z denote the presenting map of $P^{N\subset Q}$ on the label set S. We describe in Lemma 5.2 a method for showing that two tangles $T_1, T_2 \in \mathfrak{T}_k(S)$ have the same image in $P^{N\subset Q}$, i.e., $\Phi_Z(T_1) = \Phi_Z(T_2)$.

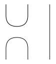
For a tangle T with standard n-boxes of size $i_1, i_2, \ldots, i_m$, define $\gamma(T) = \frac{1}{2}(i_1 + i_2 + \cdots + i_m)$.

Define functions β and K on tangles as follows:

a) For a tangle T, perform steps 1. and 2. given in the definition of Z. Look at the sub-picture consisting only of strands of type "b" that are not joined to the standard boxes of T.
b) Let $\beta(T)$ be the number of closed loops of "b" labeled strands in this sub-picture.
c) Let $K(T)$ be the underlying Kauffman $(0, k)$-diagram (Temperley-Lieb diagram) that remains once the closed loops are removed from this sub-picture.

We provide an example to clarify. For the tangle $T =$  we have



and thus $K(T) =$  , $\beta(T) = 1$.

With this notation, for a 0-tangle T we have $Z(T) = \Phi_{N \subset M}(T)[M : Q]^{\frac{1}{2}(\gamma(T) - \beta(T))}$. The following lemma follows easily.

Lemma 5.2. *Let T_1 and T_2 be two k-tangles on the labeling set S, such that $K(T_1) = K(T_2)$ and $\Phi_{N \subset M}(T_1)$ is a scalar multiple of $\Phi_{N \subset M}(T_2)$, i.e. $\Phi_{N \subset M}(T_1) = \alpha \Phi_{N \subset M}(T_2)$ for $\alpha \in \mathbb{C}$. Then*

$$\Phi_Z(T_1) = \alpha [M : Q]^{\frac{1}{2}(\gamma(T_1) - \beta(T_1) - \gamma(T_2) + \beta(T_2))} \Phi_Z(T_2)$$

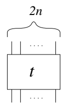
Theorem 5.3. *$P^{N \subset Q}$ is the planar algebra of the inclusion $N \subset Q$.*

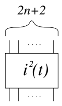
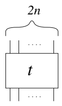
Proof: By Corollary 4.5, the tower of higher relative commutants of $N \subset Q$ is isomorphic to

$$B_0(P_0(L)) \subset_i B_1(P_1(L)) \subset_i B_2(P_2(L)) \subset_i \ldots \tag{7}$$

Note that S_{2n} is precisely $B_{2n}(P_{2n}(L))$. We will show that (7) is isomorphic to $(P_n^{N \subset Q})_n$ via the natural map. We must also show that the planar algebra structure on (7) defined by Φ_Z is the planar algebra structure of $N \subset Q$.

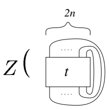
To prove (7) is isomorphic to $(P_n^{N \subset Q})_n$, we show

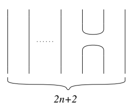
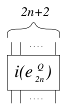
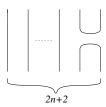
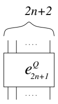
(1) The map $\Phi^n : B_{2n}(P_{2n}(L)) \to P_{2n}^{N\subset Q}$ defined by $t \mapsto \Phi_Z($  $)$ is an isomorphism.

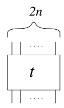
(2) Given $t \in B_{2n}(P_{2n}(L))$, $\Phi_Z($  $) = \Phi_Z($  $)$.

(3) $\Phi^n|_{i(B_{2n-1}(P_{2n-1}(L)))} = P_{2n-1}^{N\subset Q}$.

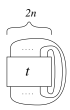
To prove that we have the right planar algebra structure, it suffices to show that

(4) $\Phi_Z($  $) = [Q : N]^n tr_{N\subset Q}(t)$ for $t \in B_{2n}(P_{2n}(L))$.

(5) $\Phi_Z($  $) = [Q : N]^{\frac{1}{2}}\Phi_Z($  $)$ and $\Phi_Z($  $) = [Q : N]^{\frac{1}{2}}\Phi_Z($  $)$.

(6) If $t \in B_{2n}(P_{1,2n}(L))$ then $\Phi_Z($  $) \in P_{1,2n}^{N\subset Q}$ and this is an isomorphism of $B_{2n}(P_{1,2n}(L))$ with $P_{1,2n}^{N\subset Q}$.

The above conditions are sufficient by Theorem 4.1 (note that (4) and (6) imply the conditions in the theorem involving conditional expectations).

Proof of 4. Let $t \in B_{2n}(P_{2n}(L))$ and let $T =$  . Then $\gamma(T) = n$ and $\beta(T) = n$. So

$$\begin{aligned}\Phi_Z(T) &= \Phi_{N\subset M}(T)\\ &= [M : N]^n tr_{N\subset M}(t)\\ &= [M : N]^n [M : Q]^{-n} tr_{N\subset Q}(t)\\ &= [Q : N]^n tr_{N\subset Q}(t)\end{aligned}$$

Proof of 1.

We begin by showing that Φ^n is a homomorphism. Let $t, u \in B_{2n}(P_{2n}(L))$ and consider their product $tu \in B_{2n}(P_{2n}(L))$. Note that

$$\Phi^n(t)\Phi^n(u) = \Phi_Z\left(\begin{array}{c}\boxed{t}\\ \boxed{u}\end{array}\right).$$

We have

$$K\left(\begin{array}{c}\boxed{t}\\ \boxed{u}\end{array}\right) = K\left(\boxed{tu}\right),$$

$$\beta\left(\begin{array}{c}\boxed{t}\\ \boxed{u}\end{array}\right) = n, \qquad \beta\left(\boxed{tu}\right) = 0,$$

$$\gamma\left(\begin{array}{c}\boxed{t}\\ \boxed{u}\end{array}\right) = 2n, \qquad \gamma\left(\boxed{tu}\right) = n,$$

and thus by Lemma 5.2

$$\Phi_Z\left(\begin{array}{c}\boxed{t}\\ \boxed{u}\end{array}\right) = \Phi_Z\left(\boxed{tu}\right) = \Phi^n(tu).$$

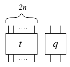
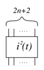
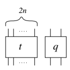
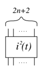
Therefore, Φ^n is a homomorphism, as desired.

Given $t, t' \in B_{2n}$, if $tr_{N\subset Q}(st) = tr_{N\subset Q}(st')$ for all $s \in B_{2n}$, then $t = t'$. Therefore, (4) implies that Φ^n is one-to-one. Given an arbitrary tangle $T \in \mathfrak{T}_{2n}(S)$, we show Φ^n onto by showing $\Phi_Z(T)$ to be the image of a standard $2n$ box labeled by an element of S_{2n}. Let $U = \Phi_{N\subset M}(B_{2n}(T)) \in S_{2n}$. It is easy to check that $K(\Phi^n(U)) = K(B_{2n}(T))$. Then by Lemma 5.2, $\Phi_Z(B_{2n}(T))$ is a scalar multiple of $\Phi^n(U)$. By Lemma 5.1, $\Phi_Z(T) = \Phi_Z(B_{2n}(T))$ and it follows that $\Phi_Z(T)$ is a scalar multiple of $\Phi^n(U)$.

Proof of 2. For $t \in B_{2n}(P_{2n}(L))$, we have $\Phi_Z\left(\overbrace{\boxed{t}}^{2n}\right) = \Phi_Z\left(\overbrace{\boxed{t}}^{2n}\ \Big|\ \Big|\right) =$ $\Phi_Z\left(\overbrace{\boxed{t}}^{2n}\ \boxed{q}\right)$, where the last equality is by Lemma 5.1.

Note that

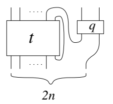
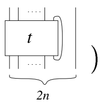
$$i^2(t) = B_{2n+2}(t) = \Phi_{N\subset M}\left(\overbrace{\boxed{t}}^{2n}\ \boxed{q}\right). \tag{8}$$

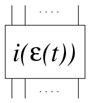
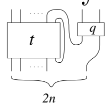
Since the functions K, γ, β and $\Phi_{N\subset M}$ are equal for the tangles  and  we have by Lemma 5.2 that $\Phi_Z($  $) = \Phi_Z($  $)$ and the result follows.

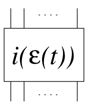
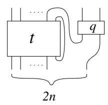
Proof of 3. We will use the following:

Claim. *For $t \in B_{2n}(P_{2n}(L))$, $\epsilon(\Phi^n(t)) = [M:Q]^{-\frac{1}{2}}\Phi^n(i(\epsilon(t)))$, ($\epsilon$ as in the proof of Corollary 4.5).*

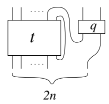
Proof of Claim. We have

$\Phi_Z($  $) = \Phi_Z($  $) = \epsilon(\Phi^n(t)),$

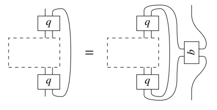
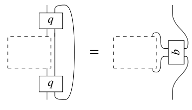
the first equality coming from Lemma 5.1. It is easy to verify that K, $\Phi_{N\subset M}$, and β are equal for the two tangles  and . In addition,

$\gamma($  $) = \gamma($  $) - 1.$

Thus by Lemma 5.2,

$\Phi^n(i(\epsilon(t))) = [M:Q]^{\frac{1}{2}}\Phi_Z($  $) = [M:Q]^{\frac{1}{2}}\epsilon(\Phi^n(t)).$ □Claim

Consider the two relations

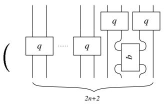
a)  b) 

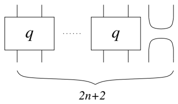
established by Corollary 4.3. Relation a) implies $\epsilon(B_{2n}(P_{2n}(L))) \subset B_{2n-1}(P_{2n-1}(L))$ while relation b) implies $\epsilon(B_{2n}(P_{2n-1}(L))) = B_{2n-1}(P_{2n-1}(L))$. Combining these two implies $\epsilon(B_{2n}(P_{2n}(L))) = B_{2n-1}(P_{2n-1}(L))$. Using this result, the claim, and the fact that $\Phi^n_{N\subset Q}$ is onto we have

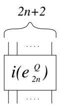
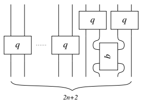
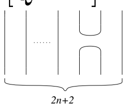
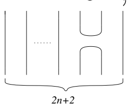
$$\Phi^n(i(B_{2n-1}(P_{2n-1}(L)))) = \Phi^n(i(\epsilon(B_{2n}(P_{2n}(L))))) = \epsilon(\Phi^n(B_{2n}(P_{2n}(L))))$$
$$= \epsilon(P^{N\subset Q}_{2n}) = P^{N\subset Q}_{2n-1}.$$

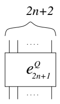
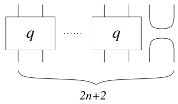
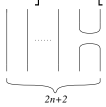
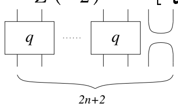
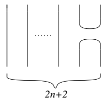
Proof of 6. The proof of 6 follows the same lines as 3 replacing the map ϵ with the map that loops together the left most top and bottom strand.

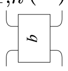
Proof of 5. In the proof of Corollary 4.5, we have shown that the $2n^{\text{th}}$ and $(2n+1)^{\text{th}}$ Jones projections, as elements of $B_{2n+2}(P_{2n+2}(L))$ are

$$e_{2n}^{Q} = [M:Q]^{1/2}[Q:N]^{-1/2}\Phi_{N\subset M}(\quad)$$


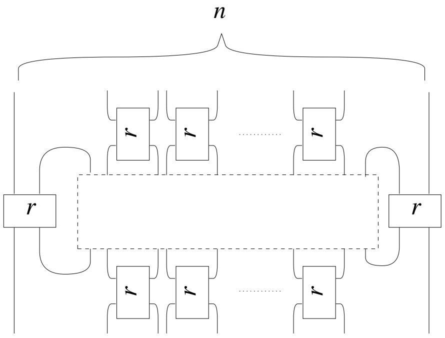
$$e_{2n+1}^{Q} = [M:N]^{-1/2}\Phi_{N\subset M}(\quad).$$


Let $T_1 =$  and $T_2 =$ . By equation (3) $\Phi_{N\subset M}(T_1) = [M : Q]^{1/2}[Q : N]^{-1/2}\Phi_{N\subset M}(T_2)$. The functions K and β are the same for T_1 and T_2, and $\gamma(T_1) = n+1 = \gamma(T_2) - 1$. So by Lemma 5.2, $\Phi_Z(T_1) = [M : Q]^{1/2}[Q : N]^{-1/2}[M : Q]^{-1/2}\Phi_Z(T_2) = [Q : N]^{-1/2}\Phi_Z(T_2)$. By Lemma 5.1, $\Phi_Z($  $) = \Phi_Z(T_2)$. Therefore, $\Phi_Z(T_1) = [Q : N]^{-1/2}\Phi_Z($  $)$.

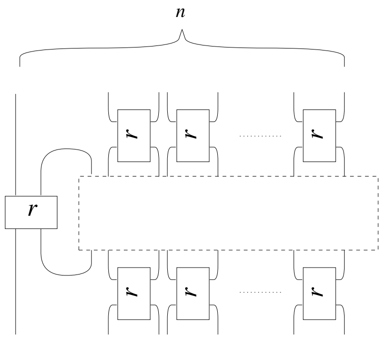
For the case of e_{2n+1}^{Q}, let $T_1 =$  and $T_2 =$ . Then $\Phi_{N\subset M}(T_1) = [M : N]^{-1/2}\Phi_{N\subset M}(T_2)$ (Corollary 4.5), $\gamma(T_1) = n+1 = \gamma(T_2)+1$, and the functions K and β take the same values on T_1 and T_2. So by Lemma 5.2, $\Phi_Z(T_1) = [M : N]^{-1/2}[M : Q]^{1/2}\Phi_Z(T_2) = [Q : N]^{-1/2}\Phi_Z(T_2)$. Since by Lemma 5.1 $\Phi_Z($  $) = \Phi_Z($  $)$, we have $\Phi_Z(T_1) = [Q : N]^{-1/2}\Phi_Z($  $)$. □

We now consider the other intermediate subfactor $Q \subset M$. We can describe its planar algebra using Theorem 5.3 and the fact that it is the dual subfactor to $M \subset Q_1$. Namely, apply Theorem 5.3 to the planar algebra $(P_{1,n}(L))_n$ of $M \subset M_1$, with respect to the projection $[M : Q]^{1/2}[Q : N]^{-1/2}($  $)$. We obtain the planar algebra $(P_n^{M\subset Q_1})_n$. The planar algebra of $Q \subset M$ is its dual, $(P_{1,n}^{M\subset Q_1})_n$. If we carry out this process, we obtain the following planar algebra.

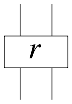
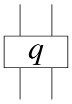
Let $B'_n \in \mathcal{A}_{n,n}$ be the annulus



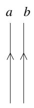
for n even,

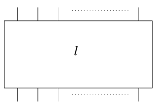


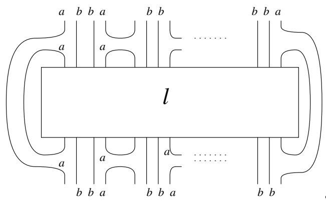
for n odd.

where  $= (\frac{[M:Q]}{[Q:N]})^{\frac{1}{2}}$  .

For $n > 0$ set $S'_{2n-1} = \emptyset$, $S'_{2n} = \{l \in P^{N\subset M}_{2n} : B'_{2n}(l) = l\}$, $S' = \coprod S'_n$. Given a zero tangle $T \in \mathfrak{T}_0(S')$ we define the partition function Z as follows:

(1) Replace every strand with two strands labeled a and b  .

(2) Replace the standard box  with

 ,

for an even n box and

(3) Set $Z(T) = \Phi_{N\subset M}(T)\cdot[Q:N]^{\frac{1}{2}c}$ where $c = \frac{1}{2}\gamma(T)-$number of loops of type "a". As usual, $\Phi_{N\subset M}$ is the presenting map for $P^{N\subset M}$ on itself.

It is straightforward to verify the following corollary:

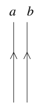
Corollary 5.4. *The planar algebra defined by the partition function Z above is the planar algebra of $Q \subset M$.*

6. Examples

The following examples serve to clarify our results in different ways. The sections on Krishnan-Sunder (6.2) subfactors, tree subfactors (6.3), and exchange relation subfactors (6.5) present new results. In all three cases we show the subfactors are compositions of well known subfactors. The section on free products (6.1) has been included to clarify the partition function defined in Section 4. The section on subgroup subfactors (6.4) provides a basic example of how to apply Theorem 4.5 as well as identifying two different descriptions of the standard invariant of a subgroup subfactor.

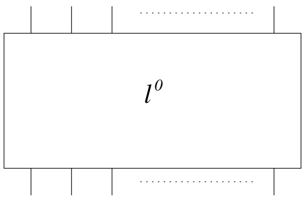
6.1. **Free Products.** The notion of a free product of planar algebras, due to D. Bisch and V. Jones, is described in [La1]. We include the definition here:

Definition 6.1 (Bisch and Jones). *Given two planar algebras $P^0 = P(L^0)$, $P^1 = P(L^1)$ define the* <u>*free product*</u> *planar algebra $P(L) = P^0 * P^1$ as follows. Let the labeling set be $\overline{L = L^0 \coprod L^1}$. Define the partition function Z of $P(L)$ as follows. Given an element $X \in \mathfrak{T}_0(L)$,*

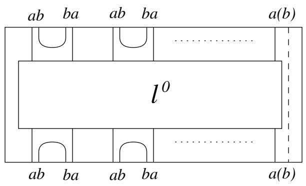
(1) *Replace all single strands*  *with two strands labeled a and b* 

.

(2) *Replace all basic boxes labeled by $l^0 \in L^0$ as follows:*



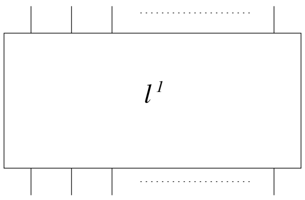
by



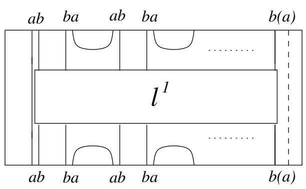
,

where the parenthetical string of type "b" is drawn if l^0 is an n box with n odd.

(3) *Replace all basic boxes labeled by $l^1 \in L^1$ as follows:*



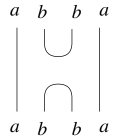
by



,

where the parenthetical string of type "a" is drawn if l^1 is an n box with n even.

(4) *The resulting picture is an overlay of two 0-tangles: an element of $\mathcal{P}_0(L^0)$ (the tangle connected by strands labeled by "a") and an element of $\mathcal{P}_1(L^1)$ (the tangle connected by strands labeled by "b"). Call these two 0-tangles S and T respectively. Define the partition function $Z(X) = Z_0(S)Z_1(T)$ where Z_0 (resp. Z_1) is the partition function for $P(L^0)$ (resp. $P(L^1)$).*

In what follows $P(L) = P^0 * P^1$ and $N \subset M$ denotes a subfactor associated to $P(L)$. It is simple to verify that a scalar times the picture 

is in $P_2(L)$ and that it satisfies the conditions of Corollary 4.3 and therefore corresponds to an intermediate subfactor Q, $N \subset Q \subset M$. A straightforward application of Corollary 4.4 shows that P^0 (resp. P^1) is a planar algebra associated to $N \subset Q$ (resp. $Q \subset M$). Thus for a free product subfactor, the computation of the planar algebra of the intermediate subfactors recovers the planar algebra components of the free product.

6.2. **Krishnan-Sunder Subfactors.** A Krishnan-Sunder subfactor $R_U \subset R$ of index k^2 is constructed from a biunitary matrix $U \in M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$ which is also a permutation matrix, i.e. the entries of U are either 0 or 1 and both U and its block transpose are unitary ([KrSu],[JoSu]). In [Bh3], the first author shows that every Krishnan-Sunder subfactor has an intermediate von Neumann algebra $R_U \subset Q_U \subset R$ by identifying the corresponding projection in the first relative commutant of $R_U \subset R$ (see [Bi1]). If $R_U \subset Q_U$ is a subfactor, we may use Corollary 3.4 to compute its standard invariant from the standard invariant of $R_U \subset R$.

The standard invariant of $R_U \subset R$ has a complicated description involving, among other things, a possibly infinite discrete group G_U that acts on the free semigroup with k generators ([KrSu], [Bh2]). However, when we compute from it the standard invariant of $R_U \subset Q_U$, we obtain a familiar lattice of algebras characterized by a subgroup $\Gamma_U \subset S_k$(the permutation group on k elements), which is easy to compute from U. This computation inspired a direct proof in [Bh1] that if Γ_U is a transitive subgroup then $R_U \subset Q_U$ is isomorphic to the *subgroup subfactor* $R^{\Gamma_U} \subset R^H$, where $H \subset \Gamma_U$ is the stabilizer subgroup of a point in $\{1, 2, \dots, k\}$.

The standard invariant of subgroup subfactors is well known, in terms of its paragroup ([KoYa]). However, our description of the standard invariant in Proposition 6.5 has a different flavor and with Theorem 5.3 defines the planar algebra of $R_U \subset Q_U$. It is easy to see that for any inclusion of groups $H \subset \Gamma$ there exists a Krishnan Sunder subfactor $R_U \subset R$ of index $|\Gamma/H|^2$ such that $R_U \subset Q_U$ is isomorphic to $R^\Gamma \subset R^H$. Thus, for any subgroup subfactor we obtain a description of its standard invariant and its planar algebra.

6.2.1. *Definition of* Γ_U. We first set some notation. Index the rows and columns of matrices in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$ by the set $\{1, 2, \dots p\} \times \{1, 2, \dots, k\}$ in the natural way. Following the notation in [KrSu], we denote elements of $\{1, 2, \dots, p\}$ with Greek letters and elements of $\{1, 2, \dots, k\}$ with Roman letters. For F in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, denote the entry of F in row (α, a) and column (β, b) by $F^{\beta b}_{\alpha a}$. Note that a permutation matrix U gives rise to a Krishnan-Sunder subfactor precisely when $\widetilde{U}$, defined by

$$\widetilde{U}^{\alpha b}_{\beta a} = \overline{U^{\beta b}_{\alpha a}},$$

is also a permutation matrix. The following lemma from [KrSu] is the starting point of our analysis.

Lemma 6.2 (Krishnan-Sunder). *If U is a permutation biunitary in $M_p(\mathbb{C}) \otimes M_k(\mathbb{C})$, then there exist permutations $\lambda_1, \lambda_2, \dots, \lambda_p$ in S_k and permutations $\rho_1, \rho_2, \dots, \rho_k$ in S_p, such that*

$$U^{\beta b}_{\alpha a} = \delta_{\beta, \rho_a(\alpha)} \delta_{b, \lambda_\alpha(a)}$$

By Lemma 6.2 there exist analogous permutations $\{\nu_\alpha\}_\alpha$ and $\{\theta_a\}_a$ for U^*:

$$U^{\beta b}_{\alpha a} = (U^*)^{\alpha a}_{\beta b} = \delta_{\alpha, \theta_b(\beta)} \delta_{a, \nu_\beta(b)}$$

(In Krishnan and Sunder's notation [KrSu], $\nu = \psi^{-1}$ and $\theta = \phi^{-1}$). Define $\Gamma_U \subset S_k$ to be the group generated by elements of the form $\nu_\beta^{-1} \nu_\alpha$, i.e.,

$$\Gamma_U = \langle \nu_\beta^{-1} \nu_\alpha : \alpha, \beta \in \{1, 2, \dots, p\} \rangle.$$

For each $a \in \{1, 2, \dots, k\}$ let $\Gamma^a \subset \Gamma_U$ be the subgroup that fixes a. If Γ_U acts transitively on $\{1, 2, \dots, k\}$, then the subgroups Γ^a are all conjugate. In this case denote Γ^a's isomorphism class by Γ^1. In general, let Ω be the set of orbits in $\{1, 2, \dots, k\}$, and for each $r \in \Omega$ set $\Gamma^r = \Gamma^a$ for some representative a in r.

Recall that $R_U \subset R$ has an intermediate von Neumann algebra Q_U. We cite Theorem 3.7 of [Bh1]; in what follows $Z(Q_U)$ denotes the center of Q_U.

Theorem 6.3. *Let Γ_U act on R by outer automorphisms. There is a canonical isomorphism of $Z(Q_U)$ with $\mathbb{C}^\Omega$. If q^r is the minimal projection in $Z(Q_U)$ corresponding to $r \in \Omega$, then $q^r R_U \subset q^r Q_U$ is isomorphic to $R^{\Gamma_U} \subset R^{\Gamma^r}$. In particular, $R_U \subset Q_U$ is a subfactor if and only if Γ_U acts transitively on $\{1, 2, \dots, k\}$, and in this case $R_U \subset Q_U$ is isomorphic to $R^\Gamma \subset R^{\Gamma^1}$. Finally, $Q_U \subset R$ and $R_{\widetilde{U}} \subset Q_{\widetilde{U}}$ are dual subfactors.*

6.2.2. *Example.* Let $k = p = 3$. We consider "Case 15" in [KrSu]. In this example,

$$U = \left[\begin{array}{ccc ccc ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

It follows that

$$(\lambda_1, \lambda_2, \lambda_3) = (\nu_1, \nu_2, \nu_3) = (id, (12), (13)),$$
$$(\rho_1, \rho_2, \rho_3) = (\theta_1, \theta_2, \theta_3) = (id, id, (12)).$$

Since $\nu_1 = id$, $\Gamma_U = \langle id, (12), (13)\rangle = S_3$. Therefore, $R_U \subset Q_U$ is isomorphic to $R^{S_3} \subset R^{S_2}$, for an outer action of S_3 on R. Now consider the dual unitary $\widetilde{U}$. Since $\widetilde{U} = U$, $R_{\widetilde{U}} \subset Q_{\widetilde{U}}$ is isomorphic to $R^{S_3} \subset R^{S_2}$ as well. Hence, $Q_U \subset R$ is isomorphic to its dual, $R \ltimes S_2 \subset R \ltimes S_3$, by Theorem 6.3.

6.2.3. *Computation of the standard invariant.* We now use Corollary 3.4 to compute the standard invariant of $R_U \subset Q_U$ in the case that $R_U \subset Q_U$ is a subfactor. We will first explore the structure of $R_U \subset R$, before retreating to the more easily understood $R_U \subset Q_U$. In Proposition 6.4 below, we state the description of the standard invariant of $R_U \subset R$ given in [Bh2]. In Proposition 6.5, we compute the standard invariant of the intermediate subfactor.

Let Σ_k denote the set of words on the alphabet $\{1, 2, \dots, k\}$, including the empty word. The standard invariant of $R_U \subset R$ is characterized by two actions: one of Σ_k on $\{1, 2, \dots, p\}$ and one of $\{1, 2, \dots, p\}$ on Σ_k. Most of the details of these actions will not be relevant to our purpose, but we define the actions for the sake of completeness.

Given a word $w = a_1 a_2 \dots a_n$ in Σ_k and α in $\{1, 2, \dots, p\}$, define

$$w(\alpha) = \theta_{a_n}^{\epsilon} \dots \theta_{a_4}^{-1} \theta_{a_3} \theta_{a_2}^{-1} \theta_{a_1}(\alpha), \tag{9}$$

where $\epsilon = -1$ if n is even and $\epsilon = 1$ if n is odd. Let $r = \lfloor \frac{n}{2} \rfloor$. For $i = 2, 4, 6, \dots, 2r$, let $w_i = a_1 a_2 \dots a_i$ and $\alpha_i = w_i(\alpha)$. The action of α on w is

$$\alpha(w) = \alpha(a_1)\alpha_2(a_2)\alpha_2(a_3)\alpha_4(a_4)\alpha_4(a_5)\alpha_6(a_6)\alpha_6(a_7) \dots \alpha_{2r}(a_n). \tag{10}$$

Define a subgroup G of permutations of Σ_k by

$$G = G_U = \langle \beta^{-1}\alpha : \alpha, \beta \in \{1, 2, \ldots, p\}\rangle \tag{11}$$

This group is the projective limit of the groups G_n in [KrSu]. A more motivated definition of G appears in [Bh2].

Let V_k be k-dimensional complex vector space over a basis indexed by $\{1, 2, \ldots, k\}$. As with all subfactors arising from vertex models, the standard invariant of $R_U \subset R$ is a sublattice (in the sense of Popa's sub-λ-lattice [Po1]) of the standard invariant of $R \subset R \otimes M_k(\mathbb{C})$, i.e. it is a sublattice of

$$\begin{array}{ccccccccc} \mathbb{C} & \subset & \mathrm{End}(V_k) & \subset & \mathrm{End}(V_k^{\otimes 2}) & \subset & \mathrm{End}(V_k^{\otimes 3}) & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \cdots \\ & & \mathbb{C} & \subset & 1 \otimes \mathrm{End}(V_k) & \subset & 1 \otimes \mathrm{End}(V_k^{\otimes 2}) & \subset & \cdots \end{array} \tag{12}$$

where inclusion is given by $F \mapsto F \otimes 1$. The planar algebra structure of $R_U \subset R$ is inherited from the tensor planar algebra (see [Jo2]). In particular, the first Jones projection $e \in \mathrm{End}(V^{\otimes 2})$ is defined by

$$e_{ab}^{cd} = \begin{cases} \frac{1}{k}, & \text{if } a = b \text{ and } c = d; \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

The nth Jones projection, denoted $e_{1,n}$ (following the notation of this paper), is

$$e_{1,n} = 1^{\otimes n-1} \otimes e \in \mathrm{End}(V^{\otimes n+1}). \tag{14}$$

Each space $V_k^{\otimes n}$ has a natural basis indexed by words of length n in Σ_k. If F is an endomorphism of $V_k^{\otimes n}$, then F may be expressed as a matrix over this basis. For words x and y of length n in Σ_k, we denote by F_x^y the entry of F in row x and column y. For $x = a_1a_2 \ldots a_n$ in Σ_k, denote its reverse, $a_na_{n-1} \ldots a_2a_1$, by x^t. The following proposition gives necessary and sufficient conditions for $F \in \mathrm{End}(V_k^{\otimes n})$ to be contained in the $n-1$st relative commutant, $R_U' \cap R_{n-1}$, where

$$R_U \subset R = R_0 \subset R_1 \subset R_2 \subset \ldots$$

is the Jones tower of $R_U \subset R$.

Proposition 6.4. *For $n = 0, 1, 2, \ldots,$*

$$R_U' \cap R_{n-1} = \left\{ F \in \mathrm{End}(V_k^{\otimes n}) : \begin{array}{l} F_x^y = 0 \text{ if } xy^t \text{ does not act identically on } \{1, 2, \ldots, p\} \\ F_x^y = F_{g(x)}^{g(y)} \text{ for all } g \in G \end{array} \right\}$$

where x and y range over all words of length n in Σ_k and the action of Σ_k and $\{1, 2, \ldots, p\}$ on one another are as defined in (9) and (10). Also, $R_0' \cap R_{n-1} = (R_U' \cap R_{n-1}) \cap (1 \otimes \mathrm{End}(V_k^{\otimes n-1}))$, for each n. The Jones *projections are $e_{1,n}$ as defined in (14) and the trace is inherited from the unique normalized trace on $\cup_n \mathrm{End}(V_k^{\otimes n})$.*

Proof. The first statement is a modified version of results in [KrSu] and is proved in [Bh2]. The statement regarding the dual tower, the Jones projections, and the trace follows immediately from planar algebra theory ([Jo2]) and the fact that $(R_U' \cap R_n)_n$ is a subplanar algebra of the tensor algebra ([Bh3]). It also follows directly from Jones and Sunder's diagrammatic description of the standard invariant in [JoSu]. □

Let $\Sigma_k(n)$ denote the words in Σ_k of length n, and let G act on $\Sigma_k(n)\times\Sigma_k(n)$ diagnonally. For future reference, note that there is a basis of $R_U' \cap R_{n-1}$ indexed by orbits of G in $\Sigma_k(n) \times \Sigma_k(n)$: the basis element indexed by the orbit X in $\Sigma_k(n) \times \Sigma_k(n)$ is

$$F_X = \sum_{(w_1,w_2)\in X} e(w_1, w_2)$$

where $e(w_1, w_2)$ denotes the matrix unit of row w_1 and column w_2.

We will now use Corollary 3.4 to compute the standard invariant of $R_U \subset Q_U$, in the case that $R_U \subset Q_U$ is a subfactor. The projection $q \in \mathrm{End}(V_k^{\otimes 2})$ corresponding to $R_U \subset Q_U$ is given by

$$q_{ab}^{cd} = \begin{cases} 1, & \text{if } a = b = c = d; \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

for all $a, b, c, d \in \{1, 2, \ldots, k\}$ (see Section 6.4 of [Bh3] and Proposition 4.9 of [Bh1]). That is, $q = e_{0,1}$ as defined at the beginning of Section 3 with $N = R_U$ and $M = R$. For each n, let q_n be the projection in Corollary 3.4, that is, q_n is the projection denoted p in Theorem 3.2. Since the planar algebra structure of $(R_U' \cap R_n)_n$ is inherited from the tensor algebra (see [Jo2], Section 6.4 of [JoSu]), we have by Lemma 4.2:

$$e_{0,n} = 1^{\otimes n-1} \otimes q \in \mathrm{End}(V^{\otimes n+1}) \tag{16}$$

$$q_n = q^{\otimes n} \in \mathrm{End}(V_k^{\otimes 2n}) \tag{17}$$

(note that in the tensor algebra q "turned sideways," as discussed in the proof of Lemma 4.2, is equal to q and $[R : Q_U] = [Q_U : R_U] = k$). Note that

$$(q_n)_{a_1a_2\ldots a_{2n}}^{b_1b_2\ldots b_{2n}} = \begin{cases} 1, & \text{if } a_{2i-1} = a_{2i} = b_{2i-1} = b_{2i} \text{ for } 1 \le i \le n; \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

and

$$(q_2e_{0,2})_{a_1a_2a_3a_4}^{b_1b_2b_3b_4} = \begin{cases} 1, & \text{if } a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4; \\ 0, & \text{otherwise} \end{cases} \tag{19}$$

Let $\mathcal{Q}_n$ be the set of words $a_1a_2 \ldots a_{2n}$ such that $a_{2i-1} = a_{2i}$ for $1 \le i \le n$. Clearly, $\mathcal{Q}_n$ is a basis of $q_nV_k^{\otimes 2n}$.

Let $A_{2n} = q_n(R'_U \cap R_{2n-1})q_n$ and $A_{2n+1} = q_n(R'_U \cap R_{2n})q_n$ for $n \geq 0$. Let $B_{2n} = q_n(R'_0 \cap R_{2n-1})q_n$, and $B_{2n-1} = q_n(R'_0 \cap R_{2n-2})q_n$ for $n \geq 1$. If $R_U \subset Q_U$ is a subfactor, then by Corollary 3.4, its standard invariant is isomorphic to:

$$\begin{array}{ccccccccccc} A_0 & \subset & A_1 & \subset & A_2 & \subset_j & A_3 & \subset & A_4 & \subset_j & \cdots \\ & & \cup & & \cup & & \cdots & & \cup & & \cdots \\ & & B_1 & \subset & B_2 & \subset_j & B_3 & \subset & B_4 & \subset_j & \cdots \end{array} \tag{20}$$

where the inclusion map $j : A_{2n} \subset A_{2n+1}$ is $x \mapsto q_{n+1}x$.

The following proposition characterizes the lattice of algebras above. If $R_U \subset Q_U$ is a subfactor, then the lattice is actually standard invariant of $R_U \subset Q_U$ and the proposition gives the Jones projections and trace of the standard invariant.

Proposition 6.5. *Let Γ be the restriction of G to words of length 1, i.e. $\{1, 2, \ldots, k\}$ Then $\Gamma = \Gamma_U$ and $A_{2n} \subset \mathrm{End}_\Gamma(V^{\otimes 2n})$ for each n. Let Γ act on Σ_k in the obvious way. Let $w_1 = a_1a_2 \ldots a_{2n}$ and $w_2 = b_1b_2 \ldots b_{2n}$ range over all words of length $2n$ in Σ_k. Then,*

$$\begin{aligned} A_{2n} &= \{F \in \mathrm{End}_\Gamma(V^{\otimes 2n}) \colon \textit{if } F^{w_2}_{w_1} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n\} \\ A_{2n-1} &= \{F \in \mathrm{End}_\Gamma(V^{\otimes 2n}) \colon \textit{if } F^{w_2}_{w_1} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n \textit{ and } a_{2n} = b_{2n}\} \\ B_{2n} &= \{F \in \mathrm{End}_\Gamma(V^{\otimes 2n}) \colon \textit{if } F^{w_2}_{w_1} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n \textit{ and } a_1 = b_1\} \\ B_{2n-1} &= \{F \in \mathrm{End}_\Gamma(V^{\otimes 2n}) \colon \textit{if } F^{w_2}_{w_1} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n,\ a_1 = b_1, \textit{ and } a_{2n} = b_{2n}\} \end{aligned}$$

The Jones projections are $e_{1,2n}q_n = q_{n-1} \otimes e \in A_{2n}$ and $e_{0,2n-1}q_n = q_{n-2} \otimes q_2e_{0,2} \in A_{2n}$ where q_n, e, and $q_2e_{0,2}$ are as in (18), (13), and (19). The trace on A_{2n} and A_{2n-1} is given by $tr_{R_U \subset Q_U}(x) = k^{-n}tr(x)$, where tr is the unique normalized trace on $\cup_n \mathrm{End}(V_k^{\otimes n})$.

Proof. If $g \in G$, let γ_g be the image of g in Γ. We claim that if $w = a_1a_2 \ldots a_{2n} \in \mathcal{Q}_n$, then w acts identically on $\{1, 2, \ldots, p\}$ for all $g \in G$ and

$$g(w) = g(a_1)g(a_2) \ldots g(a_{2n}) = \gamma_g(w) \tag{21}$$

To prove (21), it suffices to prove the first equation in (21) holds with g replaced by $\alpha \in \{1, 2, \ldots, p\}$. By (9) and (10),

$$\alpha(w) = \alpha(a_1)\alpha_2(a_2)\alpha_2(a_3)\alpha_4(a_4)\alpha_4(a_5)\alpha_6(a_6)\alpha_6(a_7) \ldots \alpha_{2n}(a_{2n})$$

where

$$\alpha_{2j} = \theta^{-1}_{a_{2j}}\theta_{a_{2j-1}} \ldots \theta^{-1}_{a_4}\theta_{a_3}\theta^{-1}_{a_2}\theta_{a_1}(\alpha)$$

for $0 < j \leq n$. Since $a_{2i-1} = a_{2i}$ for $1 \leq i \leq n$, $\alpha_{2j} = \alpha$ for each j. This proves w acts identically on $\{1, 2, \ldots, p\}$, since $w(\alpha) = \alpha_{2n} = \alpha$; it also proves (21) when g is replaced by α. So we have proved the claim.

Recall the basis $\{F_X\}_X$ of $R_U' \cap R_n$, indexed by G-orbits X in $\Sigma_k(n) \times \Sigma_k(n)$, that was mentioned after Proposition 6.4. Let w_1 and w_2 be words in $\Sigma_k(n)$. Clearly $(q_n)_{w_1}^{w_2} = 1$ if $w_1 = w_2$ and $w_1 \in \mathcal{Q}_n$; otherwise $(q_n)_{w_2}^{w_1} = 0$. Then for any $F \in \operatorname{End}(V_k^{\otimes 2n})$:

$$(q_n F q_n)_{w_1}^{w_2} = \begin{cases} F_{w_1}^{w_2}, & \text{if } w_1, w_2 \in \mathcal{Q}_n, \\ 0, & \text{otherwise} \end{cases}$$

Note that $\mathcal{Q}_n \times \mathcal{Q}_n$ is stable under the action of G. For each G-orbit $X \subset \Sigma_k(n) \times \Sigma_k(n)$, either $X \subset \mathcal{Q}_n \times \mathcal{Q}_n$ or $X \subset (\Sigma_k(n) \times \Sigma_k(n)) \setminus (\mathcal{Q}_n \times \mathcal{Q}_n)$. Then $q_n F_X q_n = F_X$ if $X \subset \mathcal{Q}_n \times \mathcal{Q}_n$ and $q_n F_X q_n = 0$, otherwise. So $q_n(R_U' \cap R_{2n})q_n$ is spanned by $\{F_X : X \subset \mathcal{Q}_n \times \mathcal{Q}_n\}$. Moreover, the G-orbits in $\mathcal{Q}_n \times \mathcal{Q}_n$ are the same as the Γ-orbits in $\mathcal{Q}_n \times \mathcal{Q}_n$, since the actions of g and γ_g on $\mathcal{Q}_n$ are the same for all $g \in G$ by the claim. This proves the characterization of A_{2n} in the proposition.

If $F \in \operatorname{End}(V_k^{2n-1})$, then for all $w_1, w_2 \in \Sigma_k(2n-1)$ and $a, b \in \{1, 2, \ldots, k\}$,

$$(q_n F q_n)_{w_1 a}^{w_2 b} = \begin{cases} F_{w_1 a}^{w_2 b}, & \text{if } w_1 a, w_2 b \in \mathcal{Q}_n \text{ and } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly Γ, and hence G (by (21)), stabilize

$$\mathcal{Q}' = \{(w_1 a, w_2 a) \colon (w_1 a, w_2 a) \in \mathcal{Q}_n \times \mathcal{Q}_n\}$$

Therefore, $q_n(R_U' \cap R_{2n-1})q_n$ is spanned by $\{F_X : X \subset \mathcal{Q}'\}$. Moreover, G and Γ have the same orbits in $\mathcal{Q}_n'$ by the claim. This proves the characterization of A_{2n-1} in the proposition.

The proofs of the statements for B_{2n} and B_{2n-1} are similar. The statements regarding the Jones projections and the trace follow from Corollary 3.4. □

If $R_U \subset Q_U$ is a subfactor, then we can use Proposition 6.5 to show that the principal and dual graphs of $R_U \subset Q_U$ have the familiar description stated in the next corollary. Of course, this result is well known given Theorem 6.3 ([KoYa],[JoSu]).

Corollary 6.6. *If Γ acts transitively on $\{1, 2, \ldots, k\}$, then $R_U \subset Q_U$ is a subfactor; its principal graph is the induction-restriction graph of $\Gamma^1 \subset \Gamma$ and its dual graph is the induction-restriction graph of the subgroups $\Gamma^1 \cap \gamma\Gamma^1\gamma^{-1} \subset \Gamma^1$, where γ ranges over representatives of the double cosets of Γ^1 in Γ.*

Proof. If Γ acts transitively on $\{1, 2, \ldots, k\}$, then $R_U \subset R$ is irreducible by Proposition 6.4. But if $R_U \subset R$ is irreducible, then $R_U \subset Q_U$ must be a subfactor.

Let $\widetilde{V}_k$ be $q_1(V_k \otimes V_k)$, the subspace of $V_k \otimes V_k$ with basis $\{aa\}_{a=1}^k \subset \Sigma_k(2)$. Let Γ act on $\widetilde{V}_k$ by permuting its basis and let Γ act on tensors of $\widetilde{V}_k$ by acting on its factors diagonally. By Proposition 6.5, A_{2n} is isomorphic to $\mathrm{End}_\Gamma(\widetilde{V}_k^{\otimes n})$. The corollary follows by an argument similar to the one in [Jo3]. Assume Γ acts transitively on $\{1, 2, \dots, k\}$. Since $\widetilde{V}_k$ is a permutation representation, we can obtain the Γ representation $\widetilde{V}_k^{\otimes n-1} \otimes \widetilde{V}_k$ from $\widetilde{V}_k^{\otimes n-1}$ by restricting Γ to the stabilizer of a point and then inducing back up, namely ${}_\Gamma\widetilde{V}_k^{\otimes n} \cong \mathrm{Ind}_{\Gamma^1}^{\Gamma}(\widetilde{V}_k^{\otimes n-1} \otimes \mathbb{C}1)$.

By the second part of Proposition 6.5, A_{2n-1} contains precisely those Γ-endomorphisms that act trivially on the nth tensor $\widetilde{V}_k$. Hence, $A_{2n-1} \cong A_{2n-1}|_{\widetilde{V}_k^{\otimes n-1} \otimes \mathbb{C}1} \cong \mathrm{End}_{\Gamma^1}(\widetilde{V}_k^{\otimes n-1})$. The Bratteli diagram of the tower (20) is determined by induction and restriction between Γ^1 and Γ. A similar argument shows that the principal graph of $(B_n)_n$ is given by induction and restriction between Γ^1 and the groups $\Gamma^1 \cap \gamma\Gamma^1\gamma^{-1}$, where γ ranges over representatives of the double cosets of Γ^1 in Γ. □

In the following corollary, we deduce from Proposition 6.5 a description of the standard invariant of subgroup subfactors.

Corollary 6.7. *Let $H \subset \Gamma$ be a finite subgroup with an outer action on R. Let $k = |\Gamma/H|$, and let Γ act on $\{1, 2, \dots, k\}$ by fixing a bijection of Γ/H with $\{1, 2, \dots, k\}$. Then the standard invariant of $R^\Gamma \subset R^H$ is given by the lattice (20) described in Proposition 6.5. The planar algebra of $R^\Gamma \subset R^H$, represented on even boxes, has the partition function Z defined at the beginning of Section 5 (with $\Phi_{N\subset M}$ being the partition function of the tensor algebra).*

Proof. By Corollary 3.8 of [Bh1] there exists a permutation biunitary U such that $R_U \subset R$ is isomorphic to $R^\Gamma \subset R^H$. The result follows from Proposition 6.5 and Theorem 6.3. □

6.3. **Tree Subfactors.** In [Bh3] (see also [Bh2]) the first author defines for each integer $k > 1$ a planar algebra $(T_n(k))_n$, which is a subplanar algebra of the tensor algebra $(\mathrm{End}(V_k^{\otimes n}))_n$. The associated standard invariant is defined by the conditions in Proposition 6.4 with respect to a certain group G. By Theorem 4.2.1 of [Jo2], there is a subfactor $N(k) \subset M(k)$ of index k^2, whose planar algebra is $(T_n(k))_n$. We call these subfactors the *tree subfactors.* In this section we apply our analysis of Krishnan-Sunder subfactors to tree subfactors to show that $N(k) \subset M(k)$ is the composition of two subfactors isomorphic to $R^{S_k} \otimes M(k) \subset R^{S_{k-1}} \otimes M(k)$ and its dual.

Fix $k > 1$ and let G be the group defined in [Bh2]. We remark on a few properties of G. First, G is a subgroup of permutations of Σ_k that preserve word length. Let $\sim$ be the equivalence relation on Σ_k generated by the relations

$$\{w_1aaw_2 \sim w_1w_2 \colon w_1, w_2 \in \Sigma_k, a \in \{1, 2, \dots, k\}\}$$

Denote the empty word in Σ_k by $\emptyset$. We list below some properties of G [Bh2]: for all $g \in G$ and words $w, w', u, u' \in \Sigma_k$,

(1) If $g(ww') = uu'$ and w has the same length as u, then $g(w) = u$.
(2) $w \sim \emptyset \iff g(w) \sim \emptyset$.
(3) If $u \sim w$, then $g(uw^t) = g(u)g(w)^t$.
(4) If we restrict G to words of length 1, i.e. to the set $\{1, 2, \dots, k\}$, the resulting quotient group is S_k.

Define

$$T_n(k) = \left\{ F \in \mathrm{End}(V_k^{\otimes n}) : \begin{array}{l} F_{w_1}^{w_2} = 0 \text{ if } w_1 w_2^t \nsim \emptyset \\ F_{w_1}^{w_2} = F_{g(w_1)}^{g(w_2)} \text{ for all } g \in G \end{array} \right\}$$

where w_1 and w_2 range over all words of length n in Σ_k. Then $(T_n(k))_n$ is a subplanar algebra of the tensor algebra, and the dual tower associated to $(T_n(k))_n$ is $(1 \otimes T_n(k))_n$ [Bh3]. Moreover, $T_1(k)$ is 1-dimensional, so $N(k) \subset M(k)$ is irreducible. The projection q defined in (15) is contained in $T_2(k)$ and corresponds to an intermediate subfactor $N(k) \subset Q(k) \subset M(k)$ of index k.

Although $(T_n(k))_n$ is constructed to appear very much like the standard invariant of a Krishnan-Sunder subfactor, in fact it is not (see Proposition 9.6 of [Bh2]).

Let q_n be as in (18). Let $A_{2n} = q_n(T_{2n}(k))q_n$ and $A_{2n-1} = q_n(T_{2n-1}(k))q_n$ for $n \geq 0$. Let $B_{2n} = q_n(1 \otimes T_{2n-1})q_n$, and $B_{2n-1} = q_n(1 \otimes T_{2n-1})q_n$ for $n \geq 1$. By Corollary 3.4, the standard invariant of $N(k) \subset Q(k)$ is as in the diagram (20). Let $\mathcal{Q}_n$ be as defined before Proposition 6.5. The following proposition characterizes the standard invariant of $N(k) \subset Q(k)$.

Proposition 6.8. *Let S_k act on Σ_k in the natural way. Let $w_1 = a_1 a_2 \dots a_{2n}$ and $w_2 = b_1 b_2 \dots b_{2n}$ range over all words of length $2n$ in Σ_k. Then,*

$$A_{2n} = \{F \in \mathrm{End}_{S_k}(V^{\otimes 2n}) \colon \textit{if } F_{w_1}^{w_2} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n\}$$
$$A_{2n-1} = \{F \in \mathrm{End}_{S_k}(V^{\otimes 2n}) \colon \textit{if } F_{w_1}^{w_2} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n \textit{ and } a_{2n} = b_{2n}\}$$
$$B_{2n} = \{F \in \mathrm{End}_{S_k}(V^{\otimes 2n}) \colon \textit{if } F_{w_1}^{w_2} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n \textit{ and } a_1 = b_1\}$$
$$B_{2n-1} = \{F \in \mathrm{End}_{S_k}(V^{\otimes 2n}) \colon \textit{if } F_{w_1}^{w_2} \neq 0 \textit{ then } w_1, w_2 \in \mathcal{Q}_n, \ a_1 = b_1, \textit{ and } a_{2n} = b_{2n}\}$$

The Jones projections are $q_{n-1} \otimes e \in A_{2n}$ and $q_{n-2} \otimes q_2 e_{0,2} \in A_{2n}$ where q_n, e, and $q_2 e_{0,2}$ are as in (18), (13), and (19). The trace on A_{2n} and A_{2n-1} is given by $tr_{N(k) \subset Q(k)}(x) = k^{-n} tr(x)$, where tr is the unique normalized trace on $\cup_n \mathrm{End}(V_k^{\otimes n})$.

Proof. Note that S_k is the restriction of G to words of length 1 by property 4 of G and of course is transitive on $\{1, 2, \dots, k\}$. For all $g \in G$, let γ_g denote the image of g in S_k. There is a natural action of γ_g on Σ_k by the diagonal action.

We may use a proof completely similar to the proof of Proposition 6.5 as long as:

(1) If w is in $\mathcal{Q}_n$ then $w \sim \emptyset$
(2) If w is in $\mathcal{Q}_n$ and g is in G, then $g(w) = \gamma_g(w)$.

Obviously 1 holds. It is easy to prove 2 by induction on n, using properties 2 and 3 of G. □

Corollary 6.9. *The intermediate subfactors $N(k) \subset Q(k)$ and $Q(k) \subset M(k)$ are isomorphic to $R^{S_k} \otimes M(k) \subset R^{S_{k-1}} \otimes M(k)$ and its dual, respectively.*

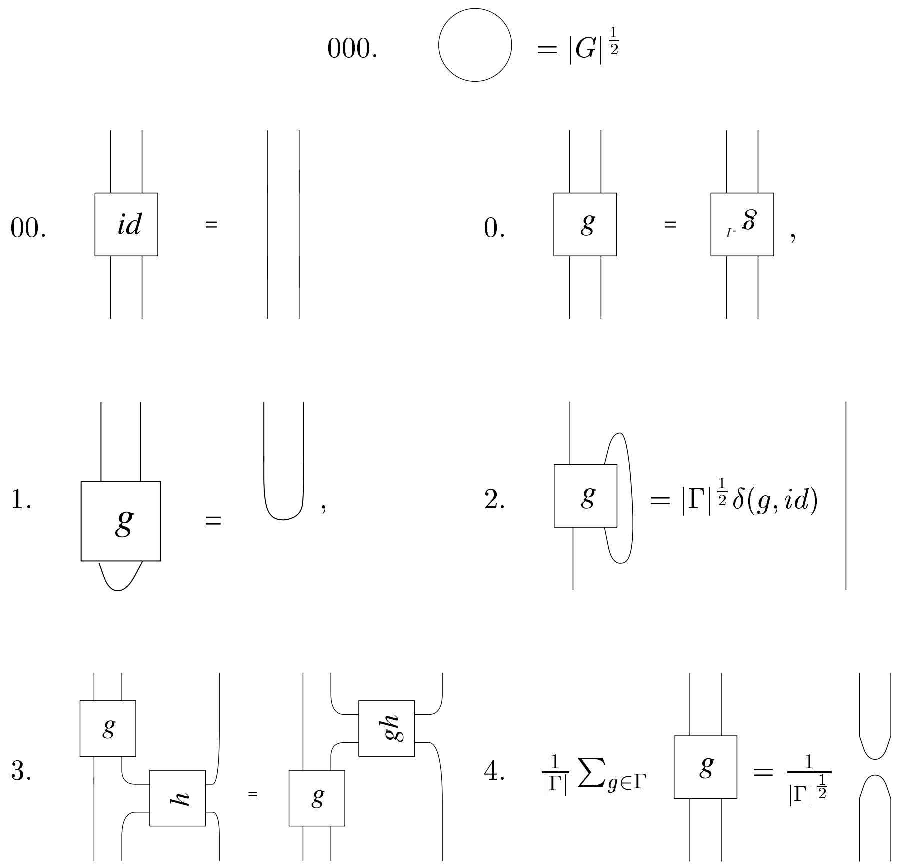
Proof. Note that the conditions in Proposition 6.8 and Proposition 6.5 are identical when $\Gamma_U = S_k$. It follows from Proposition 6.8 and Corollary 6.7 that the standard invariant of $N(k) \subset Q(k)$ is identical to the standard invariant of $R^{S_{k-1}} \subset R^{S_k}$. Since this standard invariant has finite depth, it follows that $N(k) \subset Q(k)$ is isomorphic to $R^{S_k} \otimes M(k) \subset R^{S_{k-1}} \otimes M(k)$ by [Po3].

We now prove that $N(k) \subset Q(k)$ is dual to $Q(k) \subset M(k)$. Let M_1 and Q_1 be the factors obtained by the basic construction $N(k) \subset M(k) \subset M_1$ and $Q(k) \subset M(k) \subset Q_1$. It suffices to prove that $M(k) \subset Q_1$ and $N(k) \subset Q(k)$ have the same standard invariant. First, $M(k) \subset Q_1$ is an intermediate subfactor of $M(k) \subset M_1$, and the corresponding Jones projection in M_1 is by definition $e_{0,2}$. By (16), $e_{0,2} = 1 \otimes q \in \mathrm{End}(V^{\otimes 3})$. Also, $(T_n(k))_k$ is self dual, i.e. $(1 \otimes T_{n-1}(k))_{k=1}^{\infty}$ is isomorphic to $(T_n(k))_{k=0}^{\infty}$ (see Remark 6.2.8 of [Bh3]). Since $M(k) \subset Q_1$ is the intermediate subfactor corresponding to $1 \otimes q$ in the planar algebra $(1 \otimes T_{n-1}(k))_{k=1}^{\infty}$ and $N(k) \subset Q(k)$ is the intermediate subfactor corresponding to q in the planar algebra $(T_n(k))_{k=0}^{\infty}$, the two lattices obtained in Corollary 3.4 for these two subfactors are isomorphic. Moreover, the isomorphism preserves the Jones projections and the trace specified in Corollary 3.4. Therefore, $N(k) \subset Q(k)$ and $M(k) \subset Q_1$ have the same standard invariant. □

6.4. **Group Subfactors.** Let $H \subset \Gamma$ be an inclusion of finite groups with an outer action on the hyperfinite II_1 factor R.

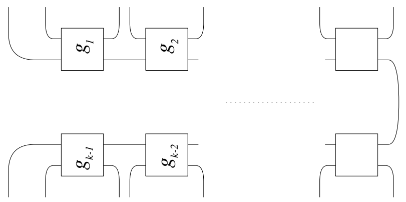
By Corollary 6.7, the results of Section 6.2.3 provide a description of the standard invariant of $R^\Gamma \subset R^H$, namely the description given in Proposition 6.5. Denote the tower of higher relative commutants of $R^\Gamma \subset R^H$ by $(A_n)_n$. Since $R^\Gamma \subset R^H$ is an intermediate subfactor of the group subfactor $R^\Gamma \subset R$, Corollary 3.4 provides an alternate description of the standard invariant of $R^\Gamma \subset R^H$, which we denote $(\widetilde{A}_n)_n$. It is natural to ask how the two descriptions are related. In this section we state without proof the morphism between the $(A_n)_n$ and $(\widetilde{A}_n)_n$ and make a few remarks on computing the principal graph from $(\widetilde{A}_n)_n$.

Recall from [La1] that the planar algebra $\mathcal{P}(\Gamma) = (P_n(\Gamma))_n$ of $R^\Gamma \subset R$ is the planar algebra with labeling set $L = L_2 = \Gamma$ with $g^* = g^{-1}$ for all $g \in \Gamma$, modulo the following relations:

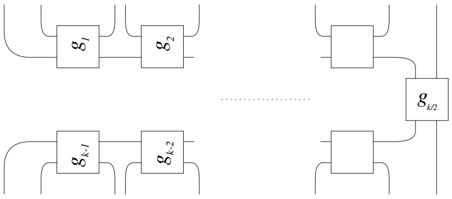


The following is Theorem 6 from [La1].

Theorem 6.10. *An orthogonal basis for $P_k(\Gamma)$, $k > 2$ is given by the pictures:*



for k odd, and



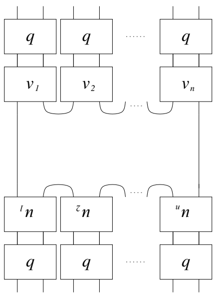
for k even, with each g_i ranging over Γ.

The projection corresponding to $R^\Gamma \subset R^H$ is

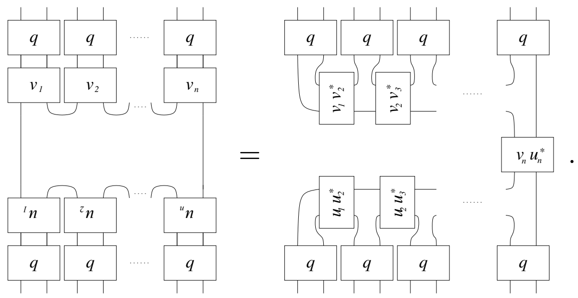
$$q = \frac{1}{|H|} \sum_{h \in H} \boxed{h}$$

Let q_n be as defined in Corollary 3.4. Choose a set of representatives of the double cosets of H in Γ and consider the subset of basis elements of $P_{2n}(\Gamma)$ (as defined in Theorem 6.10) where the g_i range over the representatives of the double cosets. Let $b \in P_{2n}(\Gamma)$ range over the elements of this subset. Then $\{q_n b q_n\}_b$ is clearly a basis of $\widetilde{A}_{2n}$.

Let $Y_1, Y_2, \ldots, Y_k$ be the cosets in $H \backslash \Gamma$. By the proof of Proposition 6.5, a basis of A_{2n} is given by F_X (see comments after Proposition 6.4), where X ranges over the orbits of Γ in $\mathcal{Q}_n \times \mathcal{Q}_n \cong (H\backslash\Gamma)^{2n}$. We define a natural map from A_{2n} to $\widetilde{A}_{2n}$ given on basis elements as follows. Fix a Γ-orbit X in $\mathcal{Q}_n \times \mathcal{Q}_n$ (Γ acts on $\mathcal{Q}_n \times \mathcal{Q}_n \cong (H\backslash\Gamma)^{2n}$ by the right action on $H\backslash\Gamma$). Choose any representative $(a_1 a_1 a_2 a_2 \ldots a_n a_n, b_1 b_1 b_2 b_2 \ldots b_n b_n) \in X$. For $1 \le i \le n$, let u_i and v_i be representatives of the cosets Y_{a_i} and Y_{b_i}, respectively. Then, supressing the presenting map, we map F_X to

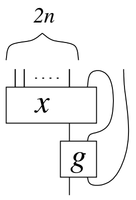


Note that:



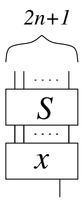
Hence, the image of F_X is independent of the choice of representative of X and is equal to one of the basis elements $q_n b q_n$ of $\widetilde{A}_{2n}$. The map from A_{2n} to $\widetilde{A}_{2n}$ is a trace preserving isomorphism compatible with the Jones projections and the dual subtowers of higher relative commutants.

The principal graph of $R^\Gamma \subset R^H$ is rather transparent from $(A_n)_n$ (see Corollary 6.6), but it can also be obtained directly from $(\widetilde{A}_n)_n$. We give a brief outline of how to do this. The essential ingredient is a sequence of representations X_n. For $n = 0, 1, 2, \dots$, let X_n be a copy of $P_{n+1}(\Gamma)$ as a vector space. We draw tangles in X_n with $2n+1$ strings on top and 1 string on the bottom. Define a right Γ-action on X_n by

$x \cdot g =$ 

$\in X_n, \quad \text{for } x \in X_n, g \in \Gamma$

There is a natural left action of $P_{2n+1}(\Gamma)$ on X_n by concatentation of tangles. It is defined by

$S \cdot x =$ 

$\in X_n, \quad \text{for } S \in P_{2n+1}(\Gamma), x \in X_n$

Note that $P_{2n}(\Gamma)$ acts on X_n as a subalgebra of $P_{2n+1}(\Gamma)$.

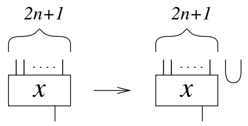
By Corollary 4.5, the tower of higher relative commutants of $R^\Gamma \subset R^H$ is isomorphic to

$$(22)\quad B_1(P_1(\Gamma)) \subset B_2(P_2(\Gamma)) \subset B_3(P_3(\Gamma)) \subset B_4(P_4(\Gamma)) \subset B_5(P_5(\Gamma)) \subset \dots$$

Let q_n denote the projection $e_{0,1}\ e_{0,3}\ e_{0,5} \cdots e_{0,2n-1}$ for the subfactor $R^\Gamma \subset R$ (see Theorem 3.2). One can almost prove that the principal graph of $R^\Gamma \subset R^H$ is the induction-restriction graph of $H \subset G$ by proving:

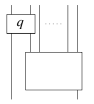
(1) $B_{2n}(P_{2n}(\Gamma)) \cong \mathrm{End}_\Gamma(q_n X_n)$, where the isomorphism is given by the left action defined above.
(2) $B_{2n+1}(P_{2n+1}(\Gamma)) \cong \mathrm{End}_H(q_n X_n)$, where the isomorphism is given by the left action defined above.
(3) $q_{n+1}X_{n+1} \cong \mathrm{Ind}_H^\Gamma\left(\mathrm{Res}_H^\Gamma\, q_n X_n\right)$

To finish the proof, one must show that the inclusion $B_{2n+1}(P_{2n+1}(\Gamma)) \subset B_{2n+2}(P_{2n+2}(\Gamma))$ is compatible with the inclusion map $i\colon q_n X_n \to q_{n+1}X_{n+1}$ defined as follows. Let $i^0 : X_n \to X_{n+1}$ by



Then we define $i(x) = q_{n+1}i^0(x)$. Note that i is H-invariant by the exchange relation.

The dual graph can be obtained by a similar argument using modules of the form

$\tilde{X}_n = \{$  $\}$

instead of X_n.

6.5. **Exchange Relation Planar Algebras.** In [La1] a new class of planar algebras is introduced. For any finite group G and automorphism group Θ of G, a subplanar algebra $P^\Theta(G)$ of the group planar algebra (see Group Subfactors above) is given by considering only those tangles generated by 2-boxes labeled by elements $v \in \mathbb{C}G$ fixed by the elements of Θ. In what follows, with the help of Corollary 4.4, we compute $P^\Theta(G)$ for $G = \mathbb{H}$ and $G = D_8$ and for Θ the set of inner automorphisms of G.

We introduce some additional notation:

Given $g \in G$, we denote the orbit of g under Θ by Θ_g.

Also, given a planar algebra P^Θ, it corresponds by Theorem 4.1 to an inclusion of subfactors $N \subset M$. Suppose there is a projection $q \in P_2^\Theta$ corresponding to an intermediate subfactor Q, $N \subset Q \subset M$. We define $P^{\Theta,1}$ and $P^{\Theta,2}$ to be the planar algebras corresponding to the subfactor inclusions $N \subset Q$ and $Q \subset M$.

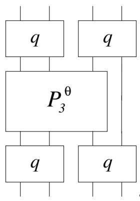
In addition we denote by $P^{Tl}(\alpha)$ the Temperley-Lieb planar algebra ([Jo2]), i.e. the planar algebra represented on the empty labeling set with parameter α; $P^{Tl}_i(\alpha)$ is the ith Temperley-Lieb algebra with parameter α.

The technique for analyzing the planar algebras arising from $G = \mathbb{H}, D_8$ will be the same. We first locate a projection $q \in P^{\Theta}_2$ corresponding to an intermediate subfactor (Corollary 4.3). We then analyze the structure of $P^{\Theta,1}$ and $P^{\Theta,2}$. We shall denote by $P^{\Theta,1} * P^{\Theta,2}$ the free product planar algebra of $P^{\Theta,1}$ and $P^{\Theta,2}$ and use the fact that $P^{\Theta,1} * P^{\Theta,2}$ is a subplanar algebra of P^{Θ} (see [Jo2] or [La1]). We show that in both cases $P^{\Theta,1} * P^{\Theta,2}$ is in fact all of P^{Θ}.

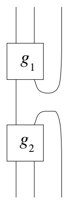
Example $G = \mathbb{H}$:

Let the quaternion group have generators $\{\pm id, \pm i, \pm j, \pm k\}$ with relations $i^2 = j^2 = k^2 = -id$ and $ij = k,\ jk = i,\ ki = j$. Letting $\Theta = \mathbb{H}$ we have $\Theta_{id} = \{id\}$, $\Theta_{-id} = \{-id\}$, $\Theta_i = \{\pm i\}$, $\Theta_j = \{\pm j\}$, $\Theta_k = \{\pm k\}$. Setting $a_i = \frac{1}{2}(-i) \oplus \frac{1}{2}(i)$, $a_j = \frac{1}{2}(-j) \oplus \frac{1}{2}(j)$, $a_k = \frac{1}{2}(-k) \oplus \frac{1}{2}(k)$, we have P^{Θ} is the subplanar algebra of $P(\mathbb{H})$ generated from the labeled two boxes $\{\pm id, a_i, a_j, a_k\}$ and thus $\dim(P^{\Theta}_2) = 5$. We have that $q = \frac{1}{2}(-id) \oplus \frac{1}{2}(id)$ satisfies the conditions of Corollary 4.3. Using Corollary 4.4 we have $P^{\Theta,1}_2 \cong \langle q\, id\, q,\ q(-id)q,\ qa_iq,\ qa_jq,\ qa_kq\rangle = \langle q,\ a_i,\ a_j,\ a_k\rangle$. It follows that $P^{\Theta,1}_2 \cong \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

We now look at $P^{\Theta,1}_3$ which by Corollary 4.4 is isomorphic to

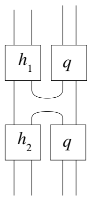


Now $P^{\Theta}_3 \subset P_3(\mathbb{H})$, and from Theorem 6.10 we have $P_3(\mathbb{H})$ is linearly spanned by



$g_1, g_2 \in \mathbb{H}.$

It follows that $P_3^{\Theta,1}$ is isomorphic to an algebra contained in the linear span of



$h_1, h_2 \in \{q, a_i, a_j, a_k\}$.

Thus $\dim(P_3^{\Theta,1}) \leq 16$. But the basic construction ensures $\dim(P_3^{\Theta,1}) \geq 16$ and thus $\dim(P_3^{\Theta,1}) = 16$ and $P^{\Theta,1}$ is a planar algebra coming from the depth two inclusion $M^{G'} \subset M$ with $G' = \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. $P^{\Theta,1} = P(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Thus $[Q : N] = 4$ which implies $[M : Q] = 2$ (since $[M : N]$ has the same index (8) as the group subfactor we started with) and thus the value of a loop in $P^{\Theta,2}$ is $\sqrt{2}$.

Although with a little work we can show that $P^{\Theta,2} = P^{Tl}(\sqrt{2})$ we shall actually be able to conclude this without analyzing the structure of $P^{\Theta,2}$. Recall that $P^{\Theta,1} * P^{\Theta,2}$ is a subplanar algebra of $P^{\Theta}(\mathbb{H})$. $P^{Tl}(\sqrt{2})$ is a sub-planar algebra of $P^{\Theta,2}$ (P^{Tl} is a subplanar algebra of any nontrivial planar algebra P). We thus have $P^{\Theta,1} * P^{Tl}(\sqrt{2})$ is a sub-planar algebra of P^{Θ}. By Theorem 8 of [La1]:

$$\dim(P^{\Theta,1} * P^{Tl}(\sqrt{2}))_2 = \dim(P_2^{\Theta,1}) + \dim(P_2^{Tl}(\sqrt{2})) - 1 = 4 + 2 - 1 = 5 = \dim P_2^{\Theta}.$$

Since P^{Θ} is generated by a labeling set consisting only of 2- boxes, we can conclude $P^{\Theta} = P^{\Theta,1} * P^{Tl}(\sqrt{2})$ and $P^{Tl}(\sqrt{2}) = P^{\Theta,2}$.

<u>Example $G = D_8$:</u>

The same technique can be used to compute P^{Θ} for $\Theta = D_8$. Let D_8 be given by generators x and y with relations $x^2 = y^4 = 1$, $xy = y^{-1}x$. We have $\Theta_1 = \{1\}$, $\Theta_y = \{y, y^{-1}\}$, $\Theta_x = \{x, xy^2\}$, $\Theta_{y^2} = \{y^2\}$, $\Theta_{xy} = \{xy, xy^{-1}\}$. Setting $q = \frac{1}{2}(1) \oplus \frac{1}{2}(y^2) \in P_2^{\Theta}$, we have that q satisfies the conditions in Corollary 4.3. The identical analysis to the case $\Theta = \mathbb{H}$ (above) gives $P^{\Theta,1} = P(\ Z_2 \times \mathbb{Z}_2)$. (Here

$$qP_2^{\Theta}q = \mathbb{C}\langle q,\ \frac{1}{2}(y) \oplus \frac{1}{2}(y^{-1}),\ \frac{1}{2}(x) \oplus \frac{1}{2}(xy^2),\ \frac{1}{2}(xy) \oplus \frac{1}{2}(xy^{-1})\rangle \cong \mathbb{C}(\mathbb{Z}_2 \times \mathbb{Z}_2).)$$

Using the same dimension considerations as in the case $\Theta = \mathbb{H}$ we conclude

$$P^{\Theta} = P(\mathbb{Z}_2 \times \mathbb{Z}_2) * P^{Tl}(\sqrt{2}) \text{ and } P^{\Theta,2} = P^{Tl}(\sqrt{2}).$$

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