Envelopes and Skin

`Topics:` adding and scaling, shrinking, envelope, skin and body, shrinking orthogonal circles, complementarity.

Adding and scaling. We revisit the vector space of circles. Explicit formulas for the sum of two circles and the scaling of a circle by a scalar can be derived from the same operations applied to the points g in $\mathbb{R}^3$.

CLAIM 5. Let $\hat{p} = (p, P^2)$ and $\hat{q} = (q, Q^2)$ be two circles in $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} \hat{p} + \hat{q} &= (p+q, P^2 + Q^2 + 2p^T q), \\ \alpha \cdot \hat{p} &= (\alpha \cdot p, \alpha P^2 + (\alpha^2 - \alpha)\|p\|^2). \end{aligned}$$

PROOF. The third coordinate of $g_{\hat{p}+\hat{q}}$ is $P^2+Q^2-\|p\|^2-\|q\|^2$. The square radius of $\hat{p}+\hat{q}$ is therefore

$$\begin{aligned} R^2 &= P^2 + Q^2 - \|p\|^2 - \|q\|^2 + \|p+q\|^2 \\ &= P^2 + Q^2 + 2p^T q. \end{aligned}$$

The third coordinate of $g_{\alpha \cdot \hat{p}}$ is $\alpha(P^2-\|p\|^2)$. The square radius of $\alpha \cdot \hat{p}$ is therefore

$$\begin{aligned} R^2 &= \alpha P^2 - \alpha\|p\|^2 + \|\alpha \cdot p\|^2 \\ &= \alpha P^2 + (\alpha^2 - \alpha)\|p\|^2. \end{aligned}$$

□

The vector space operations are used to turn a finite set of circles into an infinite family.

Shrinking. It is convenient to introduce a shrinking operation that generates smooth blends. When we shrink a circle we leave the center unchanged and decrease the radius. For a real $s \in [0, 1]$ we define

$$\hat{p}^s = (p, sP^2).$$

Observe that shrinking can be reduced to scaling and adding, namely $\hat{p}^s = s \cdot \hat{p} + (1-s) \cdot p$, where $p = (p, 0)$ is a circle with zero radius. The radius of the shrunken circle is $\sqrt{s}$ times the original radius. For $s = 1$ we get the same circle back, and for $s = 0$ we get a point. Figure 9 illustrates the operation by shrinking nine circles sampled from a line of circles.

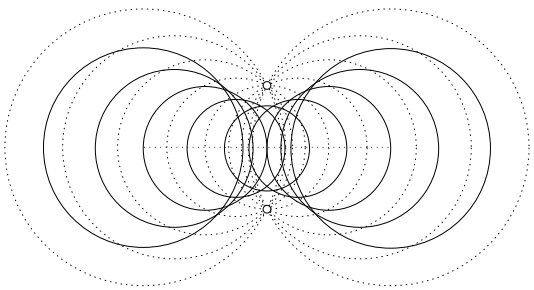

Figure 9: Each solid circle is a shrunken dotted circle sampled from the affine hull of two circles.

Envelope. We are interested in computing the envelope of families of shrunken circles. Let F be the line of circles passing through points $(0, 1)$ and $(0, -1)$ represented by the nine dotted circles in Figure 9. We parametrize the shrunken circles in $F^s = \{\hat{p}^s \mid \hat{p} \in F\}$:

$$f(\tau, x_1, x_2) = (x_1 - \tau)^2 + x_2^2 - s(\tau^2 + 1).$$

The circle with center $(\tau, 0)$ is the zero-set of the function f for fixed value of τ. The collection of all circles is the projection of the entire zero-set, $f^{-1}(0)$. It can be visualized as a slanted stack of circles in $\mathbb{R}^3$, see Figure 10. The envelope of F^s is the projection of the silhou-

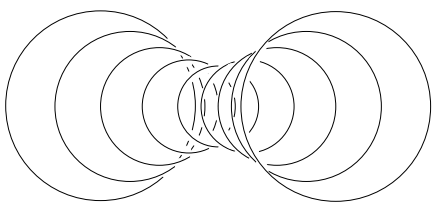

Figure 10: Stack of circles viewed from the positive τ direction. The union of the circles is the zero-set of f.

ette of the stack as viewed along the τ direction. It is the set of points where $\partial f / \partial \tau$ vanishes. The derivative

with respect to τ is

$$\begin{aligned}\frac{\partial f}{\partial \tau} &= -2(x_1 - \tau) - 2s\tau \\ &= \tau(2 - 2s) - 2x_1.\end{aligned}$$

From $\partial f/\partial \tau = 0$ we get $\tau = \frac{x_1}{1-s}$. The envelope is therefore the zero-set of

$$f(\frac{x_1}{1-s}, x_1, x_2) = -\frac{s}{1-s} \cdot x_1^2 + x_2^2 - s.$$

A more symmetric expression for this curve is

$$-\frac{1}{t} \cdot x_1^2 + \frac{1}{s} \cdot x_2^2 - 1 = 0,$$

where $t = 1 - s$. This is a hyperbola.

Skin and body. General closed curves can be constructed by taking the convex hull of a finite collection of circles, then shrinking every circle in the family, and finally taking the envelope. Let S be the finite set. The *s-skin* of S is

$$\operatorname{skn}^s S = \operatorname{env}(\operatorname{conv} S)^s.$$

The *s-body* is the region bounded by the s-skin. Equivalently, the s-body is the union of the disks bounded by shrunken circles.

The smallest non-trivial example of a skin curve is constructed from two circles, $S = \{\hat{p}, \hat{q}\}$. The convex hull of S is a line segment of circles. Specifically, the circles in $\operatorname{conv} S$ have their centers on the line segment connecting p with q and they all pass through the points where $\hat{p}$ and $\hat{q}$ intersect. This is illustrated in Figure 9. The envelope is the dumbbell shown in Figure 11, which consists of two circular arcs and a connecting hyperbola in the middle.

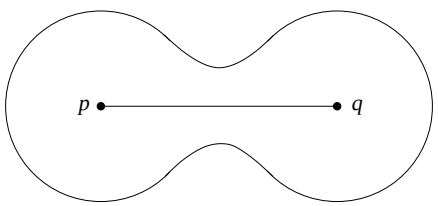

Figure 11: The envelope of the shrunken circles in a line segment of circles.

Shrinking orthogonal circles. The skin of three circles is already more difficult to understand. We first study orthogonal circles and what happens to them when they shrink. Let $\hat{p} = (p, P^2)$ and $\hat{x} = (x, X^2)$ be orthogonal and $s, t \in [0, 1]$ with $s + t = 1$. We show that shrinking $\hat{p}$ with s and shrinking $\hat{x}$ with t removes the overlap, no matter what the initial radii are. In fact, orthogonality is the limiting case where for certain radii we get touching circles.

LEMMA 6. The intersection of $\hat{p}^s$ and $\hat{x}^t$ is either empty or a single point.

PROOF. We may assume that both are real circles, with non-negative square radii P^2 and X^2. Since $\hat{p} \perp \hat{x}$ and $s + t = 1$ we have

$$\begin{aligned}\|p - x\| &= P^2 + X^2 \\ &\geq (s + t)(P^2 + X^2) - (\sqrt{s}X - \sqrt{t}P)^2 \\ &= (\sqrt{s}P + \sqrt{t}X)^2.\end{aligned}$$

Taking roots left and right implies that the radii of $\hat{p}$ and $\hat{x}$ add up to at most the distance between p and x. □

We have equality iff $\sqrt{s}X = \sqrt{t}P$. For example, for $s = t = \frac{1}{2}$ this is the case iff the two circles are equally large.

Complementarity. Let us return to coaxal systems. We construct one from four circles $\hat{p}, \hat{q}, \hat{x}, \hat{y}$, where the first two are orthogonal to the last two. Let $F = \operatorname{aff}\{\hat{p}, \hat{q}\}$, $G = \operatorname{aff}\{\hat{x}, \hat{y}\}$ we the two lines of circles of the system, as shown in Figure 8 in Section 25. Assume F is the line that contains only real circles. We already know that the envelope of F^s is a hyperbola. The family G and its shrunken circles are illustrated in Figure 12. We could turn G^t into a stack of circles and prove by differentiation that its envelope is also a hyperbola. Instead we use Lemma 6 to show that the envelopes of G^t and F^s are identical, provided $s + t = 1$.

First notice that circles $\hat{r}^s \in F^s$ and $\hat{z}^t \in G^t$ do not cross; they are either disjoint or they touch. It follows that the bodies defined by F and G do not overlap. If they intersect then at most along shared boundary. We prove the two bodies share their entire boundary:

$$\operatorname{skn}^s F = \operatorname{skn}^t G.$$

For every circle $\hat{r} \in F$ there are two circles $\hat{z} \in G$ with $\sqrt{s}Z = \sqrt{t}R$. This is the condition for which the shrinking produces touching circles. The point where $\hat{r}^s$ and $\hat{z}^t$ touch belongs to the boundary of the two bodies. Since each point of $\operatorname{skn}^s F$ lies on a shrunken circle in F^s it also lies on a shrunken circle in G^t. This implies that the two envelopes are the same. We will see later

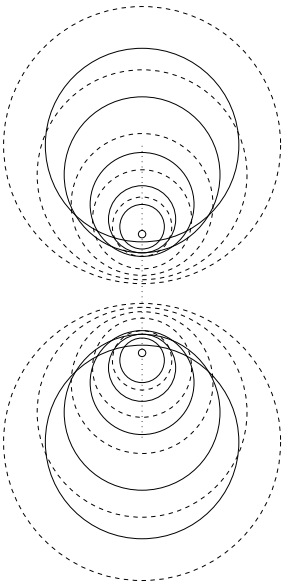

Figure 12: Each solid circle is a shrunken dotted circle sampled from a line of circles of which a segment consists of imaginary circles.

that this result generalizes. Specifically, for every finite set of circles S there is another set of circles T such that the s-skin of S is the t-skin of T. As in the case of the coaxal system, the sets S and T lie on opposite sides of the curve.

Bibliographic notes. The material of this section is taken from the paper by Edelsbrunner [3], where skin surfaces are introduced. That paper also proves a variety of properties, including the homotopy equivalence between bodies and alpha complexes. If we continuously move, grow, or shrink the defining spheres we observe smooth deformations. Software for visualizing such deformations in the plane is used in [2] to explain skin curves and their behavior under motion. Canonical deformations between two or more basic shapes can be used to construct spaces of shapes as shown in [1].

[1] H.-L. Cheng, H. Edelsbrunner and P. Fu. Shape space from deformation. *In* "Proc. 6th Pacific Conf. Comput. Graphics Appl., 1998", Singapore, 104–113.

[2] S.-W. Cheng, H. Edelsbrunner, P. Fu and P. Lam. Design and analysis of planar shape deformation. *In* "Proc. 14th Ann. Sympos. Comput. Geom., 1998", Minneapolis, Minnesota, 29–38.

[3] H. Edelsbrunner. Deformable smooth surface design. *Discrete Comput. Geom.* **21** (1999), 87–115.