Simplicial Maps

`Topics:` barycentric coordinates, vertex and simplicial maps, simplicial and PL-equivalence, link condition for manifolds.

Barycentric coordinates. Recall that an affine combination of a collection of $k+1$ points p_i is a point

$$x = \sum_{i=0}^{k} \alpha_i \cdot p_i,$$

where the α_i sum to 1. If the $k+1$ points are affinely independent then the α_i are unique and referred to as the *barycentric coordinates* of x with respect to the points p_i. The set of affine combinations of the p_i is a k-dimensional hyperplane. The number of barycentric coordinate of x is one more than the dimension, which suggests that the hyperplane can be embedded in $\mathbb{R}^{k+1}$ in such a way that the barycentric coordinates coincide with the Cartesian coordinates. Figure 9 shows how this is done for $k = 2$. Points of the hyperplane that

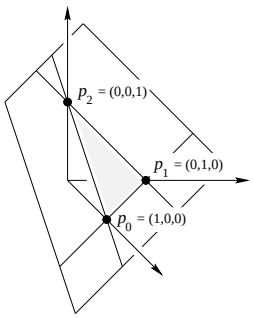

Figure 9: The plane is the affine hull of the three unit vectors, and the shaded triangle is their convex hull.

lie in the first orthant of $\mathbb{R}^{k+1}$ have all $k+1$ coordinates non-negative so they form the convex hull of the p_i, which is a k-simplex σ. A particularly important convex combination is the *barycenter*,

$$\hat{\sigma} = \frac{1}{k+1} \cdot \sum_{i=0}^{k} p_i,$$

which is also know as the centroid of σ and the average of the p_i.

Vertex and simplicial maps. We use barycentric coordinates to extend a function between vertex sets to a function between underlying spaces. It is convenient to extend the barycentric coordinates to all vertices of a simplicial complex K. Each point $x \in \|K\|$ lies in the interior of exactly one simplex $\sigma \in K$, and we suppose that $p_0, p_1, \ldots, p_k$ are the vertices of σ. For each $u \in \operatorname{Vert} K$ we define

$$\mathsf{b}_u(x) = \begin{cases} \alpha_i & \text{if } u = p_i \text{ for } 0 \le i \le k, \\ 0 & \text{if } u \text{ is not a vertex of } \sigma. \end{cases}$$

Instead of fixing x and varying u we now fix u and vary x. From this point of view we have a map $\mathsf{b}_u : |K| \to [0,1]$ which is zero outside the star of u, see Figure 10. In numerical analysis b_u would be called a base function.

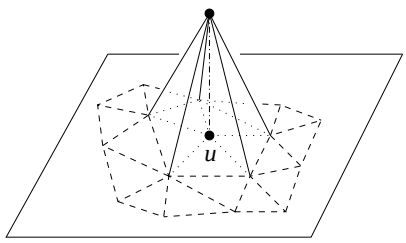

Figure 10: We have $\mathsf{b}_u(u) = 1$ and $\mathsf{b}_u(x) = 0$ for all points x outside the star of u.

A *vertex map* from K to another simplicial complex L is a function $g : \operatorname{Vert} K \to \operatorname{Vert} L$ that sends the vertices of a simplex in K to the vertices of a simplex in L. Strictly speaking, g is not a map, but it is called a map because the condition on the images is similar to continuity. The *simplicial map* that extends g is $\psi : \|K\| \to \|L\|$ defined by

$$\psi(x) = \sum_{u \in \operatorname{Vert} K} \mathsf{b}_u(x) \cdot g(u).$$

ψ is continuous because each b_u is continuous.

Simplicial and PL-equivalence. A general simplicial map has little predictable structure other than continuity because the defining vertex map is neither necessarily injective nor necessarily surjective. Even if g is bijective it is possible that ψ does not reach all points of $\|L\|$. However, if we assume in addition that its inverse is also a vertex map then vertices u_0 through u_k span a k-simplex $\sigma \in K$ iff their images $g(u_0)$ through $g(u_k)$ span a k-simplex $\tau \in L$. The corresponding simplicial map ψ sends every point $x \in \sigma$ to a unique point $\psi(x) \in \tau$ and vice versa. More generally, we have

CLAIM 3. If a vertex map $g : \operatorname{Vert} K \to \operatorname{Vert} L$ is bijective and its inverse is a vertex map then the corresponding simplicial map $\psi : \|K\| \to \|L\|$ is a homeomorphism.

If K and L have a connecting simplicial homeomorphisms then they are abstractly the same complexes. Formally, they are said to be *simplicially equivalent*.

We now have an equivalence relation for the class of simplicial complexes. A related equivalence relation for topological spaces requires that they have a common triangulation. If we start with the underlying spaces of two simplicial complexes we can generate a common triangulation by refinement. A *subdivision* of K is a simplicial complex K' such that every simplex in K' is contained in a simplex in K. Simplicial complexes K and L are *PL-equivalent* if there are simplicially equivalent subdivisions K' of K and L' of L. If K and L are PL-equivalent then by definition $\|K\|$ and $\|L\|$ are homeomorphic. As it turns out the other directions not true in general.

Link condition for manifolds. We use simplicially equivalent subdivisions to prove one direction of Theorem 2A on edge contractions for 2-manifolds. Recall that K is the triangulation of a 2-manifold, $ab \in K$ is an edge, and L is obtained from K by contracting the edge ab. The contraction can be interpreted as the simplicial map $\psi : \|K\| \to \|L\|$ defined by the vertex map

$$g(u) \quad = \quad \begin{cases} u & \text{if } u \notin \{a, b\}, \\ c & \text{if } u \in \{a, b\}. \end{cases}$$

Theorem 2A claims that $\|K\|$ and $\|L\|$ are homeomorphic iff $\operatorname{Lk} a \cap \operatorname{Lk} b = \operatorname{Lk} ab$. We prove the direction from the link condition to $\|K\| \approx \|L\|$. Since $\|K\|$ is a 2-manifold, the link of ab consists of exactly two vertices x and y as shown in Figure 11. The links of a and of b are two circles which meet at x and y. The outer pieces of the two circles glued at x and y form another circle, which is the link of $\overline{ab}$ in K and also the link of c in L. We

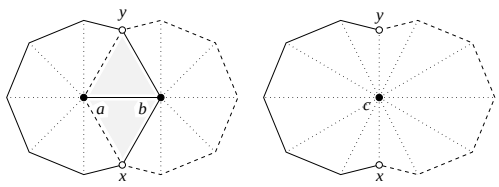

Figure 11: The links of a and b are the solid and the dashed circles around a and b. The link of c takes a piece of each and combines them to a circle around c.

construct simplicially equivalent subdivisions of K and L by mapping the common link to the boundary of a regular n-gon in the plane, as shown in Figure 12. The

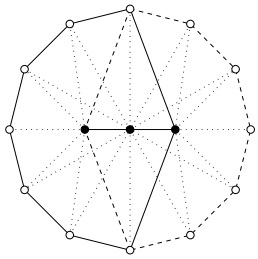

Figure 12: The superposition of the images of the stars of $\overline{ab}$ in K and of c in L.

stars of $\overline{ab}$ and of c are mapped to two triangulations of the n-gon. We superimpose the triangulations and get a decomposition into convex polygonal regions, which we further refine to another triangulation. This triangulation is mapped back to form subdivisions of the stars of $\overline{ab}$ and of c. The link has not been changed, so we can combine the subdivided star of $\overline{ab}$ with the unsubdivided rest of K, and similar for the star of c and L. The resulting subdivisions of K and L are simplicially equivalent, and hence $\|K\| \approx \|L\|$.

Bibliographic notes. A good background source on simplicial maps is the book on algebraic topology by Munkres [3]. During the early years of combinatorial topology it was conjectured that two simplicial complexes are PL-equivalent iff their underlying spaces are homeomorphic. This became known as the Hauptvermutung (German for main conjecture). It was verified for 2-complexes and for 3-complexes, see the book of Moise [2]. In 1961 the conjecture was disproved for general simplicial complexes by John Milnor who used two 7-dimensional simplicial complexes for the counterexample [1].

[1] J. Milnor. Two complexes which are homeomorphic but combinatorially distinct. *Ann. of Math.* **74** (1961), 575–590.

[2] E. Moise. *Geometric Topology in Dimensions* 2 *and* 3. Springer-Verlag, New York, 1977.

[3] J. R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Redwood City, 1984.