Homework Assignment

The solution for each question should fit on one side of a single piece of paper.

1. **Degenerate Delaunay complex.** The *face centered cube (FCC) lattice* consists of all integer points (i, j, k) with even sum $i + j + k$.
   (i) The Delaunay complex of the FCC lattice is degenerate because there are empty spheres that pass through six points. Which six points?
   (ii) The Delaunay complex of the FCC lattice has only two types of 3-dimensional cells. What are they?

2. **Intersection of unit disks.** A theorem by Jung states that if every three of a finite set of points in the plane are contained in a unit disk then the entire set is contained in a unit disk.
   (i) Use the two-dimensional version of Helly's theorem to prove Jung's theorem.
   (ii) What is the generalization of Jung's theorem to $d \geq 3$ dimensions?

3. **Non-Delaunay complex.** Exhibit a 3-complex K with vertex set $S \subseteq \mathbb{R}^3$ whose edges all belong to the Delaunay complex D of S and whose underlying space is the convex hull of S, but still $K \neq D$.

4. **Regular tetrahedron.** To prove that the regular tetrahedron does not tile $\mathbb{R}^3$ show that the dihedral angle between two of its triangles is not an integer fraction of 2π. What exactly is that dihedral angle?

5. **Induced subcomplex.** Let $S \subseteq \mathbb{R}^3$ be finite and D the Delaunay complex of S. The subcomplex $K \subseteq D$ *induced* by a subset $T \subseteq S$ consists of all simplices in D whose vertices all belong to T.
   (i) Show that K is also a subcomplex of the Delaunay complex of T.
   (ii) Use (i) to show that if there is a Delaunay complex that has an edge crossing a triangle then there is such a Delaunay complex for only five points.

6. **BCC tetrahedron.** The *body centered cube (BCC) lattice* consists of the integer points (i, j, k), where all three coordinates are either even or all three are odd. All Delaunay tetrahedra of the BCC lattice are congruent to a single tetrahedron, which we call the *BCC tetrahedron*. What is the circumradius to length of shortest edge ratio of that tetrahedron?

7. **Packing.** A collection of closed unit balls in $\mathbb{R}^3$ forms a *packing* if their interiors are pairwise disjoint. The packing is *maximal* if no unit ball can be added without distroying the packing. Let $S \subseteq \mathbb{R}^3$ such that the set of unit balls $S + \mathbb{B}^3$ is a maximal packing.
   (i) Show that increasing the balls to twice the size produces a covering of space, that is,
   $$\bigcup (S + 2 \cdot \mathbb{B}^3) \quad = \quad \mathbb{R}^3.$$
   (ii) Prove that an edge in the Delaunay complex of S has length at most 4.
   (iii) Prove that there is a constant c such that each vertex in the Delaunay complex of S belongs to at most c edges.