CS 174 Homework Assignment 4 (due Wednesday, March 6)

1. Recall the derivation of a lower bound for the probability that a minimum cut survives a run of the randomized MinCut algorithm (see Note 4). Based on this result, prove that no graph has more than $\binom{n}{2}$ minimum cuts.

2. Consider applying the randomized MinCut algorithm to a tree, a cycle, a cycle with one additional edge, and a bipartite graph. What can you say about the probability that the algorithm finds a minimum cut in each of these cases?

3. Give a randomized algorithm that finds the *second smallest* cut in a graph in $O(n^2 \log^3 n)$ time with high probability. (If a graph has two distinct minimum cuts, the second smallest cut is a minimum cut).

4. Do problem 1.8 in Motwani and Raghavan.

5. Given a graph $G = (V, E)$, an *independent set* S is a subset of vertices such that no two vertices in S have an edge between them.

   (a) Let π be a permutation of V. Let $S(\pi)$ denote the set of vertices i such that, for all neighbors j of i, we have $\pi(i) < \pi(j)$. Argue that, for any permutation π, $S(\pi)$ is always an independent set.

   (b) Now consider a random permutation π. For a fixed vertex i, what is the probability that i is placed before all of its neighbors by π? You should give your answer as a function of the degree $d(i)$ of vertex i.

   (c) Suppose now a regular graph (every vertex has the same degree d). If we choose the permutation π at random, what is the expected size of $S(\pi)$, as a function of n and d?

   (d) Conclude that, for regular graphs G, there exists an independent set in G of size at least $n/(d+1)$.

   (e) Devise a simple randomized algorithm that, with probability at least $1/2$, finds an independent set of size at least $\alpha n/(d+1)$ in such a graph, where $0 < \alpha < 1$ is a fixed constant. You should carefully justify the success probability of your algorithm, and also state its running time. [Hint: Use Markov's inequality to obtain an algorithm with a failure probability that depends on α, and then boost.]