Solutions for CS174 Homework 2

Solution 1. Assume $\lceil \log n \rceil = m$. Then to get a biased coin with probability k/n of heads, we simply toss the fair coin m times, and get the binary representation of the string from the m tosses. Let the binary string be x, if $x < k$, then we say the outcome of the biased coin is head; if $k \leq x < n$, we say the outcome of the biased coin is tail; otherwise we discard this experiment and restart.

Solution 2. The total number of random permutation is $n!$. The total number of random permutation where 2 and 3 are adjacent is equal to the total number of random permutation of 1, ... , n-1 where we glue 2 and 3 together as one element. Hence the number of this glued random permutation is $(n-1)!$. So the probability that 2 and 3 are adjacent in a random permutation is $\frac{2\cdot(n-1)!}{n!} = \frac{2}{n}$.

Solution 3a. Let $x_1, \ldots, x_n$ represent the n people. Let (x_i^1, x_j^1) denote that x_i is to the right of x_j in lunch, and (x_i^2, x_j^2) denote that x_i is to the right of x_j in dinner, and (x_i, x_j) denote that x_i is to the right of x_j in both lunch and dinner. Because the lunch setting and the dinner setting are independent, $\Pr[(x_i, x_j)] = \Pr[(x_i^1, x_j^1)] \cdot \Pr[(x_i^2, x_j^2)]$.

Let Y denote the probability that no-one is to the right of someone they were to the right of at lunch. We can see that $Y = 1 - \Pr[\bigvee_{i,j}(x_i, x_j)]$.

By using the inclusion-exclusion property, we can see that

$$\begin{aligned}
\Pr[\bigvee_{i,j}(x_i, x_j)] &= \sum_{i,j}\Pr[(x_i, x_j)] - \sum_{i_1,j_1,i_2,j_2}\Pr[(x_{i_1}, x_{j_1}) \wedge (x_{i_2}, x_{j_2})] \\
&\quad + \cdots + (-1)^{n-1}\sum_{i_1,j_1,\ldots,i_n,j_n}\Pr[(x_{i_1}, x_{j_1}) \wedge \cdots \wedge (x_{i_n}, x_{j_n})].
\end{aligned}$$

The total number of circular seating with n people is $n!/n = (n-1)!$. Similar to problem 2, the total number of circular seating with x_i to the right of x_j is $(n-2)!$. So $\Pr[(x_i, x_j)] = (\frac{(n-2)!}{(n-1)!})^2 = \frac{1}{(n-1)^2}$. There are $n(n-1)$ combinations of x_i and x_j, so

$$\sum_{i,j}\Pr[(x_i, x_j)] = n(n-1)\frac{1}{(n-1)^2} = \frac{n}{n-1} \sim 1, \text{as } n \text{ goes to large.}$$

To compute $\sum_{i_1,j_1,i_2,j_2}\Pr[(x_{i_1}, x_{j_1}) \wedge (x_{i_2}, x_{j_2})]$, there are two cases: either $x_{j_1} = x_{i_2}$ or $x_{j_1} \neq x_{i_2}$:

- For the first case where $x_{j_1} = x_{i_2}$,

  $$\Pr[(x_{i_1}, x_{j_1}) \wedge (x_{i_2}, x_{j_2})] = (\frac{(n-3)!}{(n-1)!})^2 = \frac{1}{((n-1)(n-2))^2}.$$

  And there are $n(n-1)(n-2)$ possible combinations.

- For the second case where $x_{j_1} \neq x_{i_2}$,

  $$\Pr[(x_{i_1}, x_{j_1}) \wedge (x_{i_2}, x_{j_2})] = (\frac{(n-3)!}{(n-1)!})^2 = \frac{1}{((n-1)(n-2))^2}.$$

  And there are $n(n-1)(n-2)(n-3)/2$ possible combinations.

So

$$\begin{aligned}
\sum_{i_1,j_1,i_2,j_2}\Pr[(x_{i_1}, x_{j_1}) \wedge (x_{i_2}, x_{j_2})] &= \frac{n(n-1)(n-2) + n(n-1)(n-2)(n-3)/2}{((n-1)(n-2))^2} \\
&= \frac{n + n(n-3)/2}{(n-1)(n-2)} \\
&\to \frac{1}{2}, \quad \text{as } n \text{ goes to } \infty.
\end{aligned}$$

Similarly,

$$\sum_{i_1,j_1,\ldots,i_k,j_k} \Pr[(x_{i_1},x_{j_1}) \wedge \cdots (x_{i_k},x_{j_k})] = (\frac{(n-1-k)!}{(n-1)!})^2 \cdot \{\frac{n!}{(n-2k)!k!} + o(n^{2k})$$
$$\to \frac{1}{k!}, \quad \text{as } n \text{ goes to } \infty.$$

Hence, $\lim_{n\to\infty} \Pr[\bigvee_{i,j}(x_i,x_j)] = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{k-1}\frac{1}{k!} + \cdots \sim 1 - e^{-1} \doteq 0.632$.

So the probability that no one is to the right of someone they were to the right of at lunch is $1 - 0.632 = 0.378$.

Solution 3b. Let $x_i \sim x_j$ denote that x_i is to the right of x_j at lunch and x_i and x_j are neighbours at dinner. Hence

$$\Pr[x_i \sim x_j] = \Pr[(x_i^1,x_j^1)] \cdot (\Pr[(x_i^2,x_j^2)] + \Pr[(x_j^2,x_i^2)]) = \frac{2}{(n-1)^2}.$$

Similarly,

$$\Pr[(x_{i_1} \sim x_{j_1}) \wedge \cdots (x_{i_k} \sim x_{j_k})] = 2^k \Pr[(x_{i_1},x_{j_1}) \wedge \cdots (x_{i_k},x_{j_k})].$$

So $\sum_{i_1,j_1,\ldots,i_k,j_k} \Pr[(x_{i_1} \sim x_{j_1}) \wedge \cdots (x_{i_k} \sim x_{j_k})] = \frac{2^k}{k!}$.

Hence, $\lim_{n\to\infty} \Pr[\bigvee_{i,j}(x_i \sim x_j)] = 2 - \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + (-1)^{k-1}\frac{2^k}{k!} + \cdots \sim 1 - e^{-2}$.

So the probability that no one at dinner has a neighbor on the same or other side that they had at lunch is $e^{-2} = 0.142$.

Solution 4. From the definition of a poisson distribution, we have $\Pr[X = k] = e^{-\lambda}\frac{\lambda^k}{k!}$ and $\Pr[X \geq \lambda] = e^{-\lambda}\sum_k \frac{\lambda^k}{k!}$.

Now the largest term in the sum $\sum_{k=\lambda,\ldots,\infty} \frac{\lambda^k}{k!}$ is $\lambda = k$. Later terms are multiplied by ratios such as $\lambda/(k+j)$ which is less than one, so they are less than this term. Let $t_k = \frac{k^k}{k!}$. Then

$$t_{k+1} = t_k k/(k+1) \qquad t_{k+2} = t_{k+1} k/(k+2) \cdots$$

On the other hand

$$t_{k-1} = t_k \qquad t_{k-2} = t_{k-1}(k-1)/k \qquad t_{k-3} = t_{k-2}(k-2)/k \cdots$$

From these two it follows that

$$t_k = t_{k-1} \qquad t_{k+1} \approx t_{k-2} \qquad t_{k+2} \approx t_{k-3} \cdots$$

where the error terms are $O(1/k^2)$. Now let

$$S_a = \sum_{j=0}^{k} \frac{k^j}{j!} \qquad S_b = \sum_{j=k}^{2k} \frac{k^j}{j!}$$

For each term t_{k+j} in S_b there is a corresponding term t_{k-j-1} in S_a. Since the terms are approximately the same, we have $S_a \approx S_b$. Finally using Chernoff, bounds its clear that

$$\sum_{j=2k}^{\infty} \frac{k^j}{j!} \approx 0$$

So it follows that $e^{-k}(S_a + S_b) \approx 1$, and therefore $\Pr[X \geq \lambda] = e^{-k} S_b \approx 1/2$.