Quiz #1
Solutions

1. Consider the set in $\mathbb{R}^{2n}$

$$\mathcal{E} := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & a_x \\ A_{xy}^T & A_{yy} & a_y \\ a_x^T & a_y^T & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \leq 1 \right\},$$

where $a_x, a_y \in \mathbb{R}^n$, and the symmetric matrix

$$A := \begin{pmatrix} A_{xx} & A_{xy} \\ A_{xy}^T & A_{yy} \end{pmatrix}$$

is positive-definite.

(a) What is the shape of $\mathcal{E}$? Express your answer in geometric terms, assuming that some relevant eigenvalue decomposition is available.

Solution: in the space of $z = (x,y) \in \mathbb{R}^{2n}$, the set $\mathcal{E}$ has the form

$$\mathcal{E} = \left\{ z \ : \ z^T A z + 2a^T z + \alpha \leq 1 \right\},$$

where $a = (a_x, a_y) \in \mathbb{R}^{2n}$. We can write the condition

$$z^T A z + 2a^T z + \alpha \leq 1$$

equivalently as

$$(z - z_0)^T A (z - z_0) \leq \gamma := 1 - \alpha + z_0^T A z_0$$

where $z_0 := A^{-1}a$. Thus, if $\alpha > 1 + z_0^T A z_0 = 1 + a^T A{-}1a$, the above set is empty; if $\alpha = 1 + a^T A{-}1a$, the set is the singleton $\{z_0\}$. Otherwise, the set is an ellipse having center at z_0, and with semi-axis lengths given by $\sqrt{\gamma/\lambda_i}$, $i = 1, \ldots, n$, with λ_i the inverse eigenvalues of A, and principal directions given by the eigenvectors.

(b) Likewise, determine the projection of $\mathcal{E}$ onto the space of x-variables.

Solution: The projection $\mathcal{P}$ is the set of points $x \in \mathbb{R}^n$ such that

$$\begin{aligned} 1 &\geq \min_y \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}^T \begin{pmatrix} A_{xx} & A_{xy} & a_x \\ A_{xy}^T & A_{yy} & a_y \\ a_x^T & a_y^T & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \\ &= \min_y \ x^T A_x x x + y^T A_{yy} y + 2x^T A_{xy} y + 2a_x^T x + 2a_y^T y + \alpha \end{aligned} \qquad (1)$$

Solving for the minimum simply entails taking the derivative with respect to y and setting it to zero. This leads to the optimality condition

$$A_{yy} y + A_{xy}^T x + a_y = 0.$$

Solving for y, we obtain the optimal solution

$$y^*(x) = -A_{yy}^{-1}(A_{xy}^T x + a_y)$$

Replacing y by its optimal value in (1), we conclude that the condition $x \in \mathcal{P}$ is equivalent to

$$x^T(A_{xx} - A_{xy}A_{yy}^{-1}A_{xy}^T)x + 2(a_x - A_{yy}^{-1}a_y)^T x + \beta \leq 1,$$

where

$$\beta := \alpha - a_y^T A_{yy}^{-1} a_y.$$

The above condition can be analyzed as before.

2. Find the Fenchel conjugate $f^*(y) = \max_x \ x^T y - f(x)$ for the following functions.

   (a) $f(x) = \frac{1}{2}(a^T x)^2 + b^T x$ on $\mathbb{R}^n$.

   (b) $f(x) = \max(0, 1 - c^T x)$ on $\mathbb{R}^n$.

Solution: both cases are special cases of functions of the form

$$f(x) = h(a^T x) + b^T x,$$

where h is a univariate function. The conjugate of a function $f(x) = g(x) + b^T x$ is simply $g^*(y - b)$, hence we first examine the case $b = 0$.

Assume $a \neq 0$. Decompose x as $x = ta + u$, with $a^T u = 0$, and work with variables (t, u) and the constraint $a^T u = 0$. We obtain

$$f^*(y) = \max_{t,u \,:\, a^T u = 0} (ta + u)^T y - h((ta + u)^T a) = \max_t t(y^T a) - h(t(a^T a)) + \max_{u \,:\, a^T u = 0} y^T u.$$

The first term is nothing else than $h^*((y^T a)/(a^T a))$, and the second term is finite (and zero) if and only if $y = \alpha a$ for some $\alpha \in \mathbb{R}$. We obtain

$$f^*(y) = \begin{cases} h^*(\alpha) & \text{if } y = \alpha a \text{ for some } \alpha \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

When $b \neq 0$, we have

$$f^*(y) = \begin{cases} h^*(\alpha) & \text{if } y = b + \alpha a \text{ for some } \alpha \in \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

This allows to solve the two problems at once, by using $h(t) = t^2/2$ for the first part, and $h(t) = \max(0, 1 - t)$ for the second part.

The conjugate of $h(t) = t^2/2$ is simply itself; for the second function, we have

$$\begin{aligned} h^*(\alpha) &= \max_t \ \alpha t - \max(0, 1 - t) \\ &= \max\left(\max_{t \geq 1} \ \alpha t, \max_{t \leq 1} (\alpha + 1)t - 1\right). \end{aligned}$$

The first term is $+\infty$ when $\alpha > 0$, α otherwise; the second is $+\infty$ when $\alpha < -1$, and α otherwise. Hence

$$h^*(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in [-1, 0], \\ +\infty & \text{otherwise.} \end{cases}$$