Midterm: Solutions

1. (10 points) *Convexity and conjugates.* Consider the function $f : \mathbf{R} \times \mathbf{R}_{++} \to \mathbf{R}$, with values for $x \in \mathbf{R}$, $t > 0$ given by

$$f(x,t) = \frac{x^2}{t}$$

   (a) Show that f is convex.

   (b) What is the conjugate function of f? Make sure to define the domain, and check that it is convex. *Hint:* differentiate over x first.

Solution:

(a) The function f is convex, since its epigraph is convex. Indeed, the condition $f(x,t) \le u$ on a point $(x,t,u) \in \mathbf{R}^3$ with $t > 0$, is equivalent to the LMI

$$\begin{pmatrix} t & x \\ x & u \end{pmatrix} \succeq 0.$$

Another proof relies on the Hessian of f:

$$\nabla^2 f(x,t) = \frac{2}{t^3} \begin{pmatrix} t \\ -x \end{pmatrix} \begin{pmatrix} t \\ -x \end{pmatrix}^T,$$

which is positive-definite whenever $t > 0$.
Yet another proof relies on the expression

$$\frac{x^2}{t} = \max_y \; xy - t\frac{y^2}{4}.$$

(b) The conjugate of f is the function

$$f^*(y,u) := \max_{x,t>0} \; xy + tu - \frac{x^2}{t}.$$

Differentiating over x, we obtain that $x = (ty)/2$, and the problem above reduces to

$$f^*(y,u) := \max_{t>0} \; t(u + y^2/4) = \begin{cases} 0 & \text{if } u + y^2/4 \le 0, \\ +\infty & \text{otherwise.} \end{cases}$$

2. (10 points) *Subgradients and duality.* For $\lambda > 0$, we define

$$\phi(\lambda) := \min_x \; f(x) + \lambda g(x),$$

where the functions f, g are both convex. We assume that for every $\lambda > 0$, the optimal set of the above problem is not empty (in other words, the problem is attained).

(a) Show that ϕ is concave.

(b) Show that if $x(\lambda)$ is any optimal point for the above problem, then $-g(x^*(\lambda))$ is a sub-gradient of $-\phi$ at λ. *Hint:* show that for every pair $\lambda > 0, \mu > 0$, we have

$$\phi(\mu) \leq \phi(\lambda) + (\mu - \lambda)g(x^*(\lambda)).$$

(c) For $\kappa > 0$, consider the problem

$$p^*(\kappa) := \min_x \; f(x) \; : \; g(x) \leq \kappa,$$

Assuming that $g(0) \leq 0$, show that

$$p^*(\kappa) = \max_{\lambda \geq 0} \; -\kappa\lambda + \phi(\lambda).$$

Is $p^*(\kappa)$ convex on the domain $\mathbf{R}_{++}$?

(d) Assume that the above problem is attained, and denote by λ^* an optimal point. Find a sub-gradient of p^* at κ.

Solution:

(a) The function is a point-wise minimum (over x) of affine functions of λ, hence it is concave.

(b) We start with the fact that for any $x \in \mathbf{dom} f \cap \mathbf{dom} g$, we have

$$\phi(\mu) \leq f(x) + \mu g(x) = f(x) + \lambda g(x) + (\mu - \lambda)g(x).$$

Applying this to $x = x^*(\lambda)$, we obtain the desired result.

(c) We have

$$p^* = \min_x \; \max_{\lambda \geq 0} \; f(x) + \lambda g(x) - \kappa\lambda.$$

We can apply strong duality, since 0 is strictly feasible (from $g(0) \leq 0 < \kappa$). This proves the result, which in turn shows that the function p^* is convex for $\kappa > 0$, as the pointwise maximum of affine functions.

(d) Using the same reasoning as above, a sub-gradient of p^* at κ is $-\lambda^*$.

3. (20 points) *Square-root LASSO via LASSO.* We consider the LASSO problem:

$$\phi(t) := \min_x \frac{1}{2}\|Ax - b\|_2^2 + t\|x\|_1, \tag{1}$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and $t > 0$. We consider also the so-called "square-root LASSO":

$$\eta(\lambda) := \min_x \|Ax - b\|_2 + \lambda\|x\|_1, \tag{2}$$

where $\lambda > 0$.

There are many off-the-shelf packages available to solve the LASSO problem, but much less choices for the square-root version. This motivates us to find a way to solve the square-root LASSO by making use of a LASSO code.

(a) Show that ϕ is concave on its domain, $\mathbf{R}_{++}$.

(b) Show that the function $\psi : \mathbf{R}_{++} \to \mathbf{R}$ with values for $t > 0$ given by $\psi(t) = \phi(t)/t$, is convex. *Hint:* use the result of problem 1.

(c) Show that, for every $\lambda > 0$, we have

$$\eta(\lambda) = \min_{t>0} t/2 + \lambda\psi(\lambda t).$$

Is the above problem convex?

(d) To solve the above one-dimensional problem, we can use bisection, which requires the knowledge of an upper bound on any t that is optimal for the above. Show that an upper bound on t can be set to be $1 + 2\phi(\lambda)$.

Solution:

(a) The function ϕ is the point-wise minimum of affine functions (of t), and its domain is convex, hence it is concave.

(b) For $t > 0$, we have

$$\psi(t) = \min_x \frac{\|Ax - b\|_2^2}{2t} + \|x\|_1.$$

From problem 1, and using the composition with affine mappings rule, the first term is *jointly* convex in (x, t), hence its partial minimum (over x) is also convex.

(c) We have

$$\min_{t>0} t/2 + \lambda\psi(\lambda t) = \min_{x,\, t>0} t/2 + \frac{\|Ax - b\|_2^2}{2t} + \|x\|_1$$

Minimizing over t yields the desired result, since, for any $\alpha \in \mathbf{R}$, we have:

$$\min_{t>0} t/2 + \alpha^2/2t = |\alpha|.$$

The resulting problem is convex, since ψ is.

(d) Evaluating the objective function at $t = 1$ leads to

$$\min_{t>0} t/2 + \lambda\psi(\lambda t) \leq 1/2 + \lambda\psi(\lambda) = 1/2 + \phi(\lambda).$$

This implies, in view of the non-negativity of ψ, that any optimal t satisfies

$$0 \leq t \leq 1 + 2\phi(\lambda).$$