EE 227BT 10/1/13
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Midterm: Solutions

1. (10 points) *Two vectors related to each other.* We are given two vectors $a, b \in \mathbf{R}^n$, with $b \neq 0$.

   (a) Show that the condition
   $$\forall\, x \ : \ a^T x = 0 \text{ implies } b^T x = 0$$
   is equivalent to the existence of a scalar λ such that $a = \lambda b$. *Hint*: use LP duality for an appropriately defined feasible LP. You may also use a direct linear algebra approach.

   (b) Show that $aa^T \succeq bb^T$ if and only if $a = \lambda b$, for some scalar λ such that $|\lambda| \geq 1$.

   Solution:

   (a) One direction is easy. Let us show that the condition implies $a = \lambda b$ for some $\lambda \in \mathbf{R}$. Consider the LP
   $$p^* = \min_x \ b^T x \ : \ a^T x = 0.$$
   If there exist x such that $a^T x = 0$ but $b^T x \neq 0$, then by scaling x appropriately we can get an unbounded objective, and $p^* = -\infty$. Otherwise, we have $b^T x = 0$ for any feasible point x, and thus $p^* = 0$. That is:
   $$p^* = \begin{cases} 0 & \text{if } b^T x = 0 \text{ whenever } a^T x = 0, \\ -\infty & \text{otherwise.} \end{cases}$$
   On the other hand, invoking strong duality for feasible LPs, we obtain
   $$p^* = d^* = \max_\lambda \ \min_x \ \left(b^T x - \lambda a^T x\right) = \begin{cases} 0 & \text{if } b = \lambda a \text{ for some } \lambda \in \mathbf{R}, \\ -\infty & \text{otherwise.} \end{cases}$$
   This proves the result.
   Another approach is based on linear algebra. The condition implies that the nullspace of the matrix b^T is included in that of the matrix a^T. Therefore the range of the transpose matrix b contains that of the matrix a. Since b (as a vector) is in the range of b (as a matrix), it is in the range of a, which is the desired result.

   (b) Again, one direction is obvious. To prove the converse, assume that $aa^T \succeq bb^T$. Then we have $(a^T x)^2 \geq (b^T x)^2$ for every x, which implies that the condition
   $$\forall\, x \ : \ a^T x = 0 \text{ implies } b^T x = 0$$
   holds. Therefore, we can write $a = \lambda b$ for some $\lambda \in \mathbf{R}$; plugging into the condition $aa^T \succeq bb^T$ leads, with $b \neq 0$, to $|\lambda| \geq 1$.

2. (10 points) *A result on linear inequalities.* For a matrix $A \in \mathbf{R}^{m \times n}$, and vector $b \in \mathbf{R}^m$, show that exactly one of the two following statements must hold:

(i) There exists a vector $x \in \mathbf{R}^n$, with $x \geq 0$ and $Ax = b$.

(ii) There exists a vector $s \in \mathbf{R}^m$ such that $A^T s \geq 0$ and $s^T b < 0$.

One direction is easy. For the other, you will develop one of the following two approaches, with double points if you correctly develop the two solutions.

- *Strict separation approach*: you will invoke the strict separation theorem given in Example 2.20 of [BV,p.49]: Let C be a closed convex set and $D = \{x_0\}$ a singleton that does not intersect with C: $x_0 \notin C$. Then there exists a an affine function that is positive on C and negative on D.
- *LP duality approach*: you will invoke the LP duality theorem for an appropriately defined LP. The theorem states that if an LP is feasible, then its value equals that of its dual.

Solution: The direction $(i) \implies \neg(ii)$ is easy, since (i) implies that $s^T b = s^T Ax$ for any s. So, if $A^T s \geq 0$, we then must have $s^T b = s^T Ax = (A^T s)^T x \geq 0$.

For the converse: $\neg(i) \implies (ii)$, we can have two approaches. One is based on the separation theorem, precisely the strict separation theorem given in Example 2.20 of [BV,p.49]. Assume that (i) does not hold. This means that the two sets $C := \{Ax : x \geq 0\}$ and $D := \{b\}$ do not intersect. C is convex and closed (as it is described as a set of linear, non-strict inequalities); D is a singleton. Since $b \notin C$, there exists a non-zero vector $s \in \mathbf{R}^m$ and a scalar v such that the affine function $y \to s^T y + v$ is positive on C, and negative on D. The first condition is the same as $s^T Ax + v > 0$ for every $x \geq 0$, which in turn requires $A^T s \geq 0$; otherwise the minimum value of the function over $x \geq 0$ would be $-\infty$. Applying the first condition to $x = 0$ we further observe that $v > 0$. The second condition then implies that $s^T b < -v < 0$.

Another approach is based on LP duality. Consider the LP

$$p^* = \min_s \; s^T b \; : \; A^T s \geq 0.$$

The problem is feasible, hence strong duality holds, and $p^* = d^*$, with

$$d^* = \max_{x \geq 0} \min_s \; s^T b - x^T A^T s = \begin{cases} 0 & \text{if } Ax = b \text{ for some } x \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(In the above we have used that $\min_s s^T z = -\infty$, unless $z = 0$.) If (i) does not hold, then we must have $d^* = -\infty$, so that there must exist s with $A^T s \geq 0$, and $s^T b < 0$.

3. (10 points) *Convexity of some functions.* Show that the following functions are convex.

(a) $f : \mathbf{R}^n \to \mathbf{R}$, with domain $\mathbf{dom}\, f = \mathbf{R}^n$, and values

$$f(x) = \log(\sum_{i=1}^{n} e^{x_i}).$$

Hint: use the epigraph condition.

(b) $f : \mathbf{R}^n \to \mathbf{R}$, with domain $\mathbf{dom}\, f = \mathbf{R}^n$, and values

$$f(x) = \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

Is f a norm? Proof or counterexample.

(c) The function $h : \mathbf{R}_{++} \times \mathbf{R}_{++} \to \mathbf{R}$, with values

$$h(x, y) = \max_{u>0,\, v>0} \min_{t>0} -xu - yv + 2\sqrt{u + t^2} + 2\sqrt{v + t^2} - 2t.$$

Can you find a closed-form expression for h? Make sure to justify any use of strong duality.

Solution:

(a) The condition $f(x) \le t$, for $t \in \mathbf{R}$, is equivalent to

$$g(x, t) := \sum_{i=1}^{n} e^{x_i - t} \le 1,$$

The function g is *jointly* convex on $\mathbf{R}^{n+1}$, as it is the sum of the exponential function composed with an affine function of (x, t), $(x, t) \to x_i - t$.

(b) The function is a sum of the absolute value composed with affine functions of x, $x \to x_{i+1} - x_i$. The function is *not* a norm; it does satisfy the triangle inequality, but not the definiteness condition that $f(x) = 0$ if and only if $x = 0$.

(c) The function $\mathcal{L} : \mathbf{R}^5_{++} \to \mathbf{R}$, with values

$$\mathcal{L}(x, y, t, u, v) = -xu - yv + 2\sqrt{u + t^2} + 2\sqrt{v + t^2} - 2t$$

is jointly convex with respect to (x, y, t). Hence

$$g(x, y, u, v) = \min_t \mathcal{L}(x, y, t, u, v)$$

is jointly convex with respect to (x, y). Thus,

$$\max_{u>0,\, v>0} g(x, y, u, v)$$

is convex with respect to (x, y), as claimed.

To find a closed-form expression for h, we use the fact that for every $x > 0$, $t > 0$, we have

$$\max_{u>0} -xu + 2\sqrt{u+t^2} = -x(\frac{1}{x^2} + t^2) + 2\frac{1}{x} = \frac{1}{x} - xt^2$$

Hence,

$$\begin{aligned} h(x,y) &= \min_{t>0} \max_{u>0,\, v>0} -xu - yv + 2\sqrt{u+t^2} + 2\sqrt{v+t^2} - 2t \\ &= \min_{t>0} \frac{1}{x} + \frac{1}{y} + t^2(x+y) - 2t \\ &= \frac{1}{x} + \frac{1}{y} - \frac{1}{x+y}. \end{aligned}$$

Note that the convexity of h is not immediately obvious.