EE227BT Discussion Section #4

Exercise 1 (Minimax Principle for Eigenvalues) For a symmetric matrix $M \in \mathbb{S}^n$ denote its eigenvalues by $\lambda_1(M) \geq \ldots \geq \lambda_n(M)$. Let $A, B \in \mathbb{S}^n$ be a symmetric matrices.

1. Let $\mathcal{M}$ be any k-dimensional subspace of $\mathbb{R}^n$. Show that there exist unit vectors $x, y \in \mathcal{M}$ such that $x^\top Ax \leq \lambda_k(A)$ and $y^\top Ay \geq \lambda_{n-k+1}(A)$.

2. Show that

$$\lambda_k(A) = \max_{\substack{\mathcal{M} \subseteq \mathbb{R}^n \\ \dim \mathcal{M} = k}} \min_{\substack{x \in \mathcal{M} \\ \|x\|_2 = 1}} x^\top Ax$$
$$= \min_{\substack{\mathcal{M} \subseteq \mathbb{R}^n \\ \dim \mathcal{M} = n-k+1}} \max_{\substack{x \in \mathcal{M} \\ \|x\|_2 = 1}} x^\top Ax$$

3. Show that $A \preceq B \;\Rightarrow\; \lambda_k(A) \leq \lambda_k(B)$

4. For any nondecreasing function $f : \mathbb{R} \to \mathbb{R}$ show that

$$A \preceq B \;\Rightarrow\; \operatorname{tr} f(A) \leq \operatorname{tr} f(B)$$

5. Show that

$$\sum_{j=1}^{k} \lambda_j(A) = \max_{\substack{\{x_1,\ldots,x_k\} \\ \text{orthonormal}}} \sum_{j=1}^{k} x_j^\top A x_j$$

6. Let $(A_{[11]}, \ldots, A_{[nn]})$ be the vector $(A_{11}, \ldots, A_{nn})$ sorted in non increasing order. Show that

$$\sum_{j=1}^{k} \lambda_j(A) \geq \sum_{j=1}^{k} A_{[jj]}, \quad \text{for } k = 1, \ldots, n$$

7. Show that

$$\sum_{j=1}^{k} \lambda_j(A+B) \leq \sum_{j=1}^{k} \lambda_j(A) + \sum_{j=1}^{k} \lambda_j(B)$$

Exercise 2 (Ordering Athletes) We order $N \cdot M$ athletes in a rectangular formation with N rows and M columns. From each column we choose the tallest athlete and among them we pick the shortest, whom we call A. From each row we choose the shortest athlete and among them we pick the tallest, whom we call B. There can be athletes with the same height, and in this case we pick an arbitrary athlete among them.

1. Give an example with $N = M = 2$, for which the height of B is strictly less than the height of A.

2. Give an example with $N = M = 2$, for which the height of B is equal to the height of A.

3. In general prove that A is at least as tall as B.