Decentralized Optimal Control

- Many critical infrastructures, such as electric grids, or transportation networks, require decentralized policies to efficiently implement control schemes.
- Control is often hampered by partial observability and lack of coordination/communication.
- Rather than solving a central optimization in real-time, treat agent $i$’s optimal action problem as a random variable $u_i^*$ that can be fully decentralized.
- $u_i^*$ depends on the global state variables $x = (x_1, \ldots, x_n)$, i.e. we consider $p(u_i^* | x)$.
- View decentralization as a compression problem, by letting a policy only depend on $x_i$.
- Apply classical results from information theory to analyze performance limits based on fundamental limits of compression, which is well formulated as an instance of rate distortion theory.

Rate Distortion Theory

Consider $X$ an information source and $Y$ a reconstruction after transmission over a noisy channel.

If mutual information $I(X; Y) \leq R$ then for distortion function $d(x, y)$ the minimum average distortion $D(R)$ solves a convex program.

$$D(R) = \min_{p(x | y)} \mathbb{E}[d(X, Y)], \quad \text{s.t.} \quad I(X; Y) \leq R$$

$R$ is called the “rate” of information transfer, and $D(R)$ gives a lower bound on average distortion.

Graphical Model Representation

Each agent’s optimal control $u_i^*$ depends on all state variables $x$ but the control applied in practice can only depend on local information $x_i$.

Data Processing Inequality

Suppose $X - Y - Z$ form a Markov chain. Then the following constraint holds:

$$I(X; Z) \leq I(X; Y)$$

In decentralized optimal control, the DPI applies between controls and state variables:

$$I(\hat{u}_i; u_j^*) \leq I(x_i; u_j^*) \quad \text{and} \quad I(\hat{u}_i; \hat{u}_j) \leq I(x_i; x_j)$$

Application of Rate Distortion

In distributed control, rate $R$ is effectively specified by the information contained in local state variables. We formulate the problem as:

$$D^* = \min_{p(u | X)} \mathbb{E}[d(u^*, \hat{u})], \quad \text{s.t.} \quad I(\hat{u}_i; u_j^*) \leq I(x_i; u_j^*)$$

- Constraints depend upon the problem structure.
- Mutual information must be estimated from data.
- Problem is convex in function space, but not necessarily convex for particular parameterizations.
- Solution is non-prescriptive, i.e. does not specify precisely how to perform minimum-distortion decoding (control).

For the Gaussian case with squared error distortion, there is a closed form solution for $D^*$ and a corresponding optimal decoder:

$$D^* = \sum_i \sigma_i \left(1 - \rho_i \sigma_i \right) \quad \text{and} \quad \hat{u}_i = \mathbb{E}[u_i^*] + \rho_i \sigma_i (x_i - \mathbb{E}[x_i])$$

Communication Strategy

Suppose controller $i$ is allowed to observe $k$ other state variables $S_i$.

**Theorem**

Setting $S_i$ as follows minimizes $D^*$ for any distortion $d$:

$$S_i = \arg \max I(u_i^*; x_i; \{x_j : j \in S\}) \mid |S| = k$$

The proof follows from the monotonicity of $D(R)$.

**Case: Optimal Power Flow (OPF)**

- We address voltage variability in an electric grid $\mathcal{G} := (\mathcal{N}, \mathcal{E})$ due to intermittent renewable energy.
- Optimize reactive power outputs $q_i^k, i \in \mathcal{C}$ of a set of distributed energy resources (DERs), $\mathcal{C}$.
- Adhere to physics of power flow and constraints due to energy capacity and safety.

$$u^* = \arg \min_{u_i^* \forall i \in \mathcal{N}} \sum_{i \in \mathcal{N}} [v_i - v_{\text{ref}}],$$

$$\text{s.t.} \quad p_{ij} = \sum_{(j,k) \in \mathcal{E}, j \neq i} P_{jk} + p_{ij} - p_{ij}^k$$

$$Q_{ij} = \sum_{(j,k) \in \mathcal{E}, j \neq i} Q_{jk} + q_{ij} - q_{ij}^k$$

$$e_j = v_i - 2 (r_{ij} P_{ij} + c_{ij} Q_{ij})$$

$$|q_i| \leq q_i, \quad q_i \leq v_i \leq p_i.$$

- Collect data set of offline simulations and optimize $u_i^*$.
- Generalize $u_i^*$ for all DER using an interpretable stepwise regression model $\hat{\pi}_i(x_i) = \hat{\theta}^T_i \phi_i(x_i)$.
- Use communication strategy to improve performance with additional variables $S_i$.

Voltage Variability Result

Communication Strategies