Midterm exam 1

CS70, Blum/Wagner, 9 March 2001

This is a CLOSED BOOK examination. One page of notes is permitted. Calculators are permitted. Do all your work on the pages of this examination. Give reasons for all your answers.

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

Problem 1. (Short answer grab-bag) [10 points]

For parts (a)–(c), justify your answer briefly.

(a) [2 points] True or false: $(P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$ always holds, for all propositions P, Q.

(b) [3 points] True or false: $((P \vee Q) \Rightarrow Q) \Rightarrow (Q \Rightarrow (P \vee Q))$ always holds, for all propositions P, Q.

(c) [3 points] Recall that two sets S, T are said to be *disjoint* if $S \cap T = \emptyset$. Can two events A and B be simultaneously independent and disjoint?

(d) [2 points] Recall that two random variables X and Y are *independent* just if the pair of events $X = i$ and $Y = j$ are independent no matter how you choose the values i and j. Which of the following most accurately expresses the proposition that X and Y are *not* independent? (You do not need to justify your answer.)

(i) $\forall i. \forall j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i] \Pr[Y = j]$

(ii) $\forall i. \exists j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i] \Pr[Y = j]$

(iii) $\exists j. \forall i. \Pr[X = i \wedge Y = j] \neq \Pr[X = i] \Pr[Y = j]$

(iv) $\exists i. \exists j. \Pr[X = i \wedge Y = j] \neq \Pr[X = i] \Pr[Y = j]$

Problem 2. (Permutations) [25 points]
Let S be the sample space of all permutations on the n letters $\{1, 2, \ldots, n\}$, with the uniform probability distribution. In each of the following, give reasons for all your answers. (You may find the next page, which has been left blank, useful for justifying your answers.)

(a) [1 point] The uniform probability distribution assigns to every permutation on n letters the probability: _____________

Define the random variable X_i to be the number of cycles of length i, i.e., X_i maps each permutation to an integer equal to the number of cycles in that permutation that are of length i.

(b) [1 point] For the permutation (124)(36)(5)(7) on $n = 7$ letters:

$X_1 =$ _____________, $X_2 =$ _____________, $X_3 =$ _____________, $X_4 =$ _____________.

For general positive integers n, give:

(c) [3 points] $E[X_1] =$ _____________.

(d) [7 points] $E[X_2] =$ _____________.

For an integer $k \in \{1, 2, \ldots, n\}$, what is

(e) [8 points] $E[X_k] =$ _____________.

Define the random variable X to be the number of cycles in a permutation. In other words, X maps any permutation to a positive integer equal to the number of cycles in that permutation.

(f) [1 point] For the permutation (124)(36)(5)(7), $X =$ _____________.

(g) [1 point] Is $X = X_1 + X_2 + \ldots + X_n$? _____________.

(h) [3 points] $E[X] =$ _____________.

Give reasons for all your answers below.

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Problem 3. (Tiling) [15 points]
Let D_n be the number of ways to tile a $2 \times n$ checkerboard with dominos, where a domino is a 1×2 piece. Prove that $D_n \leq 2^n$ for all positive integers n. (HINT: find a recurrence relation.)