CS 70 Discrete Mathematics for CS
Fall 2003 Wagner

Final

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your username on `cory.eecs`: ______________________________________

WRITE your section number (101 or 102): ___________________________

This exam is open-book, open-notes. *No calculators are permitted.* Do all your work on the pages of this examination. If you need more space, you may use the reverse side of the page, but try to use the reverse of the same page where the problem is stated.

You have 80 minutes. There are 6 questions, worth from 10 to 20 points each (100 points total). The questions are of widely varying difficulty, so avoid spending too long on any one question.

Do not turn this page until the instructor tells you to do so.

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total	

Problem 1. [Reasoning] (10 points)

(a) Prove or disprove: $((P \implies Q) \implies R) \implies ((R \implies Q) \implies P)$ is a tautology.

(b) Let A, B be finite sets, with $|A| = m$ and $|B| = n$. How many distinct functions $f : A \to B$ are there from A to B?

Problem 2. [Induction] (15 points)

Prove by induction: $5x \leq x^2 + 6$ for all $x \in \mathbf{N}$.

Problem 3. [Modular Arithmetic] (20 points)

(a) Let X be uniformly distributed on $\{0, 1, \ldots, 16\}$. What is $\Pr[2X + 7 \equiv 0 \pmod{17}]$?

(b) Let X be uniformly distributed on $\{0, 1, \ldots, 32\}$. What is $\Pr[2X + 7 \equiv 0 \pmod{33}]$?

(c) Let X be uniformly distributed on $\{0, 1, \ldots, 32\}$. What is $\Pr[3X + 12 \equiv 0 \pmod{33}]$?

(d) Let X be uniformly distributed on $\{0, 1, \ldots, 40\}$. What is $\Pr[X^2 + 40 \equiv 0 \pmod{41}]$?

Problem 4. [Random Variables] (20 points)

For each square of a 8×8 checkerboard, flip a fair coin, and color that square black or red according to whether you get heads or tails. Assume that all coin flips are independent.

A *same-color row* is a row on the board where all squares in the row have the same color (i.e., all red, or all black). Let the random variable X denote the number of same-color rows.

(a) If all coin tosses come up heads, what is the value of X?

(b) Calculate $\Pr[X = 0]$. (You do not need to simplify your answer.)

(c) Calculate $\mathbf{E}[X]$.

(d) Calculate $\mathrm{Var}[X]$.

(e) Show that $\Pr[X \geq 3] \leq 1/48$.

Problem 5. [Probability] (20 points)

A gambler has 4 coins in her pocket. Two are double-headed, one is double-tailed, and one is normal. The coins cannot be distinguished unless one looks at them.

(a) The gambler shuts her eyes, chooses a coin at random, and tosses it. What is the probability that the lower face of the coin is heads?

(b) She opens her eyes and sees that the upper face of the coin is a head. What is the probability that the lower face is a head?

(c) Now, after having seen that the upper face is a head, she shuts her eyes again, picks up the same coin, and tosses it a second time. What is the probability that the lower face is a head?

(d) After her second toss (as described in part (c)), she opens her eyes and sees that the upper face is a head. What is the probability that the lower face is a head?

Problem 6. [Counting] (15 points)

Choose a number uniformly at random between 0 and 999,999, inclusive. What is the probability that the digits sum to 19?