CS 70 Discrete Mathematics for CS
Fall 2000 Wagner

Final

You have three hours. The exam is open-book, open-notes.
There are XXX questions, YYY points total.

You should be able to finish all the questions, so avoid spending too long on any one question. Write your answers in the blue books. Check that you haven't skipped any part by accident. Hand in all your answers. Panic not.

1. **(?? pts.)** **A** fair 6-sided die is thrown repeatedly until a 1 appears. The expected number of throws is

   $1 \qquad 3 \qquad \frac{7}{2} \qquad 6 \qquad 36 \qquad \infty$

2. **(?? pts.)** **A** fair 6-sided die is thrown repeatedly until two *different* numbers appear. The expected number of throws is

   $\frac{6}{5} \qquad \frac{49}{40} \qquad \frac{3}{2} \qquad 2 \qquad \frac{11}{5} \qquad \infty$

3. **(?? pts.)** **A** certain casino game has a \$5 stake and three possible outcomes: with probability $\frac{1}{3}$ you lose your stake, with probability $\frac{1}{3}$ the bank returns your stake plus \$5, and with probability $\frac{1}{3}$ the bank simply returns your stake.

   (a) Let X denote your winnings in one play. The variance $\mathrm{Var}(X)$ is

   $0 \qquad \frac{10}{3} \qquad 5 \qquad \frac{50}{3} \qquad 25 \qquad \frac{125}{3}$

   (b) Now let σ^2 denote your solution to part (a), and suppose you play the game 10000 times. You would expect the *magnitude* of your win or loss to be approximately (circle the closest answer)

   $\sigma \qquad \sigma^2 \qquad 100\sigma \qquad 100\sigma^2 \qquad 10000\sigma \qquad 10000\sigma^2$

4. **(?? pts.)** **E**ach of n students is allocated a SID number drawn uniformly at random *with* replacement from a set of m numbers. The smallest value of m that will ensure that no two students get the same number with reasonably high probability is on the order of

   $\sqrt{n} \qquad n \qquad n\log n \qquad n^2 \qquad n^3 \qquad 2^n$

5. **(?? pts.)** **W**e have a group of five people. Each person picks, uniformly and independently at random, a number between 1 and 100. For each pair of distinct people i, j, let A_{ij} be the event "i, j both choose the same number", and for each triple of distinct people i, j, k, let A_{ijk} be the event "i, j, k all choose the same number." Circle each of the following pairs of events that are independent (assuming that the indices i, j, k, ℓ, m are all distinct):

   X_{ij} and $X_{k\ell}$ $\qquad$ X_{ij} and X_{ik} $\qquad$ X_{ijk} and X_{ij}

   X_{ijk} and $X_{i\ell}$ $\qquad$ X_{ijk} and $X_{ij\ell}$ $\qquad$ X_{ijk} and $X_{i\ell m}$

6. (?? pts.) **A** biased coin with Heads probability $\frac{3}{4}$ is tossed 100 times. Let the random variable H denote the number of Heads obtained, and T the number of Tails.

(a) The expectation of H is

$$\frac{3}{4} \qquad 1 \qquad 25 \qquad 50 \qquad 75 \qquad 100$$

(b) The expectation of $H - T$ is

$$\frac{1}{2} \qquad 0 \qquad 1 \qquad 25 \qquad 50 \qquad 75$$

7. (?? pts.) **L**et A and B be two arbitrary events in the same probability space. Which of the following statements *must always* be true about A and B?

$\Pr[A] > 0$ $\qquad \Pr[A \cup B] = \Pr[A] + \Pr[B]$ $\qquad \Pr[A \cup B] \leq \Pr[A] + \Pr[B]$

$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[A]}$ $\qquad \Pr[A|B] \leq \Pr[B]$ $\qquad \Pr[A \cap B] \leq \Pr[A]$

8. (19 pts.) Chris and Diane

Chris and Diane play different games (against a bank) as follows:

- Chris tosses a fair coin n times and wins if all tosses are heads;
- Diane tosses a fair n-sided die n times and wins if all tosses are different.

Answer the following questions.

(a) (5 pts) What is the probability that Chris wins, as a function of n?

(b) (5 pts) What is the probability that Diane wins, as a function of n? [Hint: How many points are there in Diane's sample space? How many of them correspond to wins for Diane?]

(c) (2 pts) For $n = 2$, who is more likely to win?

(d) (2 pts) For $n = 6$, who is more likely to win?

(e) (5 pts) When n is large (i.e., as $n \to \infty$), who is more likely to win? [Hint: You will need to use Stirling's approximation, which says that as $n \to \infty$, the value of n is approximately $(\frac{n}{e})^n\sqrt{2\pi n}$.]

9. (30 pts.) A lottery

n people take part in a lottery, in which there are a total of $2n$ tickets, n of which are winning tickets and the remaining n losing tickets. Each person buys one ticket, which is drawn uniformly at random from all those remaining. Let the random variable X denote the number of people who win a prize. Answer the following questions.

(a) (4pts) What is the sample space, and how many sample points does it contain?

(b) (6pts) In the special case $n = 2$, write down the distribution of the r.v. X, and compute its expectation $\mathrm{E}(X)$ and its variance $\mathrm{Var}(X)$.

(c) (5pts) Give a simple but rigorous argument to show that, for any value of n, the expectation $\mathrm{E}(X)$ is exactly $\frac{n}{2}$. [Hint: Write $X = \sum_{i=1}^{n} X_i$, for simpler r.v.'s X_i.]

(d) (3pts) Consider any two particular people. What is the probability that both of these people buy a winning ticket, as a function of n? Justify your answer carefully.

(e) (6pts) Now compute the variance $\mathrm{Var}(X)$, as a function of n. [Hint: Use the same representation $X = \sum_{i=1}^{n} X_i$; use part (d) to compute $\mathrm{E}(X_iX_j)$ for $i \neq j$. Check your answer for $n = 2$ against the value you computed in part (b).]

(f) (6pts) Now suppose that $n = 100$. Use Chebyshev's inequality to compute an upper bound on the probability that 60 or more people win a prize.