CS 70 Fall 2003 — Discrete Mathematics for CS — Wagner — HW 10

Due Thursday November 20

1. **(5 pts.) Any questions?**
Is there anything you'd like to see explained better in lecture or discussion sections?

2. **(10 pts.) Games**
Check out the following game: Alice and Bob will each roll a fair, six-sided die. If Alice's die comes up with a number higher than Bob's, Alice wins \$3 from Bob. If Bob's number comes up higher, or if they tie, Bob wins \$2 from Alice. Is this game a good deal for Alice? Explain.

3. **(20 pts.) Indicator variables**
An *indicator variable* I is a random variable that takes on only the values 0 or 1.

   (a) If I is an indicator variable, show that $\mathbb{E}[I] = \Pr[I = 1]$.

   Suppose we toss a fair coin n times. Let $X_i = 1$ if the i-th toss comes up heads, and $X_i = 0$ otherwise. Let Y denote the total number of heads among all n tosses.

   (b) Calculate $\mathbb{E}[X_i]$.

   (c) Express Y in terms of $X_1, \ldots, X_n$.

   (d) Calculate $\mathbb{E}[Y]$. Your answer should be a simple function of n.
   HINT: Use linearity of expectation.

   (e) Use part (d) to prove the following identity:

$$\sum_{i=1}^{n} i \cdot \binom{n}{i} = n \cdot 2^{n-1}.$$

4. **(10 pts.) St. Pedersburg Paradox**
Toss a fair coin repeatedly until it comes up heads; then stop. If it first comes up heads on the i-th toss, you win \$$2^i$. Let X denote how many dollars you win after playing this game once. Calculate $\mathbb{E}[X]$.

5. **(10 pts.) Quadruply-repeated ones**
We say that a string of bits has k *quadruply-repeated ones* if there are k positions where four consecutive 1's appear in a row. For example, the string 0100111110 has two quadruply-repeated ones.

   What is the expected number of quadruply-repeated ones in a random n-bit string, when $n \geq 3$ and all n-bit strings are equally likely? Justify your answer.

6. (20 pts.) Chopping up DNA

(a) In a certain biological experiment, a piece of DNA consisting of a linear sequence (or string) of 4000 nucleotides is subjected to bombardment by various enzymes. The effect of the bombardment is to randomly cut the string between pairs of adjacent nucleotides: each of the 3999 possible cuts occurs independently and with probability $\frac{1}{500}$. What is the expected number of pieces into which the string is cut? Justify your calculation.

[Hint: Use linearity of expectation! If you do it this way, you can avoid a huge amount of messy calculation. Remember to justify the steps in your argument; i.e., do not appeal to "common sense."]

(b) Suppose that the cuts are no longer independent, but highly correlated, so that when a cut occurs in a particular place other cuts close by are much more likely. The probability of each individual cut remains $\frac{1}{500}$. Does the expected number of pieces increase, decrease, or stay the same? Justify your answer with a precise explanation.

7. (25 pts.) The martingale

Consider a *fair game* in a casino: on each play, you may stake any amount \$$S$; you win or lose with probability $\frac{1}{2}$ each (all plays being independent); if you win you get your stake back plus \$$S$; if you lose you lose your stake.

(a) What is the expected number of plays before your first win (including the play on which you win)?

(b) The following gambling strategy, known as the "martingale," was popular in European casinos in the 18th century: on the first play, stake \$1; on the second play \$2; on the third play \$4; on the kth play \$$2^{k-1}$. Stop (and leave the casino!) when you first win.

Show that, if you follow the martingale strategy, and assuming you have unlimited funds available, you will leave the casino \$1 richer with probability 1. [Maybe this is why the strategy is banned in most modern casinos.]

(c) To discover the catch in this seemingly infallible strategy, let X be the random variable that measures your maximum loss before winning (i.e., the amount of money you have lost *before* the play on which you win). Show that $\mathbb{E}[X] = \infty$. What does this imply about your ability to play the martingale strategy in practice?

(d) Colin and Diane enter the casino with \$10 and \$1,000,000 respectively. Both play the martingale strategy (leaving the casino either when they first win, or when they lack sufficient funds to place the next bet as required by the strategy). What is the probability that Colin wins? What is the probability that Diane wins?

8. (10 pts.) Bonus question

I think of two distinct real numbers between 0 and 1, but I do not reveal them to you. Next, I secretly flip a fair coin to pick one of these two numbers, and I give it to you. Is it possible to find a procedure for guessing whether you were shown the larger or the smaller of the two numbers, such that your guess is correct with probability strictly greater than $1/2$?